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Topology
A Geometric Approach

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Foreword

This book is an introduction to general and to geometric topology. It was the authors' intention to create a book which is as far as possible not reliant on texts from other branches of mathematics. Consequently the extensive Introduction (treated here as Chapter 0) collects together the basic concepts and facts from set theory, algebra, analysis and geometry which are essential to our own development. Nevertheless we do not recommend that the Introduction should be read in advance of the main text; rather it should be made use of as and when the need arises, by way of references from the main text, or via the index of terms.

Chapter 1, which is devoted to the elementary theory of metric spaces, is also of a distinctive character. It includes much material that is presumably known to the reader from courses in mathematical analysis and geometry. We present this material in an orderly fashion so as to have the required conceptual apparatus at our disposal.

In the following chapters we try to progress gradually from spaces close to intuition and with paradigm properties to spaces which are more and more general and abstract. Chapter 2 is dedicated to polyhedra, which are in a sense the simplest spaces to be studied in topology. It includes an account of the geometric and topological properties of simplices, the theory of simplicial complexes and their subdivisions, the theory of simplicial maps and an equivalent account of the theory of polyhedra based on cell complexes. In Chapter 3 we develop homotopy theory, a body of knowledge which is used in almost all branches of modern topology. In that chapter we also consider some theories for which homotopy is the natural tool; thus we consider the problem of extending continuous maps, fibration and covering theory, and the problem of lifting continuous maps; at this juncture we also develop the theory of the fundamental group.

Chapter 4 is devoted to the topology of Euclidean spaces; in a sense this is a continuation of Section 1.10. Amongst other things we prove here the classical theorems on the invariance of separation and the invariance of the interior point, we present an elementary introduction to the theory of position and describe a series of examples of sets and mappings which every mathematician ought to know. In Chapter 5 we are concerned with topological manifolds, that is spaces which are closely linked to the Euclidean ones. Particular attention is paid to the 2-dimensional manifolds, or surfaces, and their classification.

Chapter 6 continues Chapter 1 and is devoted to metric spaces. We expand our stock of operations on metric spaces and undertake a detailed analysis of separable spaces, complete spaces and continua. We also study two classes of spaces with a regular structure: absolute retracts and absolute neighbourhood retracts, and introduce and study the concept of dimension. Chapter 7 contains an elementary treatment of general topology. After introducing basic concepts, we consider operations on topological spaces and thoroughly study the class of compact spaces. We close the chapter by considering paracompact spaces and the metrizability of topological spaces.

In view of the book's intended scope we have tried to avoid excessive generalisation in the earlier sections of our text. Adoption of such an approach leads unavoidably to some repetition. Certain results enunciated for metric spaces could have been proved outright for arbitrary topological spaces. However, we prefer to prove them first in a particular case and then, when we reach Chapter 7, which is devoted to topological spaces, we just give the appropriate references or even repeat the proof. For instance, we prove Tietze's Theorem twice: first for metric spaces and then separately for topological spaces, since the first of these proofs can be carried out with a more modest conceptual apparatus.

Every section is given a two-part number a.b, where a is the chapter number and b is the number of the section in natural order in that chapter. The last two sections of each chapter are reserved for: supplements (with a label of the form a.S) and problems (carrying a label of the form a.P). The supplements contain historical, terminological and bibliographic comments and information about concepts and results for which room could not be allotted in the main body of text as they fall outside the book's scope but nevertheless deserve mention or more thorough discussion. Some of the problems are difficult and serve to encourage the reader to provide his own proofs. However, the exercises placed at the ends of all the sections are of a different character; these are easy (though not computational) and are there to test command of the material. Figures refer to the text, but never vice versa. The captions under the illustrations are sometimes simplified versions of the theorems being illustrated.

Basic results are stated as theorems, assertions, lemmas, corollaries and examples. Assertions are distinguished from theorems by their self-evidence which permits the omission of a proof. Lemmas have ancillary status only. Examples quite apart from their construction frequently contain a proof that the construction yields the appropriate properties. Each of the units mentioned carries a three-part number a.b.c, where a is the chapter number, b the number of the section in the chapter, whilst c is the position number within the section. Units within a supplement have labels of the form a.S.c and problems are labelled a.P.c. The symbol ■ signifies the end of a section of text headed by a three-part number. We place in square brackets reference numbers to other textbooks or monographs listed in the bibliography.

In closing we wish to express our thanks to all those who helped us write and publish this book. We are particularly indebted to K. Krzyżewski and M. Gałecki, who contributed very many apt remarks and corrections. J. Łysko's observations helped us to improve the exercises and problem sections.

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Chapter 0

Introduction

We assume that the reader is familiar with the basic facts and ideas of set theory, algebra, analysis and geometry. Some of these – especially those required for this book – are recalled here in concise form. The current chapter thus also fixes terminology and notation, and suggests background reading.

0.1. Set theory

Sets will usually be denoted by upper case letters, their *elements* by lower case letters. If the element a belongs to the set A, we write $a \in A$. The set of elements belonging to X which satisfy the predicate φ will be written $\{x \in X : \varphi(x)\}$. If every element of the set A is also an element of the set B, we say that A is a *subset* of B and we write $A \subset B$. If moreover $A \neq B$, then we say that A is a *proper subset* of B. The *empty* set $\emptyset$, i.e. the set with no elements, is a subset of every set. The relation $\subset$ is called *inclusion*. We say that the sequence of sets $A_1, A_2, \ldots$ is *increasing*, if $A_n \subset A_{n+1}$ for $n = 1, 2, \ldots$, and is *decreasing* if $A_{n+1} \subset A_n$ for $n = 1, 2, \ldots$

A *map* f of the set X into the set Y is written $f: X \to Y$; here X is known as the *domain set* and Y as the *codomain*. Maps into the set of real numbers are often referred to simply as *functions*. The *image* of a set $A \subset X$ under the map f, that is, the set of elements $y \in Y$ for which there exists $a \in A$ such that $f(a) = y$, is denoted by the symbol $f(A)$. The *inverse image* or *preimage* of the set $B \subset Y$ under the map f, that is, the set of those elements $x \in X$ for which $f(x) \in B$, is denoted by the symbol $f^{-1}(B)$.

For any two maps $f: X \to Y$ and $g: Y \to Z$ their *composition* or *superposition* is the map $gf: X \to Z$ defined by the formula $(gf)(x) = g(f(x))$, for $x \in X$. If $X \subset Y$ the map $i: X \to Y$ defined by the formula $i(x) = x$ for $x \in X$ is called the *inclusion map* of X into Y. In the special case when $X = Y$ this inclusion map is called the *identity map* and is denoted by the symbol id_X, or, simply, id, when there is no danger of misunderstanding. For every map $f: X \to Y$ and every set $A \subset X$ the map $(f|A): A \to Y$ defined by the formula $(f|A)(a) = f(a)$, for $a \in A$, is called the *restriction* of the map f to the set A. If $A \subset X$ and $f_0: A \to Y$, then every map $f: X \to Y$ for which $f|A = f_0$ is called an *extension* of the map f_0 to the set X.

A map which assigns distinct values to distinct arguments is called an *injective map*. A map $f: X \to Y$ for which $f(X) = Y$ is said to be *onto* Y, or is said to be *surjective*.

An injective map f of a set X onto a set Y is said to be *bijective* or a *one-to-one map*. A bijective map $f: X \to Y$ possesses an *inverse map* $f^{-1}: Y \to X$ determined uniquely by either of the conditions $f^{-1}f = \mathrm{id}_X$, $ff^{-1} = \mathrm{id}_Y$; this map is also bijective.

A family G of bijective maps of a set X onto itself is said to be a *transformation group* if G contains the identity map, if, for each map in G its inverse is in G, and if, further, the composition of any pair of maps of G is in G. Bijective maps of the finite set of natural numbers $1, 2, \ldots, n$ onto itself are called *n-element permutations*; evidently, they form a transformation group. Transformation groups are particular instances of *groups* which we shall consider in Section 0.2.

A class K of maps will be called a *transformation category* if for every pair of maps in K of the form $f: X \to Y$ and $g: Y \to Z$, K contains the composition $gf: X \to Z$ and for each map $f: X \to Y$ of K, the class K contains both of the identity maps id_X, id_Y. Elements of the class K are called *morphisms*; the sets X and Y for which there is a morphism $f: X \to Y$ are called the *objects* of the category K. Transformation categories are instances of *categories*. A knowledge of category theory helps to detect connections between various concepts.

When writing down the elements a of a set A we often assume that they are in a bijective correspondence with the *indices* t running through a fixed set T. The element corresponding to the index $t \in T$ will be denoted a_t and we shall write $A = \{a_t\}_{t \in T}$. When employing sets with distinguished indices, one should check whether the concepts being studied depend upon the way in which the elements correspond to the indices; but we often omit such a check when it is obvious that there is no dependence of this kind. A set with indices running through the set of natural numbers $\mathbf{N}$ will be called a *sequence* and instead of using the notation $\{a_n\}_{n \in \mathbf{N}}$ we shall usually write $\{a_n\}_{n=1,2,\ldots}$, or $\{a_n\}_{n=1}^{\infty}$, or $\{a_1, a_2, \ldots\}$, or even $\{a_n\}$. In the case of finite sets we make use of the following modified notation dictated both by tradition and practicality: the finite set whose elements are $a_1, a_2, \ldots, a_n$ with ordering determined by the natural correspondence with the order of the indices $1, 2, \ldots, n$ will be called a *finite sequence*, or *n-tuple*, denoted $(a_1, a_2, \ldots, a_n)$, or $(a_j)_{j=1,2,\ldots,n}$, or $(a_j)_{j=1}^{n}$. In particular a two element sequence will be called an *ordered pair* or just a *pair*. By contrast we will write $\{a_1, a_2, \ldots, a_n\}$, or $\{a_j\}_{j=1,2,\ldots,n}$, or even $\{a_j\}_{j=1}^{n}$, when we have in mind only the set whose elements are $a_1, a_2, \ldots, a_n$. In particular the set with just one element a will be denoted $\{a\}$.

In the case of a family of sets (all of which are subsets of a fixed set X) we apply the notational conventions above as for an ordinary set. Let $\mathcal{A} = \{A_t\}_{t \in T}$, where $A_t \subset X$ for $t \in T$. The set of all elements $a \in X$, for which there exists an index $t \in T$ such that $a \in A_t$, is called the *union* of the family $\mathcal{A}$, denoted $\bigcup_{t \in T} A_t$, or more briefly $\bigcup \mathcal{A}$. In particular the formula $\bigcup \{A : \varphi(A)\}$ denotes the union of all sets satisfying the predicate φ. The set consisting of all elements $a \in X$ such that $a \in A_t$ for every $t \in T$ is called the *intersection* or *common part* of the family $\mathcal{A}$, written $\bigcap_{t \in T} A_t$, or more briefly $\bigcap \mathcal{A}$. In particular the formula $\bigcap \{A : \varphi(A)\}$ denotes the intersection of all the sets satisfying the predicate φ. The set of all maps $a: T \to \bigcup_{t \in T} A_t$ with the property that $a(t) \in A_t$ for $t \in T$ is called the *Cartesian product* of the family $\mathcal{A}$ and is denoted $\bigtimes_{t \in T} A_t$. The element $a(t)$ is usually written a_t and is called the *t^{th}-coordinate* of the

element a in the product $\bigtimes_{t\in T} A_t$. We often identify the map a with the set $\{a_t\}_{t\in T}$, where $a_t = a(t) \in A_t$ for $t \in T$. For each $t_0 \in T$ the map $p_{t_0}\colon \bigtimes_{t\in T} A_t \to A_{t_0}$ defined by the formula $p_{t_0}(\{a_t\}) = a_{t_0}$ is called the *projection* of the Cartesian product onto its t_0^{th}-factor.

In the particular case of a sequence of sets $\{A_n\}$, where $n = 1, 2, \ldots$, we write $\bigcup_{n=1}^{\infty} A_n$ for the union, $\bigcap_{n=1}^{\infty} A_n$ for the intersection, and $\bigtimes_{n=1}^{\infty} A_n$ for the product. Finally, for a finite family of sets $A_1, A_2, \ldots, A_n$ we write $\bigcup_{j=1}^{n} A_j$, or $A_1 \cup A_2 \cup \ldots \cup A_n$ for the union, $\bigcap_{j=1}^{n} A_j$, or $A_1 \cap A_2 \cap \ldots \cap A_n$ for the intersection, and $\bigtimes_{j=1}^{n} A_j$, or $A_1 \times A_2 \times \ldots \times A_n$ for the product. The product $\bigtimes_{j=1}^{n} A_j$, when $A_j = A$ for $j = 1, 2, \ldots, n$ is called the n^{th} *power* of the set A and will be written $\bigtimes^n A$.

If $f\colon X \to Y$, then the set $\{(x, y) \in X \times Y : y = f(x)\}$ is called the *graph* of the map f. For each set $A = \{a_t\}_{t\in T}$ we may wish to consider the function $\delta\colon A \times A \to \{0, 1\}$, known as the *Kronecker function* or the *Kronecker delta*, which we write for practical reasons as δ_{st} or δ_s^t instead of $\delta(a_s, a_t)$; the function is defined by the formula:

$$\delta_{st} = \begin{cases} 0, & \text{when } t \neq s, \\ 1, & \text{when } t = s, \end{cases} \qquad \text{for all} \quad s, t \in T.$$

If $f_t\colon A_t \to B_t$ for every $t \in T$, then the map $\bigtimes_{t\in T} f_t\colon \bigtimes_{t\in T} A_t \to \bigtimes_{t\in T} B_t$, defined by the formula

$$\left(\bigtimes_{t\in T} f_t\right)(\{a_t\}) = \{f_t(a_t)\}$$

for $\{a_t\} \in \bigtimes_{t\in T} A_t$, is called the *product map* of the maps $\{f_t\}_{t\in T}$. In the special case of a sequence of maps $f_n\colon A_n \to B_n$, where $n = 1, 2, \ldots$, the product is written $\bigtimes_{n=1}^{\infty} f_n$. Finally, in the case of a finite sequence of maps $f_j\colon A_j \to B_j$ where $j = 1, 2, \ldots, n$, the product map is written $\bigtimes_{j=1}^{n} f_j$, or $f_1 \times f_2 \times \ldots \times f_n$. If $f_j = f$ for $j = 1, 2, \ldots, n$, the product $\bigtimes_{j=1}^{n} f_j$ is called the n^{th} *power* of the map f and is written $\bigtimes^n f$.

Sets A and B which have empty intersection are called *disjoint*. The set of all elements of A which do not belong to B is called the *difference* of A and B and is written $A \setminus B$; if $B \subset A$, then the difference $A \setminus B$ is called the *complement* of B in A. The well known *De Morgan's Laws* hold: $X \setminus \bigcap_{t\in T} A_t = \bigcup_{t\in T}(X \setminus A_t)$ and $X \setminus \bigcup_{t\in T} A_t = \bigcap_{t\in T}(X \setminus A_t)$ for any family of sets $\{A_t\}_{t\in T}$ in X.

A relation $\leq$ on a set A is called an *ordering* if it is *reflexive*, i.e. $a \leq a$ holds for any $a \in A$, *transitive*, i.e. the conditions $a \leq b$ and $b \leq c$ imply $a \leq c$, and *weakly antisymmetric*, i.e. the conditions $a \leq b$ and $b \leq a$ imply $a = b$. For example the inclusion relation is an ordering on the family of all subsets of a fixed set X. The set A equipped with an ordering relation $\leq$ is called an *ordered set*. A subset A_0 (not necessarily proper) of an ordered set A is called *linearly ordered* if for any two elements $a, b \in A_0$ we have either $a \leq b$ or $b \leq a$. An element $a \in A$ satisfying the condition $x \leq a$ for every element x of the subset A_0 of the ordered set A is called an *upper bound* of A_0 in A. We call an element $a \in A$ *maximal* if the conditions $x \in A$ and $a \leq x$ imply $a = x$. The following holds.

0.1.1. THEOREM (Kuratowski, Zorn). *If every linearly ordered subset A_0 of a non-empty ordered set A has an upper bound in A, then the set A has a maximal element.* ■

An element a of a linearly ordered subset A_0 of an ordered set A is called the *smallest (largest)* element of A_0, when $a \leq x$ ($x \leq a$) for every $x \in A_0$. If every non-empty subset of a linearly ordered set has a smallest element, then we say that the set A is *well-ordered*, and the ordering on the set is to be a *well-ordering*. The following holds.

0.1.2. THEOREM (Zermelo). *For every set there exists a relation which well-orders it.* ■

A relation R on a set A is called an *equivalence*, if it is reflexive, *symmetric*, i.e. such that the condition aRb implies bRA, and transitive. Every equivalence relation R on a set A determines a partition of the set into pairwise disjoint subsets called the *equivalence classes* of the relation R. Two elements $a, b \in A$ belong to the same equivalence class if and only if aRb. The equivalence class of the relation R which contains the element $a \in A$ will usually be denoted $[a]$, and any element of this class will be called a *representative* of the class.

Two sets A, B are called *equinumerous*, if there exists a bijective map from the set A onto B. The relation of equinumeracy is an equivalence, and the equivalence class of a set A is called its *power* or *cardinal number* and is denoted $\operatorname{card} A$. The power of a finite set is equal to the number of its elements. The power of the set of natural numbers is denoted by the symbol $\aleph_0$ (*aleph-zero*), and that of the real numbers by the symbol $\mathbf{c}$ (the *continuum*). We say that the cardinal number $\mathbf{m} = \operatorname{card} A$ is less than or equal to the cardinal number $\mathbf{n} = \operatorname{card} B$ and we write $\mathbf{m} \leq \mathbf{n}$, if there exists a bijective map from A onto a subset of B. The following fundamental result holds.

0.1.3. THEOREM (Cantor, Bernstein). *If* $\mathbf{m} \leq \mathbf{n}$ *and* $\mathbf{n} \leq \mathbf{m}$, *then* $\mathbf{m} = \mathbf{n}$. ■

A set whose power is not larger than $\aleph_0$ is called *countable*.

If $\mathbf{m} = \operatorname{card} A$ and $\mathbf{n} = \operatorname{card} B$, the cardinal number $\mathbf{mn} = \operatorname{card}(A \times B)$ is called the *product*, whilst assuming additionally that $A \cap B = \emptyset$, the cardinal number $\mathbf{m}+\mathbf{n} = \operatorname{card}(A \cup B)$ is called the *sum* of the cardinal numbers $\mathbf{m}$ and $\mathbf{n}$. The power of the family of all subsets of A, when $\operatorname{card} A = \mathbf{m}$, is denoted by the symbol $2^{\mathbf{m}}$. It may be proved that $2^{\aleph_0} = \mathbf{c}$.

A more thorough discussion of the concepts of set theory may be found in [11].

0.2. Algebra

By a *group* we understand an arbitrary set G on which is defined an operation of *multiplication* (assigning to elements $g, g' \in G$ their *product* $gg' \in G$) satisfying the *associativity* condition (viz. $g(g'g'') = (gg')g''$ for any $g, g', g'' \in G$), and in which there is a distinguished element $1 \in G$ known as its *unity* (also denoted by the symbols e or ϵ), satisfying $g1 = g$ for every $g \in G$ and having the property that for every $g \in G$ there exists an *inverse element* g^{-1} with $gg^{-1} = 1$. One checks that under these circumstances $1g = g$ and $g^{-1}g = 1$ for $g \in G$, and that the unity of G and the assignment of inverse elements are defined uniquely. The group consisting of only a unity element is called *trivial*.

If multiplication is *commutative* (viz. $gg' = g'g$ for all $g, g' \in G$) the group is called *commutative* or *abelian*. In the case of abelian groups additive notation is often used and then the operation of multiplication is termed *addition* and is denoted by $+$; the element $g + g'$ is then called the *sum* of g and g', the unity element is called the *zero*, denoted 0, and the inverse element to g is called its *negative* and is written $-g$.

A non-empty subset G_0 of the group G is called a *subgroup* of G, if $gg' \in G_0$ for every $g, g' \in G_0$ and $g^{-1} \in G_0$ for every $g \in G_0$. The subgroup G_0 is thus itself a group under the operation of multiplication in G.

The *direct product* $G \times H$ of the groups G, H is the group obtained by equipping the Cartesian product of G and H with an operation of multiplication defined by $(g,h)(g',h') = (gg', hh')$, using $(1,1)$ as the unity element, and letting $(g,h)^{-1} = (g^{-1}, h^{-1})$.

Associativity of group multiplication allows us to consider the product of an arbitrary finite number of elements (without the need of brackets). We make the convention that the empty set of elements has product 1. A subset T of a group G *generates* the group (and its elements are called *generators* of the group) if every element of G is a product of elements of T and/or their inverses. Any equation which has on its left hand side an irreducible product of generators and/or their inverses and on the right hand side the unity, is called a *relation* of the group. We sometimes extend this term to cover equations which have products of generators and/or their inverses on both sides. A group can be specified by giving its sets of generators and their relations. A group for which there are no relations (other than the trivial equation $1 = 1$) is called *free*.

If a group G has generators $\{g_1, g_2, \ldots, g_k\}$ and relations $\{R_1, R_2, \ldots, R_l\}$ and the group H has generators $\{h_1, h_2, \ldots, h_m\}$ and relations $\{S_1, S_2, \ldots, S_n\}$, then the group generated by $\{g_1, g_2, \ldots, g_k,\ h_1, h_2, \ldots, h_m\}$ with relations $\{R_1, R_2, \ldots, R_l, S_1, S_2, \ldots, S_n\}$ is called the *free product* of the groups G and H.

A map φ of a group G into a group H is called a *homomorphism* if $\varphi(gg') = \varphi(g)\varphi(g')$ for all $g, g' \in G$. To define a homomorphism of a group G into a group H it suffices to specify its values on the generators of G in such a way that the relations of G are carried to relations of H. An injective homomorphism is called a *monomorphism*; a homomorphism that is onto is called an *epimorphism*; a bijective homomorphism is called an *isomorphism*. For a homomorphism $\varphi: G \to H$ to be an isomorphism it is necessary and sufficient that there exists an inverse homomorphism (in fact, an isomorphism) $\varphi^{-1}: H \to G$ satisfying $\varphi\varphi^{-1} = \mathrm{id}_H$ and $\varphi^{-1}\varphi = \mathrm{id}_G$. If there exists an isomorphism of G onto H, then we say that the groups are *isomorphic*.

A *ring* is an abelian group R (presented in additive notation) in which a further commutative and associative operation of *multiplication* is defined and an element $1 \neq 0$ (known as the *unity*) is distinguished which satisfies $r1 = r$ for all $r \in R$. It is, moreover, understood that addition and multiplication are connected by the *distributivity* condition $(r + s)t = rt + st$ for all $r, s, t \in R$. A ring F in which every element $r \in F$ other than 0 possesses an inverse element r^{-1} such that $rr^{-1} = 1$, is called a *field*.

A *linear space* over a field F (whose elements are called *scalars*) is an abelian group V (presented in additive notation, whose elements are referred to as *vectors*) equipped

with a further operation, that of *multiplication* of vectors by scalars (which results in vectors). The distributivity conditions $(r+s)\alpha = r\alpha + s\alpha$, $r(\alpha+\beta) = r\alpha + r\beta$ for $r,s \in F$, $\alpha,\beta \in V$ are assumed; the unity of the field F satisfies $1\alpha = \alpha$ for every $\alpha \in V$; multiplication in the field F and multiplication of vectors by scalars are connected by the condition $r(s\alpha) = (rs)\alpha$ for $r,s \in F$, $\alpha \in V$. A subset A of a linear space V over the field $\mathbf{R}$ of reals is *convex* if for every pair of vectors $\alpha,\beta \in A$ and for every real number $0 \leq r \leq 1$ we have $(1-r)\alpha + r\beta \in A$.

More details on the subject of groups, rings and fields may be found in [12].

A *Boolean algebra* is an arbitrary set A equipped with two operations: *addition*, $\cup$, and *multiplication*, $\cap$, and has two distinguished elements: *zero*, $\vee$, and *unity*, $\wedge$, moreover to every element $a \in A$, corresponds a *complement* $-a \in A$, and the following axioms are satisfied: (1) $a \cup b = b \cup a$, (2) $a \cup (b \cup c) = (a \cup b) \cup c$, (3) $a \cup \vee = a$, (4) $a \cup (-a) = \wedge$, (5) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ for all $a,b,c \in A$ and also the dual axioms obtained from (1)–(5) by interchanging the symbols $\cup$ and $\cap$ and the symbols $\vee$ and $\wedge$. An *ordering* relation $\leq$ may be introduced into a Boolean algebra by taking $a \leq b$ whenever $a \cap b = a$. Two Boolean algebras are *isomorphic*, when there exists a bijective map from one to the other which preserves the operations $\cup$ and $\cap$.

0.3. Analysis

We assume known the basic properties of the set $\mathbf{R}$ of real numbers in respect of the operations of addition and multiplication and the usual order. In particular we recall that if the set $A \subset \mathbf{R}$ has an upper bound then it has a least upper bound known as the *supremum* written $\sup A$. If the set A does not have an upper bound, we take $\sup A = \infty$. The *infimum*, $\inf A$, of a set $A \subset \mathbf{R}$ is defined similarly.

For every function $f\colon X \to \mathbf{R}$ the number (or symbol ∞) $\sup f(X)$ is called its *supremum* and is written $\sup f$. We define the *infimum* of the function f, $\inf f$, analogously. If the supremum (or infimum) of the function f lies in $f(X)$ we say that the function f *achieves its supremum* (or *infimum*).

Let a and b be real numbers or the symbols $-\infty$ or $+\infty$ and let $a \leq b$. (We conventionally assume that for all real numbers r we have $-\infty < r < +\infty$ and that $-\infty < +\infty$). We fix the following notation

$$[a,b] = \{r \in \mathbf{R} : \ a \leq r \leq b\}, \quad \text{for} \quad a,b \in \mathbf{R},$$

$$[a,b) = \{r \in \mathbf{R} : \ a \leq r < b\}, \quad \text{for} \quad a \in \mathbf{R},\ b \in \mathbf{R} \cup \{+\infty\},$$

$$(a,b] = \{r \in \mathbf{R} : \ a < r \leq b\}, \quad \text{for} \quad a \in \mathbf{R} \cup \{-\infty\},\ b \in \mathbf{R},$$

$$(a,b) = \{r \in \mathbf{R} : \ a < r < b\}, \quad \text{for} \quad a \in \mathbf{R} \cup \{-\infty\},\ b \in \mathbf{R} \cup \{+\infty\}.$$

If $a,b \in \mathbf{R}$, then each of the sets above is called an *interval with endpoints a and b*. The first of these intervals is *closed*, the second is *closed on the left*, the third is *closed on the right* and the fourth is *open*. The interval $[a,a] = \{a\}$ is said to be *degenerate*.

If $a \in \mathbf{R}$, each of the sets $[a, +\infty)$ and $(a, +\infty)$ is called a *half-line with endpoint* a; the former is a *closed* half-line the latter is *open*. If $b \in \mathbf{R}$ then each of the sets $(-\infty, b]$, $(-\infty, b)$ is called a *half-line with endpoint* b; the former is a *closed* half-line the latter is *open*. Evidently we also have $(-\infty, +\infty) = \mathbf{R}$.

From among the intervals and half-lines we single out the closed interval $[0, 1]$ which we will call the *unit interval*, to be denoted by the symbol I and the half-line $[0, +\infty)$ to be denoted by the symbol $\mathbf{R}_+$.

We also assume that the reader is familiar with *complex numbers*; in particular we shall be using the *modulus* $|c| = \sqrt{a^2 + b^2}$ and the *complex conjugate* $\bar{c} = a - bi$ of the complex number $c = a + bi$.

We shall be freely making use of the *convergence of sequences* and *series* of numerical terms and in the examples and supplements we shall avail ourselves of *derivatives* and *integrals*.

All the required information on this subject may be found in any text on mathematical analysis, for example see [13].

0.4. Geometry

If m is any positive integer, the set of all finite sequences of real numbers with m terms is called *m-dimensional Euclidean space* and is denoted $\mathbf{R}^m$. The 1-dimensional Euclidean space $\mathbf{R}^1$ is also called the *Euclidean line*. Since a finite sequence consisting of one number can be identified with that number, we may regard the Euclidean line $\mathbf{R}^1$ as being the set of real numbers $\mathbf{R}$; it is in this sense that we talk of the *real line* $\mathbf{R}$. We also refer to the 2-dimensional Euclidean space $\mathbf{R}^2$ as the *Euclidean plane*. For convenience we also consider the 0-dimensional Euclidean space $\mathbf{R}^0$ which we take to be a one-element set consisting of the empty sequence of real numbers.

The elements of the Euclidean space $\mathbf{R}^m$ are called the *points* of the space. If $x = (x^1, x^2, \ldots, x^m) \in \mathbf{R}^m$, the numbers $x^1, x^2, \ldots, x^m$ are called the *coordinates* of the point x. Let us observe that the superscripts do not denote exponentiation, but are the coordinate indices. The advantages of such a notation become clear when we have to index a sequence of points; we then use the appropriate subscript to denote a point.

Let $x = (x^1, x^2, \ldots, x^m)$, $y = (y^1, y^2, \ldots, y^m) \in \mathbf{R}^m$ and let r be a real number. We define the points:

$$x + y = (x^1 + y^1, x^2 + y^2, \ldots, x^m + y^m),$$
$$rx = (rx^1, rx^2, \ldots, rx^m).$$

The point $x + y$ is called the *sum* of the points x and y while the point rx is called the scalar *multiple* of the point x by r. We also put $-x = (-1)x$ and instead of $x + (-y)$ we write $x - y$. The point $-x$ is called the *negative* of the point x and $x - y$ is called the *difference* of x and y. The point $(0, 0, \ldots, 0) \in \mathbf{R}^m$ is called the *origin of the coordinate system* of the Euclidean space $\mathbf{R}^m$ and is written 0. Instead of $\left(\frac{1}{r}\right) x$, where $r \neq 0$, we also write $\frac{x}{r}$ and call this the *quotient* of the point x by the number r.

The following is a direct consequence of the definitions above.

0.4.1. ASSERTION. *For all points $x, y, z \in \mathbf{R}^m$ and all real numbers $r, s \in \mathbf{R}$ the following equations hold:*

(1) $(x+y)+z = x+(y+z)$,
(2) $x+0 = x$,
(3) $x+(-x) = 0$,
(4) $x+y = y+x$,
(5) $(r+s)x = rx+sx$,
(6) $r(x+y) = rx+ry$,
(7) $1x = x$,
(8) $r(sx) = (rs)x$. ■

In other words, the Euclidean space $\mathbf{R}^m$ is a linear space over the field $\mathbf{R}$.

When adding a larger number of points in place of the expression $x_1+x_2+\ldots+x_n$ we shall also use the notation $\sum_{j=1}^n x_j$. The sum $\sum_{j=1}^n r^j x_j$, where $x_j \in \mathbf{R}^m$ and $r^j \in \mathbf{R}$ for $j = 1, 2, \ldots, n$, is called the *linear combination of the points $x_1, x_2, \ldots, x_n$ with coefficients $r^1, r^2, \ldots, r^n$*.

Let $x = (x^1, x^2, \ldots, x^m)$, $y = (y^1, y^2, \ldots, y^m) \in \mathbf{R}^m$. The number $x \cdot y = \sum_{i=1}^m x^i y^i$ is called the *scalar product* of x and y. A consequence of the definition is the following.

0.4.2. ASSERTION. *For arbitrary $x, y, z \in \mathbf{R}^m$ and any number $r \in \mathbf{R}$ we have the equations:*

(1) $x \cdot y = y \cdot x$,
(2) $(x+y) \cdot z = x \cdot z + y \cdot z$,
(3) $(rx) \cdot y = r(x \cdot y)$,
(4) $x \cdot x > 0$ if $x \neq 0$. ■

The properties listed in Assertions 0.4.1 and 0.4.2 permit the development of a calculus of points and the use of rules which from the formal point of view are identical with the rules of arithmetic.

Instead of $x \cdot x$ we shall write x^2. The number $\|x\| = \sqrt{x^2}$ is called the *norm* of the point $x \in \mathbf{R}^m$. From this definition we have the following.

0.4.3. ASSERTION. *For any point $x \in \mathbf{R}^m$ and any number $r \in \mathbf{R}$ we have:*

(1) $\|x\| = 0$ *if and only if* $x = 0$,
(2) $\|rx\| = |r|\|x\|$. ■

A set of the form $\{(x^1, x^2, \ldots, x^m) \in \mathbf{R}^m\colon\ a \leq x^i \leq b \text{ for } i = 1, 2, \ldots, m\}$, where $a, b \in \mathbf{R}$ and $a \leq b$ will be called an *m-dimensional cube* and will be denoted $[a, b]^m$. The cube $[0, 1]^m$ will be called the *m-dimensional unit cube*, denoted I^m.

Let $c \in \mathbf{R}^m$ and $r > 0$. The set $B(c; r) = \{x \in \mathbf{R}^m : \|x - c\| < r\}$ is called the *m-dimensional open ball centred at c with radius r*. Replacing $<$ by $\leq$ we obtain the definition of an *m-dimensional closed ball $\bar{B}(c; t)$ centred at c with radius r*. Finally, replacing the inequality by an equation we obtain the definition of an *$(m-1)$-dimensional sphere $S(c; r)$ centred at c with radius r*. The balls $B(0; 1)$, $\bar{B}(0; 1)$ and the sphere $S(0; 1)$ are known, respectively, as the *m-dimensional open unit ball*, the *m-dimensional closed unit ball* and the *$(m-1)$-dimensional unit sphere* and will be denoted by, respectively, B^m, $\bar{B}^m$ and S^{m-1}. 2-dimensional balls are also known as *discs* and the 1-dimensional spheres as *circles*.

A finite sequence of points $a_0, a_1, \ldots, a_n \in \mathbf{R}^m$ is said to be *affinely dependent* if there exists numbers $r^0, r^1, \ldots, r^n$, not all zero, such that $\sum_{j=0}^n r^j a_j = 0$ and $\sum_{j=0}^n r^j = 0$. Otherwise the sequence is *affinely independent*. Evidently, affine independence does

not depend on the order in which the points are listed and the property is inherited by every subsequence of the affinely independent sequence.

The following holds.

0.4.4. THEOREM. *In Euclidean space* $\mathbf{R}^m$ *every affinely independent set contains at most* $m+1$ *points; moreover, any such set may be extended to an affinely independent set consisting of precisely* $m+1$ *points.* ■

We say that the set $A \subseteq \mathbf{R}^m$ is in *general position*, if every set consisting of at most $m+1$ points of the set A is affinely independent.

If $a, b \in \mathbf{R}^m$ and $a \neq b$, the set of points of the form $x = (1-r)a + rb$, where $r \in \mathbf{R}$ is called a *line* through a and b. If the line L passes through the distinct points a and b, while the line L' passes through the distinct points a' and b', then we say that L' is *parallel* to L if $b' - a' = r(b-a)$ for some number r. It is readily verified that this definition does not depend on the choice of points a, b of L nor on the choice of a', b' on L' and that the relation of parallelism between lines is an equivalence. The equivalence classes of this relation are called *directions.*

A set $H \subset \mathbf{R}^m$ with the property that for every pair of distinct points of H the line through the two points lies in H, is called an *affine subspace.* Thus the empty set, single points, all lines and the whole space $\mathbf{R}^m$ are affine subspaces in $\mathbf{R}^m$.

0.4.5. ASSERTION. *The intersection of any family of affine subspaces in* $\mathbf{R}^m$ *is an affine subspace in* $\mathbf{R}^m$. ■

For any set $A \subset \mathbf{R}^m$ the intersection of all the affine subspaces containing A is called the *affine hull* of A and will be denoted $H(A)$.

0.4.6. THEOREM. *If* $A \subset \mathbf{R}^m$, *then* $H(A)$ *consists of all points* $x \in \mathbf{R}^m$ *of the form* $x = \sum_{j=0}^{n} r^j a_j$, *where* $a_j \in A$, $r^j \in \mathbf{R}$ *for* $j = 0, 1, \ldots, n$ *and* $\sum_{j=0}^{n} r^j = 1$. ■

0.4.7. THEOREM. *Any affine subspace is the hull of some affinely independent subset of points. Any two such subsets have an equal number of elements.* ■

The number, diminished by 1, of points in an affinely independent set whose hull is the affine subspace H is called the *dimension* of H, denoted $\dim H$. Thus the dimension of the empty set is -1, that of a single point is 0, and that of a line is 1. Affine subspaces of dimension 2 are called *planes.* We note the following theorem concerning the dimension of an affine subspace.

0.4.8. THEOREM. *If* H, H' *are affine subspaces and* $H' \subset H$ *then* $\dim H' \leq \dim H$; *if moreover* $H \neq H'$, *then* $\dim H' < \dim H$. ■

0.4.9. THEOREM. *The points* $a_0, a_1, \ldots, a_n$ *form an affinely dependent set, if and only if, they are contained in an affine subspace of dimension less than* n. ■

Affine subspaces of dimension $m-1$ in $\mathbf{R}^m$ may be described as the solution sets of single linear equations. These are called *hyperplanes* of $\mathbf{R}^m$.

The following in fact holds.

0.4.10. THEOREM. *A hyperplane of* $\mathbf{R}^m$ *is a set of points* $(x^1, x^2, \ldots, x^m)$ *satisfying an equation of the form*

$$\alpha_0 + \sum_{i=1}^{m} \alpha_i x^i = 0,$$

where the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m$ *are not all zero.* ■

If a hyperplane H of $\mathbf{R}^m$ is described by the equation $\alpha_0 + \sum_{i=1}^{m} \alpha_i x^i = 0$, then each of the two sets defined by the inequalities $\alpha_0 + \sum_{i=1}^{m} \alpha_i x^i \geq 0$ and $\alpha_0 + \sum_{i=1}^{m} \alpha_i x^i \leq 0$ is called a *closed half-space* of the space $\mathbf{R}^m$ *determined* by H; similarly each of the two sets defined by the inequalities $\alpha_0 + \sum_{i=1}^{m} \alpha_i x^i > 0$ and $\alpha_0 + \sum_{i=1}^{m} \alpha_i x^i < 0$ is called an *open half-space* of $\mathbf{R}^m$ *determined* by H. The two closed half-spaces determined by the one hyperplane are said to be *complementary.*

We say that the finite set of points $a_0, a_1, \ldots, a_n \in \mathbf{R}^m$ is *orthonormal* if $(a_j - a_0) \cdot (a_k - a_0) = \delta_{jk}$ for $j, k = 1, 2, \ldots, n$. In this definition the point a_0 plays a distinguished role. The following theorems hold:

0.4.11. THEOREM. *Every orthonormal set is affinely independent.* ■

0.4.12. THEOREM. *Every affine subspace contains an orthonormal set whose affine hull is the given affine subspace.* ■

0.4.13. THEOREM. *In Euclidean space* $\mathbf{R}^m$ *any orthonormal set may be extended to an orthonormal set consisting of* $m + 1$ *points.* ■

A set of points of the form $x = (1 - r)a + rb$ where $r \in \mathbf{R}_+$ is called a *half-line* with origin a passing through b. The set of points of the form $x = (1 - r)a + rb$ where $r \in I$ is called the *line segment with endpoints* a and b and will be denoted $\overline{ab}$. We say that the set $A \subset \mathbf{R}^m$ is *convex*, if $\overline{ab} \subset A$ for all points $a, b \in A$.

0.4.14. EXAMPLE. The following sets are convex: the whole space $\mathbf{R}^m$, every half-space of $\mathbf{R}^m$ (open or closed), every affine subspace in $\mathbf{R}^m$, every m-dimensional cube, every m-dimensional ball (open or closed). The $(m-1)$-dimensional sphere is not convex for any $m > 0$. ■

0.4.15. ASSERTION. *The intersection of any family of convex sets in* $\mathbf{R}^m$ *is convex.* ■

For any set $A \subset \mathbf{R}^m$ the intersection of all the convex sets of $\mathbf{R}^m$ which contain A is called the *convex hull* of A, denoted conv A.

0.4.16. THEOREM. *If* $A \subset \mathbf{R}^m$, *then* conv A *consists of all points* $x \in \mathbf{R}^m$ *of the form* $x = \sum_{j=0}^{n} r^j a_j$, *where* $a_j \in A$, $r^j \geq 0$ *for* $j = 0, 1, \ldots, n$ *and* $\sum_{j=0}^{n} r^j = 1$. ■

If an affinely independent set of points $a_0, a_1, \ldots, a_n$ has affine hull H and $x \in H$, then it may readily be verified that the numbers $r^0, r^1, \ldots, r^n$ appearing in the equation $x = \sum_{j=0}^{n} r^j a_j$ and satisfying the relation $\sum_{j=0}^{n} r^j = 1$ (cf. Theorem 0.4.6) are uniquely determined by the point x; we call them the *barycentric coordinates* of the point x relative to the sequence $a_0, a_1, \ldots, a_n$.

The following is readily verified.

0.4.17. ASSERTION. *If relative to the sequence of points $a_0, a_1, \ldots, a_n$ the point p has barycentric coordinates $r^0, r^1, \ldots, r^n$ and the point q has barycentric coordinates s^0, $s^1, \ldots, s^n$, then $p - q = \sum_{j=1}^{n}(r^j - s^j)(a_j - a_0)$.* ■

Let $H \subset \mathbf{R}^m$ be the affine hull of an affinely independent sequence of points $a_0, a_1, \ldots, a_n$. Every map $f: H \to \mathbf{R}^p$ with the property

$$f\left(\sum_{j=0}^{n} r^j a_j\right) = \sum_{j=0}^{n} r^j f(a_j),$$

where $\sum_{j=0}^{n} r^j = 1$ is called an *affine transformation.* It may readily be checked that this definition does not depend on the choice of the affinely independent sequence whose affine hull is H.

0.4.18. THEOREM. *An affine transformation preserves the affine dependence of a set of points and carries any affine subspace into an affine subspace of no greater dimension.* ■

A bijective, affine transformation of an affine subspace onto an affine subspace is called an *affine isomorphism.* The inverse map of an affine isomorphism is also an affine isomorphism. Two sets A, A' lying, respectively, in the affine subspaces H, H' are called *affinely isomorphic* if there exists an affine isomorphism of H onto H' which takes A onto A'.

0.4.19. THEOREM. *An affine isomorphism preserves affine dependence and independence, and also the dimension of affine subspaces.* ■

0.4.20. THEOREM. *For any two affinely independent sets in $\mathbf{R}^m$, both consisting of the same number of points, there exists an affine transformation of $\mathbf{R}^m$ onto itself which takes one set onto the other. For any two affine subspaces of equal dimension lying in $\mathbf{R}^m$ there exists an affine transformation of $\mathbf{R}^m$ onto itself which takes one affine subspace onto the other.* ■

The *m-dimensional projective space* P^m is the set of equivalence classes on the set $\mathbf{R}^{m+1} \setminus \{0\}$ defined by the equivalence relation: $a \sim b$ whenever $b = ra$ for some real number r. These equivalence classes are regarded as the points of the space P^m. The coordinates of a representative of a point $x \in P^m$, which are determined up to a constant of proportionality, are called the *homogenous coordinates* of the point; the coordinates are $m + 1$ in number and are indexed in order from 0 to m. To simplify notation we will just write $x = [x^0, x^1, \ldots, x^m]$.

The point $[x^0, x^1, \ldots, x^m] \in P^m$ will be called *proper* or *improper* depending on whether $x^0 \neq 0$, or $x^0 = 0$. For proper points we may as well assume that $x^0 = 1$. By treating the remaining coordinates $x^1, x^2, \ldots, x^m$ of the proper point $[1, x^1, x^2, \ldots, x^m] \in P^m$ as the coordinates of some point of the space $\mathbf{R}^m$, we can say that the space P^m may be obtained from the space $\mathbf{R}^m$ by adding the improper points. On the other hand, each improper point $[0, x^1, x^2, \ldots, x^m] \in P^m$ may be identified with a direction

in the space $\mathbf{R}^m$, in fact with the direction of the line passing through the points 0 and $(x^1, x^2, \ldots, x^m)$.

We mention that each point of the space P^m has precisely two representatives on the sphere S^m; they are of the form x and $-x$. We may therefore regard P^m as being obtained from the sphere S^m after identification of each point with its negative.

More on the geometry of Euclidean and projective spaces may be found in [1].

Chapter 1
Metric spaces

One of the most obvious features of the space we live in is its susceptibility to the measurement of distance. This fact lay at the heart of the development of geometry, which was initially the science concerned with making measurements on the earth's surface and with tracing their interdependences. Also, as the physical sciences, particularly astronomy and mechanics, progressed, it was found useful to study the very notion of space as a conceptual framework encompassing various measurements: distances between material points, changes in these distances (that is to say movements), and also dimensions of rigid bodies (rigid in the sense that the distances between their constituent points stay fixed). An examination of the properties that distance possesses in the setting of Euclidean space leads to the observation that some of them are consequences of certain others which are particularly simple to state and are intuitively obvious. Many theorems of elementary geometry may be proved using only these basic properties of distance.

In such circumstances, it is natural to introduce a notion of space more general than Euclidean by taking as primitive the distance between a pair of points, and as axioms some of the obvious properties that distance enjoys in Euclidean space. This idea turns out to be fruitful; it leads to the concept of a metric space, which holds an important place in geometry and in geometric topology, and constitutes a point of departure for the further generalizations of the notion of space in general topology.

In Section 1.1 we give the definition of metric spaces, their simplest properties and various examples which are important in geometry, topology and analysis. A number of other examples of metric spaces may also be found in the exercises for that section and separately in the Problems Section. Section 1.2 describes two basic operations on metric spaces: metric subspace and metric product. The introduction of these operations allows us to give further examples of metric spaces in the section. In order to study categories whose objects are metric spaces, we distinguish in Section 1.3 certain classes of maps between these spaces. These comprise non-expansive maps (which do not expand distances), Lipschitz maps, uniformly continuous maps, and continuous maps. The corresponding classes of isomorphisms are: isometries, similarities, uniform homeomorphisms, homeomorphisms. On the basis of these maps we explain the classification principles for geometric notions underlying what is known as the Erlangen programme.

In the subsequent sections we proceed to a more detailed study of metric concepts. In Section 1.4 we restrict ourselves to those which are strictly metric. We thus introduce the open and closed balls, the diameter of a space, bounded spaces, and bounded maps. Section 1.5 is devoted to the introduction of limits in metric spaces, their basic properties, the characterization of continuous maps by means of limits, and also of pointwise

convergence and uniform convergence for sequences of maps. In Section 1.6 we discuss the concept of open set, closed set, dense set, boundary set, we give the basic properties of closed and of open sets and also characterize continuous maps by means of open sets, closed sets and neighbourhoods.

In the next part of the Chapter we distinguish certain classes of metric spaces. Thus in Section 1.7 we are concerned with connected spaces, we give their definition, their simplest properties, some examples and also some sufficiency conditions for connectedness. Section 1.8 is devoted to compact spaces. We begin with a proof of the Bolzano-Weierstrass Theorem, which is followed by the definition of compact spaces and their simplest properties. Then we prove a number of classic theorems about compact spaces (Lebesgue's Lemma, the Borel-Lebesgue Theorem, the Theorems of Cantor, Heine and Weierstrass). Finally in Section 1.9 we examine the class of complete spaces. The section commences with a study of Cauchy sequences; next, we prove Cauchy's Theorem for numeric sequences and give the definition of completeness. Thereafter we give examples and the simplest properties of this concept.

Section 1.10 applies metric and topological notions to the study of Euclidean spaces and their subsets. We will be concerned with the characterization of compact and of complete subspaces of Euclidean spaces, the interdependence of connectedness and convexity, the characterization of regions by means of broken lines and the topological classification of certain convex sets.

We shall return to the discussion of metric spaces in Chapter 6. Here we limit ourselves to information of a basic character merely to gain the conceptual apparatus needed in the succeeding chapters.

1.1. Concept of a metric space

By a *metric space* we mean an arbitrary set X together with a function ρ which associates to every pair x, y of elements of X a real number $\rho(x, y)$ in such a way that the following axioms are obeyed:

(M1) $\rho(x, y) = 0$ *if and only if* $x = y$,

(M2) $\rho(x, y) = \rho(y, x)$ *for every* $x, y \in X$,

(M3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ *for every* $x, y, z \in X$.

The members of the set X are conventionally called *points*, the function ρ is known as the *metric* and the value $\rho(x, y)$ of the metric corresponding to the points $x, y \in X$ is said to be the *distance* between these points (see Supplement 1.S.1). We draw attention to the fact that a metric space is a pair (X, ρ). The same set X may in general support many functions $\rho: X \times X \to \mathbf{R}$ satisfying the axioms (M1)–(M3); each of them is said to *metrize* the set X. If a metric ρ on the set X is fixed, or its prescription is beyond doubt, then the metric space (X, ρ) will be denoted for simplicity by the single symbol X.

Axiom (M2) is put more briefly by saying that the metric is a *symmetric function.* Axiom (M3) goes by the name of the *triangle inequality* in view of the obvious geometric

interpretation in the case when the three points are the vertices of a triangle in the Euclidean plane.

The distance between two points in a Euclidean space is always a non-negative number. However, there is no need to assume this in the form of a separate axiom, because of the following.

1.1.1. THEOREM. *If (X,ρ) is a metric space, then $\rho(x,y) \geq 0$ for any pair of points $x,y \in X$.*

PROOF. Using the axioms (M2), (M3) and (M1), in that order, we infer that

$$\rho(x,y) = \frac{1}{2}(\rho(x,y) + \rho(y,x)) \geq \frac{1}{2}\rho(x,x) = 0. \blacksquare$$

Observe that from axiom (M3) follows the next theorem, which may be called the *polygon inequality.*

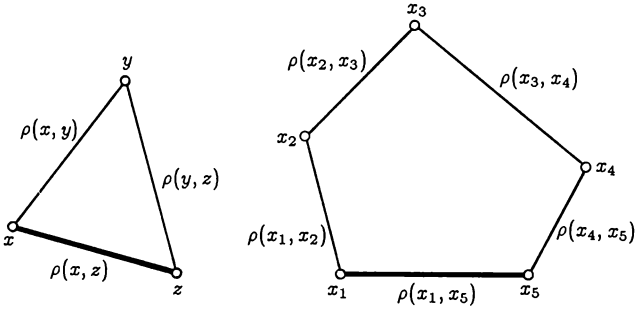


Fig.1. The triangle inequality (axiom (M3)) and the polygon inequality (Theorem 1.1.2) for $n = 5$.

1.1.2. THEOREM. *If (X,ρ) is a metric space and $x_1, x_2, \ldots, x_n \in X$, then*

$$\rho(x_1, x_n) \leq \sum_{j=1}^{n-1} \rho(x_j, x_{j+1}).$$

PROOF. The proof is by induction on the number of points. In the case $n = 2$ the given inequality is obvious. Suppose, that

$$\rho(x_1, x_k) \leq \sum_{j=1}^{k-1} \rho(x_j, x_{j+1})$$

for any set of points $x_1, x_2, \ldots, x_k \in X$ where $k \geq 2$. Then, if $x_1, x_2, \ldots, x_{k+1} \in X$, we have

$$\begin{aligned} \rho(x_1, x_{k+1}) &\leq \rho(x_1, x_k) + \rho(x_k, x_{k+1}) \\ &\leq \sum_{j=1}^{k-1} \rho(x_j, x_{j+1}) + \rho(x_k, x_{k+1}) = \sum_{j=1}^{k} \rho(x_j, x_{j+1}), \end{aligned}$$

which completes the proof. $\blacksquare$

We now give some examples of metric spaces.

1.1.3. EXAMPLE. *The discrete metric space.* This consists of an arbitrary set X and a metric ρ defined by the formula:

$$\rho(x,y) = \begin{cases} 0 & \text{for } x = y, \\ 1 & \text{for } x \neq y. \end{cases}$$

Axioms (M1) and (M2) are satisfied for obvious reasons. To check the triangle inequality, suppose that $\rho(x,z) > \rho(x,y) + \rho(y,z)$ for some points $x, y, z \in X$. Then it must be the case that $\rho(x,z) = 1$ and $\rho(x,y) = \rho(y,z) = 0$, so that $x \neq z$ and $x = y = z$, which is a contradiction.

The metric defined above is called the *zero-one metric* or the *discrete metric* on the set X. ■

1.1.4. EXAMPLE. *The real line* $\mathbf{R}$. This is the space consisting of the set $\mathbf{R}$ of real numbers with metric defined by the formula $\rho(x,y) = |x-y|$ for $x, y \in \mathbf{R}$. The axioms (M1) and (M2) are obviously satisfied. The triangle inequality follows from the well-known property of the modulus function for real numbers, thus

$$\rho(x,z) = |x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z| = \rho(x,y) + \rho(y,z)$$

for any three real numbers $x, y, z \in \mathbf{R}$. ■

1.1.5. EXAMPLE. *The m-dimensional Euclidean space* $\mathbf{R}^m$. This is the space whose points are m-tuples of real numbers, distance between the points x and y being defined by the formula $\rho(x,y) = \|x-y\|$. It follows immediately from Assertion 0.4.3 that the axioms (M1) and (M2) are obeyed. To verify the triangle inequality we first prove an inequality connecting the scalar product and the norm of points of the space $\mathbf{R}^m$.

Let $a, b \in \mathbf{R}^m$ with $a \neq 0$. Then

$$\|a\|^2\|b\|^2 - (a \cdot b)^2 = \|a\|^2\|b\|^2 - 2(a \cdot b)^2 + (a \cdot b)^2 = (\|a\|b - \frac{a \cdot b}{\|a\|}a)^2 \geq 0.$$

Checking the trivial case $a = 0$ separately, we obtain the inequality

$$(a \cdot b)^2 \leq \|a\|^2\|b\|^2,$$

known as the *Cauchy-Schwartz inequality.* It is obviously equivalent to the inequality $|a \cdot b| \leq \|a\|\|b\|$ from which it follows that $a \cdot b \leq \|a\|\|b\|$. Using this inequality we have $\|a+b\|^2 = (a+b)\cdot(a+b) = \|a\|^2 + 2(a \cdot b) + \|b\|^2 \leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 = (\|a\| + \|b\|)^2$. We thus obtain the inequality

$$\|a+b\| \leq \|a\| + \|b\|$$

known as *Minkowski's inequality.*

Substituting into Minkowski's inequality $a = x - y$ and $b = y - z$ we obtain $a + b = x - z$ and so $\|x-z\| \leq \|x-y\| + \|y-z\|$ or $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. ■

1.1.6. EXAMPLE. *The sphere* S^{m-1} *with angular metric.* For any pair of points x, y of the $(m-1)$-dimensional sphere S^{m-1} we have by the Cauchy-Schwartz inequality (of

Example 1.1.5) that $(x\cdot y)^2 \le \|x\|^2\|y\|^2 = 1$. It follows that there is exactly one number $\hat\rho(x,y)$ satisfying $0 \le \hat\rho(x,y) \le \pi$ and $\cos\hat\rho(x,y) = x\cdot y$. We show that $\hat\rho$ is a metric on S^{m-1}.

The condition $\hat\rho(x,y) = 0$ is obviously equivalent to the equation $x\cdot y = 1$. Since $\|x-y\|^2 = \|x\|^2 - 2x\cdot y + \|y\|^2 = 2 - 2x\cdot y$, we have $x\cdot y = 1$ if and only if $x = y$. This proves that axiom (M1) is satisfied. Axiom (M2) follows from the commutativity of the scalar product.

To prove the triangle inequality consider the points $x, y, z \in S^{m-1}$ and let $\cos a = x\cdot y$, $\cos b = y\cdot z$, $\cos c = z\cdot x$. Substituting $p = \frac{1}{2}(-a+b+c)$, $q = \frac{1}{2}(a-b+c)$, $r = \frac{1}{2}(a+b-c)$, $s = \frac{1}{2}(a+b+c)$ and applying some well-known trigonometric formulas we have:

$$4\sin p\sin q\sin r\sin s = 1 + 2\cos a\cos b\cos c - \cos^2 a - \cos^2 b - \cos^2 c,$$

thus

$$\begin{aligned} 4\sin p\sin q\sin r\sin s &= (1-\cos^2 c)(1-\cos^2 b) - (\cos a - \cos b\cos c)^2 \\ &= (1-(x\cdot z)^2)(1-(y\cdot z)^2) - ((x\cdot y) - (x\cdot z)(y\cdot z))^2 \\ &= \|x-(x\cdot z)z\|^2\|y-(y\cdot z)z\|^2 - ((x-(x\cdot z)z)\cdot(y-(y\cdot z)z))^2. \end{aligned}$$

Using the Cauchy-Schwartz inequality for the points $x-(x\cdot z)z$, $y-(y\cdot z)z$ we deduce that $\sin p\sin q\sin r\sin s \ge 0$. But $0 \le a,b,c \le \pi$, so $-\frac{1}{2}\pi \le p,q,r \le \pi$, $0 \le s \le \frac{3}{2}\pi$. Now $p+q = c$, $q+r = a$, $r+p = b$ and $a,b,c \ge 0$, so at most one of the numbers p,q,r can be negative.

If even one of them were negative, then, since the other two in sum do not exceed π and $p+q+r = s$, we would have $0 \le s < \pi$, whence $\sin s \ge 0$. Thus among the numbers p,q,r,s exactly one would have negative sine, contradicting what we proved about their sines having a non-negative product. Thus $p,q,r \ge 0$.

In particular from $r \ge 0$ we obtain $c \le a+b$, that is $\hat\rho(x,z) \le \hat\rho(x,y)+\hat\rho(y,z)$, which completes the proof of the triangle inequality. The obvious geometric interpretation of the formula $\cos\hat\rho(x,y) = x\cdot y$ when $\|x\| = \|y\| = 1$ suggests the name of *angular metric* on the sphere S^{m-1}. ■

1.1.7. EXAMPLE. *The m-dimensional projective space* P^m. Following the remark of Section 0.4 the projective space P^m may be regarded as being the sphere S^m in which every pair of points $x, -x$ has been identified. This naturally permits the introduction of a metric on P^m by means of the angular metric $\hat\rho$ described in Example 1.1.6.

It is obvious that for any pair of points $x, y \in S^m$ there is exactly one number $\sigma(x,y)$ satisfying the conditions $0 \le \sigma(x,y) \le \frac{1}{2}\pi$, $\cos\sigma(x,y) = |x\cdot y|$. We then have

$$\sigma(x,y) = \begin{cases} \hat\rho(x,y), & \text{if } 0 \le \hat\rho(x,y) \le \frac{1}{2}\pi, \\ \pi - \hat\rho(x,y), & \text{if } \frac{1}{2}\pi \le \hat\rho(x,y) \le \pi. \end{cases}$$

It follows that $\sigma(x,y) = 0$ if and only if $x = y$ or $x = -y$ and also that $\sigma(x,y) = \sigma(y,x)$.

Put

$$\xi = \begin{cases} x, & \text{if } \sigma(x,y) = \hat\rho(x,y), \\ -x, & \text{if } \sigma(x,y) = \pi - \hat\rho(x,y); \end{cases}$$

$$\zeta = \begin{cases} z, & \text{if } \sigma(y,z) = \hat\rho(y,z), \\ -z, & \text{if } \sigma(y,z) = \pi - \hat\rho(y,z). \end{cases}$$

Then $\sigma(x,y) = \hat{\rho}(\xi,y)$, $\sigma(y,z) = \hat{\rho}(y,\varsigma)$. By the triangle inequality for the metric $\hat{\rho}$ we have $\hat{\rho}(\xi,\varsigma) \leq \hat{\rho}(\xi,y) + \hat{\rho}(y,\varsigma) = \sigma(x,y) + \sigma(y,z)$. Since of course $\sigma(x,z) \leq \hat{\rho}(\xi,\varsigma)$ we have $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

Letting $\rho([x],[y]) = \sigma(x,y)$ for $x,y \in S^m$ we obtain a metric ρ on P^m, where $[x]$ denotes the equivalence class of x under the relation identifying x with $-x$. ■

1.1.8. EXAMPLE. *The Hilbert space* $\mathbf{R}^\omega$. This is the space whose points are the infinite sequences of real numbers $x = \{x^1, x^2, \ldots\}$ for which $\sum_{i=1}^{\infty}(x^i)^2$ converges and the distance between the points $x = \{x^1, x^2, \ldots\}$ and $y = \{y^1, y^2, \ldots\}$ is defined by the formula

$$\rho(x,y) = \sqrt{\sum_{i=1}^{\infty}(x^i - y^i)^2}.$$

Note first that the series appearing under the square root sign is convergent. This follows from the inequalities $0 \leq (x^i - y^i)^2 = (x^i)^2 - 2x^iy^i + (y^i)^2 \leq 2((x^i)^2 + (y^i)^2)$ for $i = 1,2,\ldots$ and the fact that the series $\sum_{i=1}^{\infty}(x^i)^2$ and $\sum_{i=1}^{\infty}(y^i)^2$ are assumed convergent.

Checking axioms (M1) and (M2) presents no difficulty. To prove the triangle inequality, suppose that $x = \{x^1, x^2, \ldots\}$, $y = \{y^1, y^2, \ldots\}$, $z = \{z^1, z^2, \ldots\}$ are points of the space $\mathbf{R}^\omega$ and for $m = 1,2,\ldots$ take $x_m = (x^1, x^2, \ldots, x^m)$, $y_m = (y^1, y^2, \ldots, y^m)$, $z_m = (z^1, z^2, \ldots, z^m)$. Then, by the triangle inequality in the space $\mathbf{R}^m$ obtained in Example 1.1.5 we have $\rho_m(x_m, z_m) \leq \rho_m(x_m, y_m) + \rho_m(y_m, z_m)$, where ρ_m denotes the metric in the space $\mathbf{R}^m$ for $m = 1,2,\ldots$ Taking limits over m we obtain $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. ■

1.1.9. EXAMPLE. *The space of maps.* Suppose X is a non-empty set and Y is a metric space with the property that $\sup\{\rho(y', y'') : y', y'' \in Y\} < \infty$. Consider the set P of all maps $f: X \to Y$. In P define the distance between two points f and g by the formula

$$\tilde{\rho}(f,g) = \sup\{\rho(f(x), g(x)) : x \in X\}.$$

Observe that from the assumption about the space Y it follows that $\tilde{\rho}(f,g) < \infty$ for any two maps $f, g \in P$.

To check that $(P, \tilde{\rho})$ is a metric space it is enough to verify the triangle inequality, since the axioms (M1) and (M2) are obviously satisfied. Suppose therefore that $f, g, h \in P$. From the triangle inequality in the space Y we have that $\rho(f(x), h(x)) \leq \rho(f(x), g(x)) + \rho(g(x), h(x))$ for each member $x \in X$. It follows that

$$\begin{aligned}\rho(f(x), h(x)) &\leq \sup\{\rho(f(x), g(x)) + \rho(g(x), h(x)) : x \in X\} \\ &\leq \sup\{\rho(f(x), g(x)) : x \in X\} + \sup\{\rho(g(x), h(x)) : x \in X\} \\ &= \tilde{\rho}(f,g) + \tilde{\rho}(g,h), \quad \text{for each} \quad x \in X.\end{aligned}$$

Hence $\tilde{\rho}(f,h) = \sup\{\rho(f(x), h(x)) : x \in X\} \leq \tilde{\rho}(f,g) + \tilde{\rho}(g,h)$. ■

Exercises

a) For $j = 1, 2, 3$ give an example of a function ρ_j which associates to each pair from a three-element set X a real number in such a way that axiom (M_j) is not satisfied while the other two axioms for a metric are.

b) Show that the axiom system (M1), (M2), (M3) is equivalent to the axiom system consisting of (M1) and (M3'), where

(M3') $$\rho(z,x) \leq \rho(x,y) + \rho(y,z) \quad \textit{for every} \quad x,y,z \in X.$$

c) Let $S^\omega = \{x = \{x^1, x^2, \ldots\} \in \mathbf{R}^\omega : \sum_{i=1}^\infty (x^i)^2 = 1\}$. Examine whether the function $\hat{\rho}$ defined by the conditions: $0 \leq \hat{\rho}(x,y) \leq \pi$, $\cos \hat{\rho}(x,y) = \sum_{i=1}^\infty x^i y^i$ for $x = \{x^1, x^2, \ldots\}$, $y = \{y^1, y^2, \ldots\} \in S^\omega$ is a metric.

1.2. Operations on metric spaces

We now pass to a discussion of certain operations which will allow us to expand on the number of examples of metric spaces.

If (X, ρ) is a metric space and $A \subset X$, then taking $\rho_A(x,y) = \rho(x,y)$ for $x, y \in A$ we obtain a function which is obviously a metric on A. The pair (A, ρ_A) is then called a *metric subspace* of the space (X, ρ). If when referring to the space (X, ρ) we omit the symbol ρ then we say that A is a metric subspace, or more briefly a subspace, of the space X, meaning to say that the metric ρ_A defined above is used to metrize A. We may sometimes come across situations, where for practical or traditional reasons one speaks of subsets rather than metric subspaces of X. To avoid possible confusion and also unnecessary formality we agree the convention that whenever concepts which refer to the set $A \subset X$ are metric in character, we tacitly treat A as a metric subspace of the space X.

1.2.1. EXAMPLE. *Metric subspaces of the space* $\mathbf{R}^m$. The real intervals and half-lines introduced in Section 0.3 may be regarded as metric subspaces of the real line $\mathbf{R}$. Similarly the m-dimensional cubes and balls, the $(m-1)$-dimensional spheres, affine subspaces, half-spaces, half-lines and line segments introduced in Section 0.4 may be regarded as metric subspaces of the Euclidean space $\mathbf{R}^m$. ■

1.2.2. EXAMPLE. *The Hilbert cube* I^ω. This is the metric subspace of the Hilbert space $\mathbf{R}^\omega$ defined by the formula

$$I^\omega = \{\{x^1, x^2, \ldots\} \in \mathbf{R}^\omega : 0 \leq x^i \leq 1/i \quad \text{for} \quad i = 1, 2, \ldots\}.$$

Since $0 \leq (x^i)^2 \leq (1/i)^2$ for $i = 1, 2, \ldots$ and the series $\sum_{i=1}^\infty (1/i)^2$ is convergent, we do indeed have $I^\omega \subset \mathbf{R}^\omega$. ■

Suppose given a finite sequence of metric spaces (X_i, ρ_i) for $i = 1, 2, \ldots, m$. We may define on the set $X = \bigtimes_{i=1}^m X_i$ a metric ρ by means of the formula

$$\rho(x,y) = \sqrt{\sum_{i=1}^m \rho_i^2(x_i, y_i)},$$

where $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m) \in X$.

Certainly the axioms (M1) and (M2) follow directly from the respective axioms applied to the metric ρ_i for $i = 1,2,\dots,m$. To prove the triangle inequality suppose that the points $x = (x_1, x_2, \dots, x_m)$, $y = (y_1, y_2, \dots, y_m)$, $z = (z_1, z_2, \dots, z_m)$ lie in X; put $a^i = \rho_i(x_i, y_i)$, $b^i = \rho_i(y_i, z_i)$, $c^i = \rho_i(x_i, z_i)$ for $i = 1,2,\dots,m$ and then consider in the Euclidean space $\mathbf{R}^m$ the points $a = (a^1, a^2, \dots, a^m)$, $b = (b^1, b^2, \dots, b^m)$, $c = (c^1, c^2, \dots, c^m)$. From the triangle inequality in the space (X_i, ρ_i) we obtain $c^i \leq a^i + b^i$ for $i = 1,2,\dots,m$, hence $\|c\| \leq \|a + b\|$. Using the Minkowski inequality proved in Example 1.1.5 we thus have $\rho(x,z) = \|c\| \leq \|a + b\| \leq \|a\| + \|b\| = \rho(x,y) + \rho(y,z)$.

The set X together with the metric ρ defined above is called the *metric product* of the spaces (X_i, ρ_i) for $i = 1,2,\dots,m$ and we write $(X, \rho) = (X_1, \rho_1) \times (X_2, \rho_2) \times \dots \times (X_m, \rho_m)$, or just $X = X_1 \times X_2 \times \dots \times X_m$. We shall also use the brief notation $(X, \rho) = \bigtimes_{i=1}^m (X_i, \rho_i)$ for the metric product, or simply $X = \bigtimes_{i=1}^m X_i$.

Note now that the m-dimensional Euclidean space $\mathbf{R}^m$ may be regarded as the metric product of m copies of the real line. Similarly, the m-dimensional cube I^m may be treated as a metric product of m copies of the unit interval. We also see that the following is obvious.

1.2.3. ASSERTION. *If A_i is a metric subspace of metric space X_i for $i = 1,2,\dots,m$, then $\bigtimes_{i=1}^m A_i$ is a metric subspace of the space $\bigtimes_{i=1}^m X_i$.* ■

In future we shall wish to make use of the following estimate for the distance in the metric product.

1.2.4. LEMMA. *If $(X, \rho) = \bigtimes_{i=1}^m (X_i, \rho_i)$, $x = (x_1, x_2, \dots, x_m)$, and $y = (y_1, y_2, \dots, y_m)$, then*

$$\max\{\rho_i(x_i, y_i) : i = 1,2,\dots,m\} \leq \rho(x,y) \leq \sqrt{m}\max\{\rho_i(x_i, y_i) : i = 1,2,\dots,m\}.$$

PROOF. It is enough to prove that for $i = 1,2,\dots,m$ the inequalities

$$\rho_i^2(x_i, y_i) \leq \sum_{i=1}^m \rho_i^2(x_i, y_i) \leq m\max\{\rho_i^2(x_i, y_i) : i = 1,2,\dots,m\},$$

hold. The one on the left follows from the non-negativity of the summands, that on the right from the definition of the max function. ■

Exercises

a) Suppose that (X_i, ρ_i) is a metric space for $i = 1,2,\dots,m$ and let $X = \bigtimes_{i=1}^m X_i$. Show that the function ρ which associates with every pair of points $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$ of the set X the number $\rho(x,y) = \sum_{i=1}^m \rho_i(x_i, y_i)$ is a metric on X.

b) Suppose that (X_i, ρ_i) is a metric space for $i = 1,2,\dots,m$ and let $X = \bigtimes_{i=1}^m X_i$. Show that the function ρ which associates with every pair of points $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$ of the set X the number $\rho(x,y) = \max\{\rho_i(x_i, y_i) : i = 1,2,\dots,m\}$ is a metric on X.

c) Suppose $X = \bigcup_{j=1}^{n} X_j$ with $X_j \cap X_k = \{x_0\}$ for $j \neq k$ and suppose ρ_j is a metric on X_j for $j = 1, 2, \ldots, n$. Show that the function ρ defined by the formula:

$$\rho(x,y) = \begin{cases} \rho_j(x,y), & \text{if } x,y \in X_j, \\ \rho_j(x,x_0) + \rho_k(x_0,y), & \text{if } x \in X_j,\ y \in X_k,\ j \neq k, \end{cases}$$

is a metric on X.

1.3. Maps on metric spaces

In this section we distinguish certain classes of maps on metric spaces which will enable us to treat metric spaces as the objects of certain categories. Suppose that X and Y are metric spaces and $f: X \to Y$. We shall denote the metrics on the spaces X and Y by the same symbol ρ on the understanding however that the symbol has a different meaning when applied to the points of the space X than when it is applied to the points of the space Y.

We say that the map f is *non-expansive* when $\rho(f(x), f(x')) \leq \rho(x,x')$ for every pair of points $x, x' \in X$.

1.3.1. EXAMPLE. *Inclusion of a metric subspace.* Let A be a metric subspace of a metric space X. The map $i_A: A \to X$ defined by the formula $i_A(a) = a$ for $a \in A$ is called the *inclusion* map of the subspace A into the space X. From the definition of the metric on a subspace it follows immediately that inclusion is non-expansive. ■

1.3.2. EXAMPLE. *Projection of the metric product onto a factor.* Let the space X be the metric product of the metric spaces $X_1, X_2, \ldots, X_m$. For $i = 1, 2, \ldots, m$ define the map $p_i: X \to X_i$ by the formula $p_i(x_1, x_2, \ldots, x_m) = x_i$ for $(x_1, x_2, \ldots, x_m) \in \bigtimes_{i=1}^{m} X_i$. It follows from Lemma 1.2.4 that for $i = 1, 2, \ldots, m$ we have $\rho(p_i(x), p_i(x')) \leq \rho(x,x')$; the map p_i for $i = 1, 2, \ldots, m$, which we call the *projection of the product* $\bigtimes_{i=1}^{m} X_i$ *onto its* i^{th}*-factor*, is thus a non-expansive map. ■

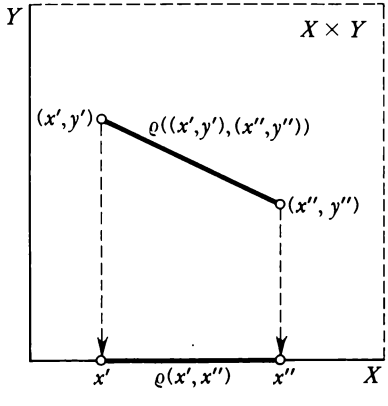


Fig.2. Projection of the metric product $X \times Y$ onto the factor X is non-expansive (Example 1.3.2).

1.3.3. EXAMPLE. *Orthogonal projection of the space* $\mathbf{R}^m$ *onto an affine subspace.* Let H be the affine hull of an affinely independent set of points $a_0, a_1, \ldots, a_n \in \mathbf{R}^m$ and let $x \in \mathbf{R}^m$. Any point $y \in H$ such that $(x-y)\cdot(p-q) = 0$ for any $p, q \in H$ is called an *orthogonal projection* of the point x onto H. We prove below the existence and uniqueness of the orthogonal projection. By Assertion 0.4.17 it follows that a point $y \in H$ is an orthogonal projection of the point x onto H if and only if $(x-y)\cdot(a_j-a_0) = 0$ for $j = 1, 2, \ldots, n$.

To prove the existence of the orthogonal projection y of the point x into H replace (using Theorem 0.4.12) the set of points $a_0, a_1, \ldots, a_n$ by an orthonormal set $b_0, b_1, \ldots, b_n$ whose hull is also H. Let $r^j = (x-b_0)\cdot(b_j-b_0)$ for $j = 1, 2, \ldots, n$, $r^0 = 1 - \sum_{j=1}^n r^j$. Putting $y = \sum_{j=0}^n r^j b_j \in H$ we have $x - y = (x-b_0) + (b_0 - \sum_{j=0}^n r^j b_j) = (x-b_0) - \sum_{j=1}^n r^j (b_j - b_0)$. Hence $(x-y)\cdot(b_k-b_0) = r^k - \sum_{j=1}^n r^j \delta_{jk} = r^k - r^k = 0$ for $k = 1, 2, \ldots, n$. Hence y is indeed an orthogonal projection of x onto H.

To prove uniqueness of this projection observe that if $y, \bar{y} \in H$ and $(x-y)\cdot(a_0 - a_j) = 0$ and $(x-\bar{y})\cdot(a_0-a_j) = 0$ for $j = 1, 2, \ldots, n$, then by subtracting the equations we have $(y-\bar{y})\cdot(a_0-a_j) = 0$ for $j = 1, 2, \ldots, n$. If $y = \sum_{j=0}^n r^j a_j$ and $\bar{y} = \sum_{j=0}^n \bar{r}^j a_j$ where $\sum_{j=0}^n r^j = \sum_{j=0}^n \bar{r}^j = 1$ then by Assertion 0.4.17 we have $y - \bar{y} = \sum_{j=1}^n (r^j - \bar{r}^j)(a_j - a_0)$. Hence $\|y-\bar{y}\|^2 = \sum_{j=1}^n (r^j - \bar{r}^j)(y-\bar{y})\cdot(a_j - a_0) = 0$, so that $y - \bar{y} = 0$ or $y = \bar{y}$.

We now prove that if y and $\bar{y}$ denote respectively the orthogonal projections of the points x and $\bar{x}$ onto H, then $\rho(y, \bar{y}) \le \rho(x, \bar{x})$. For, if $y = \sum_{j=0}^n r^j a_j$, $\bar{y} = \sum_{j=0}^n \bar{r}^j a_j$ where $\sum_{j=0}^n r^j = \sum_{j=0}^n \bar{r}^j = 1$, then by Assertion 0.4.17 we have $y - \bar{y} = \sum_{j=1}^n (r^j - \bar{r}^j)(a_j - a_0)$, hence $(x-y)\cdot(y-\bar{y}) = 0$ and $(\bar{x}-\bar{y})\cdot(y-\bar{y}) = 0$ and so $((x-\bar{x}) - (y-\bar{y}))\cdot(y-\bar{y}) = 0$. From this it follows that $\|x-\bar{x}\|^2 = \|(x-\bar{x}) - (y-\bar{y})\|^2 + \|y-\bar{y}\|^2$ and so $\|y-\bar{y}\|^2 \le \|x-\bar{x}\|^2$, or $\|y-\bar{y}\| \le \|x-\bar{x}\|$. ■

1.3.4. EXAMPLE. *The norm.* The map $\nu: \mathbf{R}^m \to \mathbf{R}$ defined by $\nu(x) = \|x\|$ is non-expansive. For, if $x, x' \in \mathbf{R}^m$, then $\|x\| = \|x' + (x-x')\| \le \|x'\| + \|x-x'\|$ and $\|x'\| = \|x + (x'-x)\| \le \|x\| + \|x'-x\|$, hence $\big|\|x\| - \|x'\|\big| \le \|x-x'\|$. ■

1.3.5. EXAMPLE. *The map* $p: S^m \to P^m$. Associate with each point x of the sphere S^m its equivalence class $p(x) = [x]$ in the projective space P^m. This is a non-expansive map in the sense of the metrics $\hat{\rho}$ and ρ defined in Examples 1.1.6 and 1.1.7, since $\rho([x],[y]) = \hat{\rho}(x,y)$ when $0 \le \hat{\rho}(x,y) \le \frac{1}{2}\pi$ and $\rho([x],[y]) = \pi - \hat{\rho}(x,y)$ when $\frac{1}{2}\pi \le \hat{\rho}(x,y) \le \pi$. Thus in both cases $\rho([x],[y]) \le \hat{\rho}(x,y)$. ■

It is obvious that the identity map is non-expansive and that the composition of two non-expansive maps is itself non-expansive.

If there exists a constant $c \ge 0$ with the property that $\rho(f(x), f(x')) \le c\rho(x, x')$ for any pair of points $x, x' \in X$, then f is said to be a *Lipschitz map with constant* c. Obviously if a map is non-expansive then it is a Lipschitz map with constant 1. The Lipschitz maps thus form a wider class than the non-expansive maps. The composition of two Lipschitz maps with constants c and c' is a Lipschitz map with constant cc'. (See also the Supplements 1.S.7 and 1.S.8).

1.3.6. EXAMPLE. *Every real differentiable function* $f: \mathbf{R} \to \mathbf{R}$ *which has derivative bounded by* c *is a Lipschitz map with constant* c. For, by the Mean Value Theorem we

deduce that for any two points $x, x' \in \mathbf{R}$ there is a point ξ such that

$$|f(x) - f(x')| = |f'(\xi)||x - x'| \leq c|x - x'|. \blacksquare$$

1.3.7. EXAMPLE. *The metric.* The metric ρ of a metric space (X, ρ) may be treated as a map $\rho: X \times X \to \mathbf{R}$. It is then a Lipschitz map with constant $\sqrt{2}$. Indeed, if $x, x', y, y' \in X$ then using Theorem 1.1.2 we have $\rho(x, x') \leq \rho(x, y) + \rho(y, y') + \rho(y', x')$ and $\rho(y, y') \leq \rho(y, x) + \rho(x, x') + \rho(x', y')$. Thus

$$|\rho(x, x') - \rho(y, y')| \leq \rho(x, y) + \rho(x', y') \leq \sqrt{2}\sqrt{\rho^2(x, y) + \rho^2(x', y')}. \blacksquare$$

1.3.8. EXAMPLE. *Addition of points.* The operation of addition for points in the Euclidean space $\mathbf{R}^m$ may be viewed as a map $\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^m$. It is then a Lipschitz map with constant $\sqrt{2}$. For, if $x, x', y, y' \in \mathbf{R}^m$, then

$$\begin{aligned}\rho(x + y, x' + y') &= \|(x + y) - (x' + y')\| = \|(x - x') + (y - y')\| \\ &\leq \|x - x'\| + \|y - y'\| \leq \sqrt{2}\sqrt{\|x - x'\|^2 + \|y - y'\|^2}. \blacksquare\end{aligned}$$

1.3.9. EXAMPLE. *Every affine map is a Lipschitz map.* From Theorem 0.4.4 it follows that every affine map $f: H \to \mathbf{R}^p$ where H is an affine subspace of $\mathbf{R}^m$ may be extended to an affine map of the space $\mathbf{R}^m$ into $\mathbf{R}^p$. It suffices to examine maps of this type.

If a map $f: \mathbf{R}^m \to \mathbf{R}^p$ is affine and we take $e_0 = 0$, $e_i = (\delta_i^1, \delta_i^2, \ldots, \delta_i^m) \in \mathbf{R}^m$ for $i = 1, 2, \ldots, m$, then for each $x = (x^1, x^2, \ldots, x^m) \in \mathbf{R}^m$ we have

$$x = (1 - \sum_{i=1}^{m} x^i)e_0 + \sum_{i=1}^{m} x^i e_i,$$

hence

$$f(x) = (1 - \sum_{i=1}^{m} x^i)f(e_0) + \sum_{i=1}^{m} x^i f(e_i) = f(e_0) + \sum_{i=1}^{m} x^i(f(e_i) - f(e_0)).$$

If moreover $\bar{x} = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m)$, then

$$\begin{aligned}\|f(x) - f(\bar{x})\| &= \|\sum_{i=1}^{m}(x^i - \bar{x}^i)(f(e_i) - f(e_0))\| \\ &\leq \sum_{i=1}^{m}|x^i - \bar{x}^i|\|f(e_i) - f(e_0)\| \leq c\sum_{i=1}^{m}|x^i - \bar{x}^i|\end{aligned}$$

where $c = \max\{\|f(e_i) - f(e_0)\| : i = 1, 2, \ldots, m\}$. Applying Lemma 1.2.4 we obtain

$$\|f(x) - f(\bar{x})\| \leq cm \max\{|x^i - \bar{x}^i| : i = 1, 2, \ldots, m\} \leq cm\|x - \bar{x}\|. \blacksquare$$

We now present a class of metric maps more general than the Lipschitz maps. We shall say that a map f is *uniformly continuous* if for every positive real number ϵ there is a positive real number δ such that if $x, x' \in X$ and $\rho(x, x') < \delta$ then $\rho(f(x), f(x')) < \epsilon$.

Every Lipschitz map is uniformly continuous, for if $c \neq 0$ is its constant then for given $\epsilon > 0$ it is enough to take $\delta = \epsilon/c$.

1.3.10. EXAMPLE. *Any map f defined on a discrete metric space X is uniformly continuous.* Evidently, for any $\epsilon > 0$ it is enough to take $\delta = 1$. For if $\rho(x, x') < \delta$ with $x, x' \in X$ then $x = x'$ and so $f(x) = f(x')$ and so $\rho(f(x), f(x')) = 0 < \epsilon$. It easily follows that there are uniformly continuous maps which are not Lipschitz for any constant c. ■

1.3.11. THEOREM. *The composition of two uniformly continuous maps is uniformly continuous.*

PROOF. If the maps $f: X \to Y$ and $g: Y \to Z$ are uniformly continuous then for every positive real number ϵ there is a positive real number δ such that if $y, y' \in Y$ and $\rho(y, y') < \delta$ then $\rho(g(y), g(y')) < \epsilon$. Corresponding to δ there is a real number $\eta > 0$ such that if $x, x' \in X$ and $\rho(x, x') < \eta$ then $\rho(f(x), f(x')) < \delta$. Hence if $x, x' \in X$ and $\rho(x, x') < \eta$, we have $\rho(gf(x), gf(x')) < \epsilon$. ■

We now expand the class of uniformly continuous maps as follows. We say that a map f is *continuous at the point $x \in X$* if for every positive real number ϵ there is a positive real number δ such that if $x' \in X$ and $\rho(x, x') < \delta$ then $\rho(f(x), f(x')) < \epsilon$. A map $f: X \to Y$ that is continuous at every point of the space X is called *continuous* (see Supplement 1.S.9). It follows immediately from this definition that every uniformly continuous map is continuous.

1.3.12. EXAMPLE. *Scaling points by reals.* The operation of scaling a point of the Euclidean space $\mathbf{R}^m$ by a real number may be viewed as a map $\mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$. This map is continuous though not uniformly continuous. Observe that if $r, r' \in \mathbf{R}$ and $x, x' \in \mathbf{R}^m$, then

$$\begin{aligned}\rho(rx, r'x') = \|rx - r'x'\| &= \|r(x - x') + (r - r')x - (r - r')(x - x')\| \\ &\leq |r|\|x - x'\| + |r - r'|\|x\| + |r - r'|\|x - x'\|.\end{aligned}$$

Thus if $\epsilon > 0$, then choosing δ so that $\delta(|r| + \|x\|) < \frac{1}{2}\epsilon$ and $\delta^2 < \frac{1}{2}\epsilon$ we infer that if $\sqrt{(r - r')^2 + \|x - x'\|^2} < \delta$, then $|r - r'| < \delta$ and $\|x - x'\| < \delta$, and so $\rho(rx, r'x') \leq \delta(|r| + \|x\|) + \delta^2 < \epsilon$. The scaling operation is thus continuous.

On the other hand for any positive number δ and fixed point $p \in \mathbf{R}^m$ with $p \neq 0$ we may take $r = 1/\delta$, $r' = 1/\delta + \delta/2$, $x = rp$, $x' = r'p$, then $\rho(rx, r'x') = \|rx - r'x'\| = (r'^2 - r^2)\|p\| = (1 + \frac{1}{4}\delta^2)\|p\| > \|p\|$. Thus the scaling operation is not uniformly continuous. ■

1.3.13. EXAMPLE. *The scalar product.* The scalar product of points in the Euclidean space $\mathbf{R}^m$ may be viewed as a map $\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$. It is continuous but not uniformly. This may be checked by an argument analogous to that of the last Example. ■

By an argument similar to the proof of Theorem 1.3.11 we obtain

1.3.14. THEOREM. *If a map $f: X \to Y$ is continuous at the point $x_0 \in X$ and the map $g: Y \to Z$ is continuous at the point $y_0 = f(x_0)$, then the composition $gf: X \to Z$ is continuous at the point x_0.* ■

In the discussion above we successively picked out more and more general classes of maps: non-expansive, Lipschitz, uniformly continuous and continuous. Metric spaces taken as objects form a category with each of these classes of maps as morphisms. We now study the isomorphisms in these categories.

A map $f: X \to Y$ which is non-expansive, bijective and has an inverse $f^{-1}: Y \to X$ which is non-expansive is called an *isometric* map or simply an *isometry*. An isometry $f: X \to Y$ is thus a map of a space X onto a space Y characterized by the condition

$$\rho(f(x), f(x')) = \rho(x, x')$$

for any two points $x, x' \in X$. The isometries of a fixed metric space onto itself form a transformation group.

1.3.15. EXAMPLE. *Translations.* Fix a point $a \in \mathbf{R}^m$ and define a map $t_a: \mathbf{R}^m \to \mathbf{R}^m$ by the formula $t_a(x) = x + a$ for $x \in \mathbf{R}^m$. Maps of this form are called *translations*. These are of course isometric, since $\rho(t_a(x), t_a(x')) = \|x + a - x' - a\| = \|x - x'\| = \rho(x, x')$ for any points $x, x' \in \mathbf{R}^m$.

Note moreover that $t_0 = \mathrm{id}$, $t_b t_a = t_{a+b}$, $t_a^{-1} = t_{-a}$ for every $a, b \in \mathbf{R}^m$. Translations thus form an abelian subgroup of the group of isometries of the space $\mathbf{R}^m$. The subgroup of translations has also the additional property that for any two points $x, y \in \mathbf{R}^m$ there is exactly one translation which takes the point x onto y; this is the translation t_{y-x}. ■

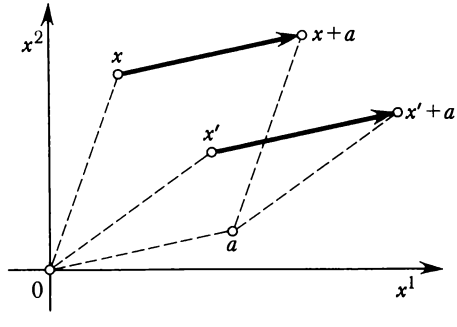


Fig.3. The translation t_a defined by the formula $t_a(x) = x + a$.

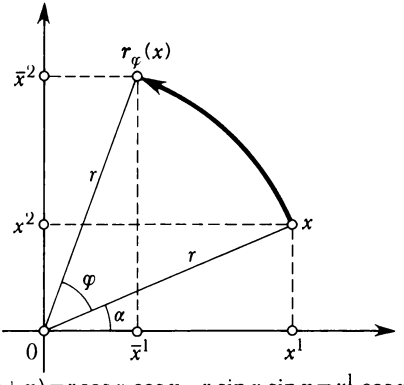
1.3.16. EXAMPLE. *Rotations.* Fix a real number φ and define a map $r_\varphi: \mathbf{R}^2 \to \mathbf{R}^2$ by the formula $r_\varphi(x^1, x^2) = (x^1 \cos\varphi - x^2 \sin\varphi, x^1 \sin\varphi + x^2 \cos\varphi)$ for $(x^1, x^2) \in \mathbf{R}^2$. The map is called an *elementary rotation* through an angle of φ. We note that we do not define the angle φ as such but only a rotation through an angle φ; however despite this formality the sense of the definition agrees with the intuitive notion of rotation. Obviously $r_\varphi = r_{\varphi + 2\pi k}$ for $k = 0, \pm 1, \ldots$

Every elementary rotation is an isometry. We have

$$\begin{aligned}\rho^2(r_\varphi(x), r_\varphi(x')) &= ((x^1 - x'^1)\cos\varphi - (x^2 - x'^2)\sin\varphi)^2 \\ &\quad + ((x^1 - x'^1)\sin\varphi + (x^2 - x'^2)\cos\varphi)^2 \\ &= (x^1 - x'^1)^2 + (x^2 - x'^2)^2 = \rho^2(x, x'),\end{aligned}$$

where $x = (x^1, x^2)$, $x' = (x'^1, x'^2)$. Note also that $r_0 = \mathrm{id}$, $r_\psi r_\varphi = r_{\psi+\varphi}$, $r_\varphi^{-1} = r_{-\varphi}$ for each $\varphi, \psi \in \mathbf{R}$. The elementary rotations thus form an abelian subgroup of the group of isometries of the plane $\mathbf{R}^2$.

Let $1 \leq i < j \leq m$. Consider a map $f: \mathbf{R}^m \to \mathbf{R}^m$ defined so that if $y = f(x)$ with $x = (x^1, x^2, \ldots, x^m)$, $y = (y^1, y^2, \ldots, y^m)$, then $y^k = x^k$ for $k \neq i, j$, and the induced map taking $(x^i, x^j) \in \mathbf{R}^2$ to the point $(y^i, y^j) \in \mathbf{R}^2$ is an elementary rotation. The composition of a finite number of maps of this type (allowing all possible choices of i, j) is called a *rotation* of the space $\mathbf{R}^m$. For $m = 1$ the rotations are taken to be just the two maps: identity and the map $f(x) = -x$ for $x \in \mathbf{R}$. It is easily checked that every rotation of the space $\mathbf{R}^m$ is an isometry fixing the point 0 and that the rotations form an abelian subgroup of the group of isometries of the space $\mathbf{R}^m$.



$$\bar{x}^1 = r\cos(\alpha+\varphi) = r\cos\alpha\cos\varphi - r\sin\alpha\sin\varphi = x^1\cos\varphi - x^2\sin\varphi,$$
$$\bar{x}^2 = r\sin(\alpha+\varphi) = r\cos\alpha\sin\varphi + r\sin\alpha\cos\varphi = x^1\sin\varphi + x^2\cos\varphi.$$

Fig.4. An elementary rotation through an angle of φ.

Note moreover that for any two points $a, b \in \mathbf{R}^m$ satisfying $\|a\| = \|b\|$ there is a rotation $f: \mathbf{R}^m \to \mathbf{R}^m$ such that $f(a) = b$. To prove this it suffices to show that for every $a \in \mathbf{R}^m$ there is a rotation $f: \mathbf{R}^m \to \mathbf{R}^m$ such that $f(a) = (\|a\|, 0, \ldots, 0)$. This is obvious when $m = 1$ and when $m = 2$ may be checked by a simple calculation. Thereafter we proceed by induction. Let $a = (a^1, a^2, \ldots, a^m) \in \mathbf{R}^m$ and consider $a' = (a^2, a^3, \ldots, a^m) \in \mathbf{R}^{m-1}$. There is a rotation $f': \mathbf{R}^{m-1} \to \mathbf{R}^{m-1}$ such that $f'(a') = (\|a'\|, 0, \ldots, 0)$. Let $a'' = (a^1, \|a'\|) \in \mathbf{R}^2$. There is a rotation $f'': \mathbf{R}^2 \to \mathbf{R}^2$ such that $f''(a'') = (\|a''\|, 0)$. Putting $f = (f'' \times \mathrm{id}_{\mathbf{R}^{m-2}})(\mathrm{id}_{\mathbf{R}} \times f'): \mathbf{R}^m \to \mathbf{R}^m$ we obtain a rotation such that $f(a) = (f'' \times \mathrm{id}_{\mathbf{R}^{m-2}})(a^1, \|a'\|, 0, \ldots, 0) = (\|a''\|, 0, \ldots, 0) = (\|a\|, 0, \ldots, 0)$.

The property of rotations proved above together with the corresponding property of translations allow us to deduce that for any two points $a, b \in \mathbf{R}^m$ there is an isometry $f: \mathbf{R}^m \to \mathbf{R}^m$ (namely the composition of a translation and a rotation) such that $f(a) = 0$ while the point $f(b)$ has all its coordinates zero with the exception possibly of the first. This observation simplifies some geometric arguments as it allows us to assume outright that certain points in the space $\mathbf{R}^m$ have convenient positions. ■

1.3.17. EXAMPLE. *The antipodal map.* The map $a: \mathbf{R}^m \to \mathbf{R}^m$ taking each point $x \in \mathbf{R}^m$ to its opposite $a(x) = -x$ is called the *antipodal* map or an *antipodism*. It is obviously

an isometry since $\rho(a(x), a(x')) = \| - x + x'\| = \|x - x'\| = \rho(x, x')$ for $x, x' \in \mathbf{R}^m$. ■

Two metric spaces X and Y for which there exists an isometry map of X onto Y are said to be *isometric* or *congruent*. The class of all metric spaces which are isometric to a space X is called the *metric type* of the space X. The theory of isometry invariants, that is the theory concerned with those properties of metric spaces which if enjoyed by one space are enjoyed by all spaces isometric to it, is called *metric geometry* or simply *geometry*.

1.3.18. EXAMPLE. *Any n-dimensional affine subspace of* $\mathbf{R}^m$ *is isometric with the space* $\mathbf{R}^n$. Let $H \subset \mathbf{R}^m$ be an n-dimensional affine subspace. By Theorem 0.4.12 we may assume that H is the hull of an orthonormal set of points $a_0, a_1, \ldots, a_n$. For every point $x = \sum_{j=0}^n r^j a_j \in H$ with $\sum_{j=0}^n r^j = 1$ associate the point $f(x) = (r^1, r^2, \ldots, r^n) \in \mathbf{R}^n$. To show that f is an isometry consider also the point $\bar{x} = \sum_{j=0}^n \bar{r}^j a_j \in H$ with $\sum_{j=0}^n \bar{r}^j = 1$. By Assertion 0.4.17 we have $x - \bar{x} = \sum_{j=1}^n (r^j - \bar{r}^j)(a_j - a_0)$, whence

$$\begin{aligned}\|x - \bar{x}\|^2 &= \sum_{j=1}^n \sum_{k=1}^n (r^j - \bar{r}^j)(r^k - \bar{r}^k)(a_j - a_0) \cdot (a_k - a_0) \\ &= \sum_{j=1}^n \sum_{k=1}^n (r^j - \bar{r}^j)(r^k - \bar{r}^k)\delta_{jk} \\ &= \sum_{j=1}^n (r^j - \bar{r}^j)^2 = \|f(x) - f(\bar{x})\|^2. \blacksquare\end{aligned}$$

1.3.19. EXAMPLE. *Commutativity of the metric product.* Consider the metric spaces (X_1, ρ_1), (X_2, ρ_2) and their metric products $(X_1, \rho_1) \times (X_2, \rho_2)$ and $(X_2, \rho_2) \times (X_1, \rho_1)$. We may map the former product onto the latter by sending the point (x_1, x_2) to the point (x_2, x_1). This map is easily seen to be an isometry. From the point of view of metric geometry we may thus identify the two products and in this sense claim that the metric product is *commutative*. ■

1.3.20. EXAMPLE. *Associativity of the metric product.* Consider a sequence of metric spaces $(X_1, \rho_1), (X_2, \rho_2), \ldots, (X_m, \rho_m)$, a number k such that $1 \leq k \leq m$, and the two metric products $(\bigtimes_{i=1}^k (X_i, \rho_i)) \times (\bigtimes_{i=k+1}^m (X_i, \rho_i))$ and $\bigtimes_{i=1}^m (X_i, \rho_i)$. We may map the former product onto the latter by sending the point $((x_1, x_2, \ldots, x_k), (x_{k+1}, x_{k+1}, \ldots, x_m))$ to the point $(x_1, x_2, \ldots, x_m)$. This map is easily seen to be an isometry. From the point of view of metric geometry we may again identify the two products and in this sense claim that the metric product is *associative*. ■

We have the following conclusion from an inductive argument which appeals to Example 1.3.20.

1.3.21. COROLLARY. *An arbitrary family of metric spaces which for any pair of members contains their metric product, also contains the metric product of any finite number of its member.* ■

A Lipschitz map $f: X \to Y$ with constant c which is bijective and has as its inverse $f^{-1}: Y \to X$ a Lipschitz map with constant $1/c$ is called a *similarity map with*

coefficient c. A similarity $f: X \to Y$ with coefficient c is thus a map of the space X onto Y characterized by condition

$$\rho(f(x), f(x')) = c\rho(x, x')$$

for any pair of points $x, x' \in X$. The similarities of a fixed metric space onto itself form a transformation group.

1.3.22. EXAMPLE. *Homotheticity.* The map $h_c: \mathbf{R}^m \to \mathbf{R}^m$ defined by the formula $h_c(x) = cx$ for $x \in \mathbf{R}^m$ is known as a *homotheticity with coefficient* c, where c is any positive real number. Note that every homotheticity with coefficient c is a similarity with coefficient c. Clearly, $\rho(h_c(x), h_c(x')) = \|cx - cx'\| = c\|x - x'\| = c\rho(x, x')$ for all $x, x' \in \mathbf{R}^m$.

Observe moreover that $h_1 = \mathrm{id}$, $h_{c_2} h_{c_1} = h_{c_1 c_2}$, $h_c^{-1} = h_{1/c}$ for all positive c_1, c_2, c. It follows that the homotheticities constitute an abelian subgroup of the group of all similarities of the space $\mathbf{R}^m$.

We also see that every similarity of the space $\mathbf{R}^m$ may be expressed as the composition of an isometry with a homotheticity. Indeed, if $f: \mathbf{R}^m \to \mathbf{R}^m$ is a similarity with coefficient c then the composition $g = h_{1/c} f$ is a similarity with coefficient $(1/c)c = 1$, so is an isometry; thus $f = h_c g$ where g is an isometry. ■

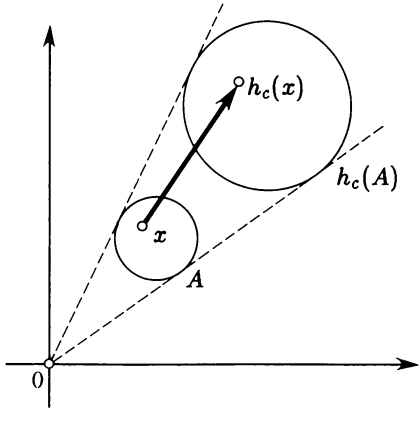


Fig.5. A homotheticity with coefficient $c > 1$.

1.3.23. EXAMPLE. *The diagonal map.* Let X be a metric space. Consider the power $\bigtimes^m X = X \times X \times \ldots \times X$ where m is any natural number. The map $d: X \to \bigtimes^m X$ defined by the formula $d(x) = (x, x, \ldots, x) \in \bigtimes^m X$ for $x \in X$ is called the *diagonal map.* It is a similarity with coefficient $\sqrt{m}$ of the set X onto the set $\Delta = \{(x_1, x_2, \ldots, x_m) \in \bigtimes^m X : x_1 = x_2 = \ldots = x_m\}$ known as the *diagonal* of the power $\bigtimes^m X$. We have

$$\rho((x, x, \ldots, x), (x', x', \ldots, x')) = \sqrt{m\rho^2(x, x')} = \sqrt{m}\rho(x, x')$$

for every $x, x' \in X$. (See also Supplement 1.S.12). ■

Two metric spaces X and Y for which there exists a similarity map from X onto Y are called *similar*.

1.3.24. EXAMPLE. *All non-degenerate closed intervals on the real line are similar.* If $a < b$, the map $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = (b-a)x + a$ is a similarity with coefficient $(b-a)$ mapping the unit interval I onto the closed interval $[a, b]$. We may similarly prove that *all non-empty open intervals on the real line are similar* and also that *all non-empty intervals which are left-closed or right-closed are similar.* ■

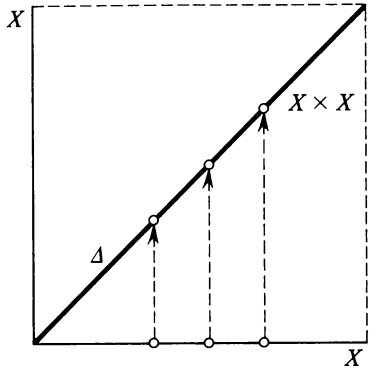


Fig.6. The diagonal map $d: X \to \Delta \subset X \times X$ is a similarity (Example 1.3.23).

The theory of similarity invariants, that is of those properties of metric spaces which if enjoyed by one space are enjoyed by all spaces similar to it is known as *similarity geometry.*

A uniformly continuous map (respectively, a continuous map) $f: X \to Y$ which is bijective and has as its inverse $f^{-1}: Y \to X$ a map that is likewise uniformly continuous (respectively, continuous) is called a *uniform homeomorphism* (respectively, a *homeomorphism*). The uniform homeomorphisms (respectively, the homeomorphisms) of a fixed space onto itself form a transformation group. Two metric spaces X and Y for which there is a uniform homeomorphism (respectively, a homeomorphism) mapping X onto Y are said to be *uniformly homeomorphic* (respectively, *homeomorphic*).

1.3.25. EXAMPLE. Let X denote the real line with discrete metric and Y the same set with the usual real line metric. The map $\mathrm{id}: X \to Y$ is continuous (even, uniformly continuous, cf. Example 1.3.10) and bijective but is not a homeomorphism since the inverse map is not continuous. More may be proved, namely that the spaces X and Y are not homeomorphic. ■

1.3.26. EXAMPLE. *Any non-empty open interval and the real line are homeomorphic.* By the property given in Example 1.3.24 we may consider the interval $(-\pi, \pi)$. Then the map $f: (-\pi, \pi) \to \mathbf{R}$ defined by the formula $f(x) = \tan x$ for $x \in (-\pi, \pi)$ is continuous and bijective and the inverse function $f^{-1} = \arctan$ is also continuous. The homeomorphism f is however not uniform since the map f is not uniformly continuous. More may be proved, namely that the interval $(-\pi, \pi)$ (or any other open interval) and the real line $\mathbf{R}$ are not uniformly homeomorphic (cf. Corollary 1.9.10). ■

From Example 1.3.9 we obtain the following.

1.3.27. COROLLARY. *Every affine isomorphism is a uniform homeomorphism.* ■

1.3.28. EXAMPLE. *Inversion.* Observe that if r is any fixed positive real number, then to every point $x \in \mathbf{R}^m\backslash\{0\}$ there corresponds just one point $i(x)$ lying on the half-line which has its endpoint at the origin and passes through x, such that $\|i(x)\|\|x\| = r^2$. In fact, if $i(x) = (1-t)0 + tx = tx$, where $t \in \mathbf{R}_+$, then since $r^2 = \|i(x)\|\|x\| = \|tx\|\|x\| = t\|x\|^2$, we have $t = r^2/\|x\|^2$, whence $i(x) = (r^2/\|x\|^2)x$ for $x \in \mathbf{R}^m\backslash\{0\}$.

The map $i\colon \mathbf{R}^m\backslash\{0\} \to \mathbf{R}^m\backslash\{0\}$ is known as the *inversion* in the sphere centred at 0 of radius r. It takes the sphere onto itself while the open ball centred at 0 of radius r with its centre removed is taken to the complement of the closed ball with the same centre and radius.

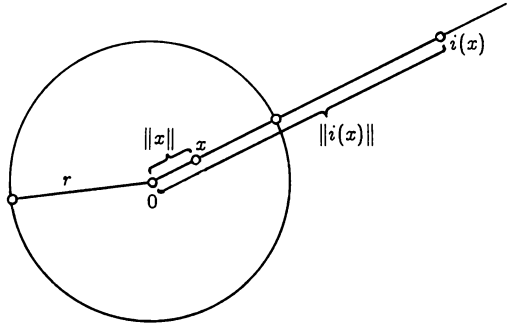


Fig.7. Inversion i in the circle centred at 0 and of radius r in the plane $\mathbf{R}^2$ defined by requiring $\|i(x)\|\|x\| = r^2$ for $x \in \mathbf{R}^2\backslash\{0\}$.

Inversion is of course a bijective map of $\mathbf{R}^m\backslash\{0\}$ onto itself and it follows from Examples 1.3.4 and 1.3.12 that it is continuous. Since it satisfies the condition $ii = \mathrm{id}$ (maps with such a property are called *involutions*) it is also a homeomorphism of the set $\mathbf{R}^m\backslash\{0\}$ onto itself. ■

1.3.29. EXAMPLE. *Stereographic projection.* Let S be an m-dimensional sphere centred at $c = (0,0,\ldots,0,r) \in \mathbf{R}^{m+1}$ of radius $r > 0$. Then of course $0 \in S$. Let H be the hyperplane defined by the equation $x^{m+1} = 2r$. Evidently H meets S in precisely the one point $a = (0,0,\ldots,0,2r)$.

Observe that every line joining the origin 0 to an arbitrary point $x \in S$, where $x = (x^1, x^2, \ldots, x^{m+1}) \neq 0$ meets H in exactly one point, which we denote by $s(x)$. For, if the point $s(x) = (1-t)0 + tx = tx$ is also to belong to H then $tx^{m+1} = 2r$. Thus t is determined uniquely by $t = 2r/x^{m+1}$ and so $s(x) = (2r/x^{m+1})x$ for $x \in S\backslash\{0\}$. The map $s\colon S\backslash\{0\} \to H$ is called the *stereographic projection* of the sphere S from the pole 0 onto the hyperplane H. It is easy to see that s takes $S\backslash\{0\}$ onto H.

Recall from Example 1.3.28 that the inversion i in the sphere centred at 0 of radius $2r$ in the space $\mathbf{R}^{m+1}$ is given by the formula $i(x) = (4r^2/\|x\|^2)x$ for $x \in \mathbf{R}^{m+1}\backslash\{0\}$. If we also assume that $x = (x^1, x^2, \ldots, x^{m+1}) \in S\backslash\{0\}$ then $\|x-c\|^2 = r^2$, that is $\|x\|^2 - 2x\cdot c + \|c\|^2 = r^2$. Since $x\cdot c = x^{m+1}r$ and $\|c\|^2 = r^2$ we have $\|x\|^2 = 2x^{m+1}r$ so the

inversion i restricted to the sphere S is given by $i(x) = (4r^2/2x^{m+1}r)x = (2r/x^{m+1})x = s(x)$. Thus the stereographic projection s is the restriction of the inversion i to the sphere S. It follows therefore that s is a homeomorphism of the set $S\setminus\{0\}$ onto H. ■

1.3.30. COROLLARY. *The m-dimensional sphere with one point removed and the m-dimensional Euclidean space* $\mathbf{R}^m$ *are homeomorphic.* ■

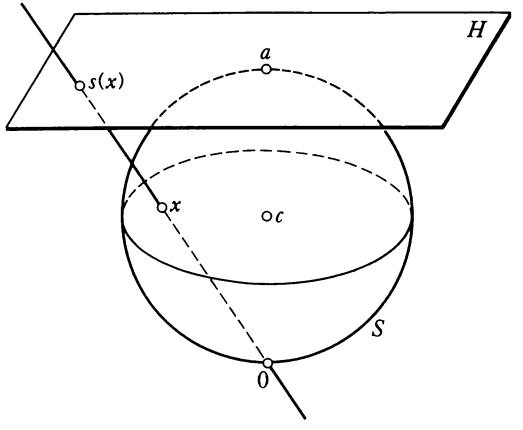


Fig.8. The stereographic projection of the sphere S from the pole 0 onto the hyperplane H (Example 1.3.29).

The theory of uniform homeomorphism (respectively, homeomorphism) invariants, that is the properties of metric spaces which if enjoyed by one such space are enjoyed by all those which are uniformly homeomorphic (respectively, homeomorphic) to it is called the *uniform topology* (respectively, the *topology*) of metric spaces. The class of all metric spaces homeomorphic to a space X is called the *topological type* of the space X.

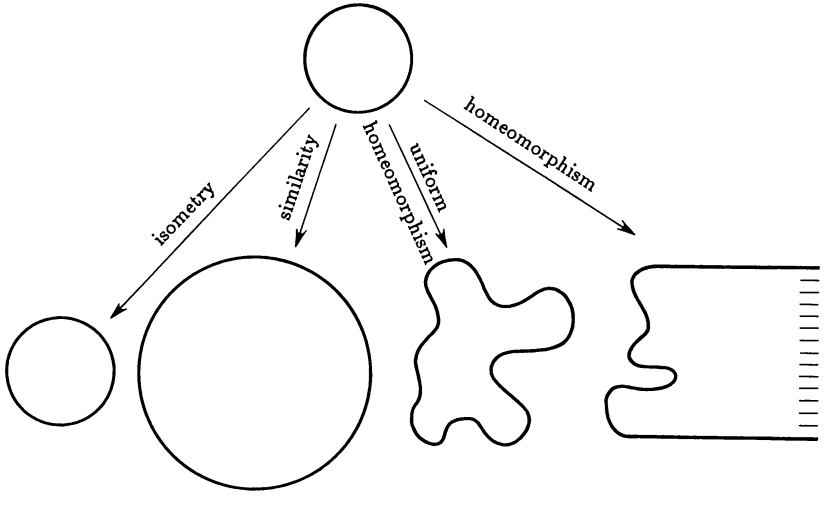


Fig.9. What may be obtained from the open disc by various maps.

We have successively distinguished more and more general groups of maps between metric spaces: isometries, similarities, uniform homemorphisms and homeomorphisms. Their corresponding successively smaller sets of invariants are studied by metric geometry, similarity geometry, uniform topology and topology. Every property which is an invariant of a wider group of maps is in a sense more fundamental than one which is preserved only by a narrower group of maps. In this sense the properties studied in topology (which are more briefly called *topological properties*) are more fundamental than properties studied for example in metric geometry (which are more briefly known as *metric properties*). In the sequel when introducing properties connected with metric spaces we shall always try to establish as wide a group of maps as possible under which the property considered is preserved. This will allow us to carry out a classification of the properties under study. The classification scheme for geometric properties just presented is due to F. Klein and is known as the *Erlangen programme* (see Supplement 1.S.11).

Sometimes the Erlangen programme is taken to mean a more general approach as follows. Apart from properties of spaces (which can be identified with the classes of spaces possessing the given property), various concepts connected with these spaces are studied. With each concept is connected a class of objects to which the concept may be applied. In describing the nature of the concept, we study the images of the objects corresponding to the concept under various maps (the definition of the image is usually obvious). Concepts which are invariant under the maps of a given group are then said to belong to the appropriate geometry. Concepts belonging to metric geometry are briefly called *metric concepts*, those belonging to topology - *topological concepts.*

Exercises

a) Show that for $j = 0, 1, \ldots$ the maps $p_j: \mathbf{R}^\omega \to \mathbf{R}$ defined by the formula $p_j(x^1, x^2, \ldots) = x^j$ are non-expansive.

b) Prove that inversion and stereographic projection are not uniform homeomorphisms.

c) Give an example of a bijective map $f: \mathbf{R} \to \mathbf{R}$ which is uniformly continuous but is not a uniform homeomorphism.

d) Show that every continuous bijective map of the real line $\mathbf{R}$ onto itself is a homeomorphism.

1.4. Metric concepts

We will be concerned in this section with certain elementary concepts which may be defined using only the concept of distance; they will therefore be metric concepts. Moreover, we restrict our attention to concepts which are either strictly metric in character or at most invariant under similarity.

We suppose that (X, ρ) is a fixed metric space.

If $c \in X$ and r is a positive real number then by the *open ball centred at c of radius r* we mean the set $B(c; r) = \{x \in X : \rho(x, c) < r\}$. Of course $c \in B(c; r)$, but it might

happen that the open ball does not contain any points other than its centre; such is the case for balls of radius not exceeding 1 in the discrete metric space. However, in that space a ball of radius greater than 1 will contain all the points of the space, we thus see that in an arbitrary metric space a ball in general does not determine its centre and radius (cf. Problem 1.P.27).

The set $\bar{B}(c;r) = \{x \in X : \rho(c,x) \leq r\}$ is called the *closed ball centred at c of radius r*.

The real number (or possibly the symbol ∞) defined by

$$\operatorname{diam} X = \sup\{\rho(x,x') : x, x' \in X\}$$

is called the *diameter* of the non-empty metric space X. Additionally we put $\operatorname{diam} \emptyset = 0$. A space X whose diameter $\operatorname{diam} X$ differs from ∞ is called a *bounded* space. For example, a discrete metric space is bounded and has diameter 1. The real line is not bounded. The unit interval is and has diameter 1.

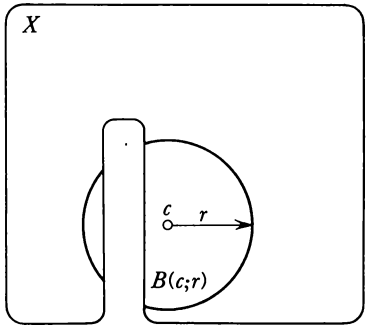


Fig.10. The open ball $B(c;r)$ in the metric space X.

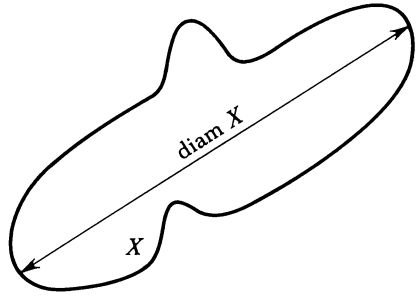


Fig.11. The diameter $\operatorname{diam} X$ of a metric space X.

Frequently subsets of a fixed metric spaces are studied; we then speak of the diameter $\operatorname{diam} A$ of the set $A \subset X$, or say that a set $A \subset X$ is or is not bounded, on the understanding that the appropriate concept refers to A as a metric subspace.

We now list some simple properties of the concepts just presented.

1.4.1. ASSERTION. *If* $B \subset A$, *then* $\operatorname{diam} B \leq \operatorname{diam} A$. ■

1.4.2. COROLLARY. *A subset of a bounded space is bounded.* ■

1.4.3. THEOREM. $\operatorname{diam} \bigtimes_{i=1}^{m} A_i \leq \sqrt{m} \max\{\operatorname{diam} A_i : i = 1,2,\ldots,m\}$.

PROOF. The inequality follows from Lemma 1.2.4. ■

1.4.4. COROLLARY. *The metric product of finitely many bounded spaces is bounded.* ■

We thus obtain the following.

1.4.5. EXAMPLE. *Every m-dimensional cube is a bounded set.* ■

1.4.6. THEOREM. *Every closed ball is a bounded set and moreover* $\operatorname{diam} \bar{B}(c;r) \leq 2r$.

PROOF. If $x, x' \in \bar{B}(c;r)$, then $\rho(x,x') \leq \rho(x,c) + \rho(c,x') \leq r + r = 2r$. ■

1.4.7. LEMMA. *For every set $A \subset \mathbf{R}^m$ the equation* $\operatorname{diam}\operatorname{conv} A = \operatorname{diam} A$ *holds.*

PROOF. Put $d = \operatorname{diam} A$ and $d' = \operatorname{diam}\operatorname{conv} A$. We may of course assume that $d < \infty$. Observe that if $a \in A$, then $A \subset \bar{B}(a; d)$, hence $\operatorname{conv} A \subset \operatorname{conv} \bar{B}(a; d) = \bar{B}(a; d)$. So if $a' \in \operatorname{conv} A$, then $\rho(a, a') \leq d$. It follows that if $a' \in \operatorname{conv} A$, then $A \subset \bar{B}(a'; d)$ and so $\operatorname{conv} A \subset \operatorname{conv} \bar{B}(a'; d) = \bar{B}(a'; d)$. So if $a', a'' \in \operatorname{conv} A$, then $\rho(a', a'') \leq d$ and so $d' \leq d$. But the inequality $d \leq d'$ is obvious so the proof is complete. ■

1.4.8. LEMMA. *For a set $A \subset X$ to be bounded it is necessary and sufficient that it is contained in an open (or, closed) ball in the space X.*

PROOF. Sufficiency of the condition follows from Corollary 1.4.2 and Theorem 1.4.6. To prove its necessity suppose that $a \in A$. Then $A \subset B(a; 1 + \operatorname{diam} A)$, since if $a' \in A$ then $\rho(a, a') \leq \operatorname{diam} A < 1 + \operatorname{diam} A$. ■

1.4.9. LEMMA. *If $A_j \subset X$ is a bounded set for $j = 1, 2, \ldots, n$, then the set $\bigcup_{j=1}^{n} A_j$ is bounded.*

PROOF. Pick $a_j \in A_j$ for $j = 1, 2, \ldots, n$ and let $x, x' \in \bigcup_{j=1}^{n} A_j$. Suppose for the sake of argument that $x \in A_j$ and $x' \in A_{j'}$. Then using Theorem 1.1.2 we infer that

$$\begin{aligned}\rho(x, x') &\leq \rho(x, a_j) + \rho(a_j, a_{j'}) + \rho(a_{j'}, x') \\ &\leq 2 \max\{\operatorname{diam} A_j : j = 1, 2, \ldots, n\} + \max\{\rho(a_j, a_{j'}) : j, j' = 1, 2, \ldots, n\},\end{aligned}$$

which completes the proof. ■

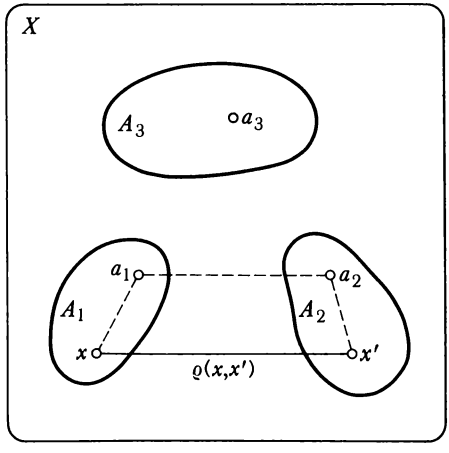


Fig.12. The sets A_1, A_2, A_3 are bounded, hence so is their union (see Lemma 1.4.9).

The concepts of open and closed balls and the diameter of a metric space have a strictly metric character. The concept of a bounded space belongs to similarity geometry but not to uniform topology. For, if by X we denote the set of natural numbers with discrete metric and by Y the same set with the subspace metric of the real line, then the identity map of X onto Y is a uniform homeomorphism, but the space X is bounded, whereas Y is not.

A map $f: X \to Y$ of a set X into a metric space is said to be *bounded*, if the image $f(X)$ is a bounded set in Y (cf. Supplement 1.S.13). In the particular case when Y is the real line $\mathbf{R}$ this signifies by Lemma 1.4.8 that there is a number M such that $|f(x)| < M$ for all $x \in X$. Then of course, $\inf f$, $\sup f \in [-M, M]$.

Another concept which can be defined in metric spaces is the distance of a point from a set. Suppose that $\emptyset \neq A \subset X$ and $x \in X$. The *distance* of the point x from the set A is the number

$$\rho(x, A) = \inf\{\rho(x, a) : a \in A\}.$$

It is convenient to agree the convention $\rho(x, \emptyset) = 1$. Evidently for a singleton set $\{a\}$ we have $\rho(x, \{a\}) = \rho(x, a)$. Moreover, if $x \in A$, then $\rho(x, A) = 0$, but not conversely, since for instance the distance of the point 0 from the open interval $(0, 1)$ on the real line $\mathbf{R}$ is zero. The set $B(A; r) = \{x \in X : \rho(x, A) < r\}$ for $A \subset X$ and $r > 0$ is called a *generalized open ball centred on* A (or, *about* A) *of radius* r.

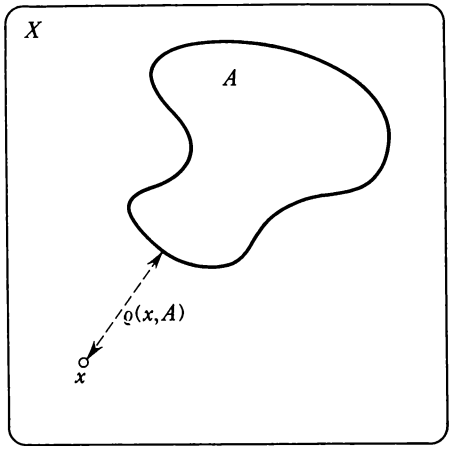


Fig.13. Distance $\rho(x, A)$ of a point x from a set A.

Observe that the following holds.

1.4.10. THEOREM. *The map which sends a point $x \in X$ to its distance from a fixed set $A \subset X$ is a non-expansive map of the space X into the real line.*

PROOF. We may of course assume that $A \neq \emptyset$. If $x, x' \in X$ and $a \in A$, then $\rho(x, a) \leq \rho(x', a) + \rho(x, x')$; thus $\rho(x, A) \leq \rho(x', A) + \rho(x, x')$. Similarly $\rho(x', A) \leq \rho(x, A) + \rho(x, x')$, so $|\rho(x, A) - \rho(x', A)| \leq \rho(x, x')$. ■

Exercises

a) Give an example of a metric space X, a point $c \in X$ and a positive real number r with the property that $\operatorname{diam} \bar{B}(c; r) < 2r$.

b) Prove that if every proper subset of a metric space X is bounded, then the space X itself is bounded.

1.5. Convergence and limits

We now introduce the important concept of convergence for a sequence of points in a metric space. Suppose that (X, ρ) is a fixed metric space and suppose we are given a sequence of points $x_n \in X$, where $n = 1, 2, \ldots$, and a point $x_0 \in X$. We say that the sequence $\{x_n\}$ *converges to the point* x_0, symbolically written $\lim_n x_n = x_0$, if the sequence of real numbers $\rho(x_n, x_0)$ converges to zero, that is if for every positive real number ϵ there is an index k such that $\rho(x_n, x_0) < \epsilon$ for $n \geq k$. Using the concept of an open ball we may say that the sequence $\{x_n\}$ converges to x_0 if and only if for every positive real number ϵ there exists an index k such that $x_n \in B(x_0; \epsilon)$ for all $n \geq k$. Any point x_0 satisfying the condition $\lim_n x_n = x_0$ is called *limit* of the sequence $\{x_n\}$. The definitions above easily imply the following results (see also Supplement 1.S.17):

1.5.1. THEOREM. *A sequence may have at most one limit.*

PROOF. If $\lim_n x_n = x_0$ and $\lim_n x_n = x_0'$, then for every positive real number ϵ there is an index k such that

$$\rho(x_0, x_0') \leq \rho(x_n, x_0) + \rho(x_n, x_0') < \epsilon + \epsilon = 2\epsilon \quad \text{for all} \quad n \geq k.$$

Hence $\rho(x_0, x_0') = 0$, that is $x_0 = x_0'$. ■

1.5.2. THEOREM. *The constant sequence* $\{x_n\}$, *where* $x_n = x_0$ *for* $n = 1, 2, \ldots$, *converges and* $\lim_n x_n = x_0$.

PROOF. Since in this case $\rho(x_n, x_0) = 0$ for $n = 1, 2, \ldots$ we have $\lim_n \rho(x_n, x_0) = 0$. ■

1.5.3. THEOREM. *Every subsequence of a sequence converging to a point* x_0, *also converges to the point* x_0.

PROOF. If $\lim_n x_n = x_0$, then $\lim_n \rho(x_n, x_0) = 0$, hence for every subsequence $\{x_{k_n}\}$ we have $\lim_n \rho(x_{k_n}, x_0) = 0$, so $\lim_n x_{k_n} = x_0$. ■

1.5.4. THEOREM. *If every subsequence of the sequence* $\{x_n\}$ *contains a subsequence converging to the point* x_0, *then* $\lim_n x_n = x_0$.

PROOF. If the sequence $\{x_n\}$ did not converge to x_0, there would exist a positive real number ϵ and a subsequence $\{x_{k_n}\}$ such that $x_{k_n} \notin B(x_0, \epsilon)$ for $n = 1, 2, \ldots$ The subsequence $\{x_{k_n}\}$ would then not contain any subsequence converging to the point x_0. ■

From Theorems 1.5.2 and 1.5.3 it follows in particular that any sequence $\{x_n\}$ for which there is an index k such that $x_n = x_0$ for all $n \geq k$, is convergent to the point x_0; sequences with this property are called *almost constant*. It is easily observed that in discrete metric spaces the almost constant sequences are the only convergent sequences.

We prove the following straightforward result.

1.5.5. LEMMA. *If for a sequence of points* $x_n \in X$, *where* $n = 1, 2, \ldots$, *there is a positive real number* η *such that* $\rho(x_n, x_k) \geq \eta$ *for all* $n \neq k$, *then no subsequence of this sequence converges.*

PROOF. Since the property of the sequence $\{x_n\}$ mentioned in the hypothesis passes down to all its subsequences, it is enough to prove that the sequence $\{x_n\}$ itself does not converge. If there were a point $x_0 \in X$ with $\lim_n x_n = x_0$, then starting from some index onwards the terms of the sequence $\{x_n\}$ would have to lie in the ball $B(x_0; \eta/3)$. However, by Theorem 1.4.6 it follows that $\operatorname{diam} B(x_0; \eta/3) \leq 2\eta/3 < \eta$. This contradiction completes the proof. ■

We now prove that the concept of the limit of a sequence of points in a metric space belongs not only to metric geometry, but also to topology. We prove moreover that the concept is not only invariant under homeomorphisms but also under arbitrary continuous maps and as such is characteristic of the latter (cf. Supplement 1.S.9).

1.5.6. THEOREM. *A map $f: X \to Y$ is continuous at the point $x_0 \in X$ if and only if for every sequence of points $x_n \in X$, for $n = 1, 2, \ldots$, satisfying the equation $\lim_n x_n = x_0$ the equation $\lim_n f(x_n) = f(x_0)$ holds.*

PROOF. To prove that the condition is necessary, suppose given a sequence of points $\{x_n\}$ convergent to the point x_0 of the space X and a positive real number ϵ. Choose a positive real number δ such that for every point $x \in X$ satisfying $\rho(x_0, x) < \delta$ we have $\rho(f(x_0), f(x)) < \epsilon$. Since $\lim_n x_n = x_0$, there is an index k such that $\rho(x_0, x_n) < \delta$ for all $n \geq k$. Then $\rho(f(x_0), f(x_n)) < \epsilon$ for all $n \geq k$. Thus $\lim_n f_n(x_n) = f(x_0)$.

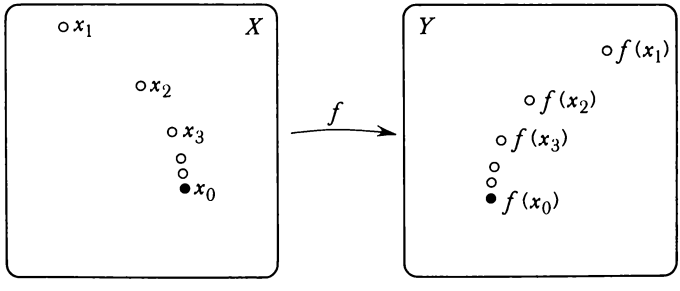


Fig.14. The map f is continuous at x_0 if the condition $\lim_n x_n = x_0$ implies $\lim_n f(x_n) = f(x_0)$ for every sequence $\{x_n\}$ (Theorem 1.5.6).

Suppose now that the map f is not continuous at x_0. There exists therefore a positive real number ϵ such that for each $n = 1, 2, \ldots$ there is a point $x_n \in X$ satisfying $\rho(x_0, x_n) < 1/n$ and $\rho(f(x_0), f(x_n)) \geq \epsilon$. Since $\lim_n 1/n = 0$ we have also $\lim_n \rho(x_0, x_n) = 0$, so that $\lim_n x_n = x_0$. On the other hand in view of the inequality $\rho(f(x_n), f(x_0)) \geq \epsilon > 0$ for $n = 1, 2, \ldots$ we conclude that $\{f(x_n)\}$ is not convergent to $f(x_0)$. Thus the stated condition is also sufficient. ■

1.5.7. COROLLARY. *A map f of a space X onto a space Y is a homeomorphism if and only if for every sequence of points $x_n \in X$, where $n = 0, 1, \ldots$, the conditions $\lim_n x_n = x_0$ and $\lim_n f(x_n) = f(x_0)$ are equivalent.*

PROOF. Bijectivity of the map f follows from Theorems 1.5.1 and 1.5.2. Continuity of the maps f and f^{-1} follows from Theorem 1.5.6. ■

We now examine how convergence in subspaces is related to convergence in the original space and how convergence in the metric product is related to convergence of the coordinates in the factor spaces. The following is obvious.

1.5.8. ASSERTION. *If A is a metric subspace of a space X and $x_n \in A$ for $n = 0, 1, 2, \ldots$, then $\lim_n x_n = x_0$ in A if and only if $\lim_n x_n = x_0$ in X.* ■

1.5.9. THEOREM. *If $(X, \rho) = \bigtimes_{i=1}^{m}(X_i, \rho_i)$ and $x_n = (x_n^1, x_n^2, \ldots, x_n^m) \in X$ for $n = 0, 1, 2, \ldots$, then $\lim_n x_n = x_0$ in the space X if and only if $\lim_n x_n^i = x_0^i$ in each of the spaces X_i for $i = 1, 2, \ldots, m$.*

PROOF. If $\lim_n x_n = x_0$, then $\lim_n \rho(x_n, x_0) = 0$. By Lemma 1.2.4 we infer that $\lim_n \max\{\rho_i(x_n^i, x_0^i) : i = 1, 2, \ldots, m\} = 0$ and so $\lim_n \rho_i(x_n^i, x_0^i) = 0$, thus $\lim_n x_n^i = x_0^i$ for $i = 1, 2, \ldots, m$.

Conversely, if $\lim_n x_n^i = x_0^i$, that is $\lim_n \rho_i(x_n^i, x_0^i) = 0$ for $i = 1, 2, \ldots, m$, then $\lim_n \max\{\rho_i(x_n^i, x_0^i) : i = 1, 2, \ldots, m\} = 0$, so by Lemma 1.2.4 we obtain $\lim \rho(x_n, x_0) = 0$, or $\lim_n x_n = x_0$. ■

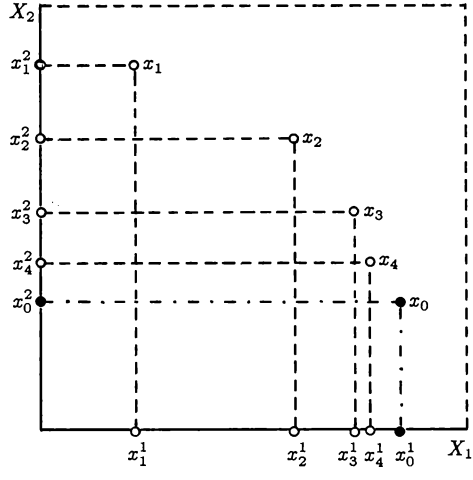


Fig.15. Convergence in a metric product is equivalent to coordinatewise convergence (Theorem 1.5.9).

We next prove a theorem on convergence in the projective space P^m.

1.5.10. THEOREM. *If $x_n \in P^m$ for $n = 0, 1, \ldots$, then $\lim_n x_n = x_0$ in the space P^m if and only if for $n = 0, 1, \ldots$ there is a representative $\xi_n \in \mathbf{R}^{m+1}$ of the point x_n such that $\lim_n \xi_n = \xi_0$ in the space $\mathbf{R}^{m+1}$.*

PROOF. Say $\lim_n x_n = x_0$. Suppose $x_n = [\xi_n]$ where $\xi_n \in \mathbf{R}^{m+1}$ and $\|\xi_n\| = 1$ for $n = 0, 1, \ldots$ We thus have $\lim_n \rho([\xi_n], [\xi_0]) = 0$. We infer from the continuity of the function cos that $\lim_n |\xi_n \cdot \xi_0| = 1$. Denoting by $\epsilon_n = 0, \pm 1$ the sign of the scalar product $\xi_n \cdot \xi_0$, we may suppose that $\epsilon_n \neq 0$ for $n = 1, 2, \ldots$ Now $\|\epsilon_n \xi_n - \xi_0\|^2 = 2 - 2\epsilon_n \xi_n \cdot \xi_0$ and $\lim_n \epsilon_n \xi_n \cdot \xi_0 = 1$, so $\lim_n \epsilon_n \xi_n = \xi_0$. The desired representative is thus $\epsilon_n \xi_n$ for $n = 1, 2, \ldots$ and ξ_0 for $n = 0$.

Conversely, let $0 \neq \xi_n \in \mathbf{R}^{m+1}$ for $n = 0, 1, \ldots$ and let $\lim_n \xi_n = \xi_0$, that is

$\lim_n \|\xi_n - \xi_0\| = 0$. But $\|\xi_n - \xi_0\|^2 = \|\xi_n\|^2 - 2\xi_n \cdot \xi_0 + \|\xi_0\|^2$ for $n = 1,2,\ldots$ and $\lim_n \|\xi_n\|^2 = \|\xi_0\|^2$, so $\lim_n(\xi_n \cdot \xi_0) = \|\xi_0\|^2$, that is $\lim_n(\xi_n/\|\xi_n\| \cdot \xi_0/\|\xi_0\|) = \lim_n \|\xi_0\|/\|\xi_n\| = 1$. Hence $\lim_n \rho([\xi_n],[\xi_0]) = 0$, or $\lim_n x_n = x_0$, where $x_n = [\xi_n]$ for $n = 0,1,\ldots$ ■

We obtain the following Corollary of the theorem above.

1.5.11. COROLLARY. *So far as proper points are concerned, convergence of sequences of points in the projective space P^m is identical with convergence in the Euclidean space $\mathbf{R}^m$.* ■

1.5.12. COROLLARY. *If L is the straight line through the two points $a = (a^1, a^2, \ldots, a^m)$ and $b = (b^1, b^2, \ldots, b^m)$ of Euclidean space $\mathbf{R}^m$ and $x_n = (1 - r_n)a + r_n b \in L$ for $n = 1,2,\ldots$ with $\lim_n |r_n| = \infty$, then $\lim_n x_n = [0, b^1 - a^1, \ldots, b^m - a^m]$.*

PROOF. We have $x_n = [1, (1-r_n)a^1 + r_n b^1, \ldots, (1-r_n)a^m + r_n b^m] = [\frac{1}{r_n}, \frac{a^1}{r_n} + (b^1 - a^1), \ldots, \frac{a^m}{r_n} + (b^m - a^m)]$ when $r_n \neq 0$. So if $\lim_n |r_n| = \infty$ then $\lim_n \frac{1}{r_n} = 0$ and by Theorem 1.5.10 we have $\lim_n x_n = [0, b^1 - a^1, \ldots, b^m - a^m]$. ■

Corollary 1.5.12 justifies viewing the improper points of the projective space P^m as the directions of lines contained in $\mathbf{R}^m$. The discussion above motivates the construction of the following model of the projective space P^m.

1.5.13. EXAMPLE. *A topological model of the projective space P^m.* Consider the m-dimensional closed unit ball $\bar{B}^m$, the m-dimensional open unit ball B^m and the $(m-1)$-dimensional unit sphere S^{m-1}. Define a map of the ball $\bar{B}^m$ onto the projective space P^m. To each point $x \in B^m$ assign the point $tan(\frac{1}{2}\pi\|x\|)x \in \mathbf{R}^m$. To each point $x \in S^{m-1}$ assign the direction of the line through the points 0 and x. We thus obtain a continuous map of the ball $\bar{B}^m$ onto the space P^m which bijectively maps the open ball B^m onto the set of proper points of the space P^m whilst the inverse image of every improper point of this space is a pair of antipodal points of the sphere S^{m-1}. We may thus view the space P^m as the closed ball $\bar{B}^m$ on whose boundary we have identified every pair of antipodal points. (This identification is to be understood in the everyday, intuitive sense as a kind of "pasting together"; a precise method for defining a topology in a space with identifications will be given in Section 7.4).

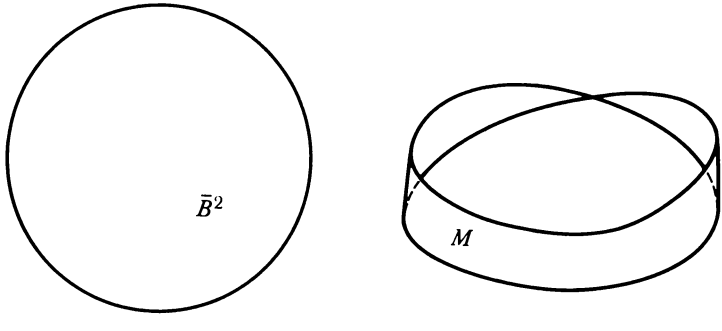


Fig.16. The disc $\bar{B}^2$ and the Möbius band M. The projective plane P^2 is homeomorphic to the set obtained by pasting together $\bar{B}^2$ and M along their boundaries (Example 1.5.13).

In particular the projective line P^1 is homeomorphic with the closed interval whose two endpoints have been identified; it is in effect homeomorphic with a circle.

The construction for the case of the projective plane is not so easy to carry out. It is easier to picture if we start with the space obtained from the unit square $I^2 \subset \mathbf{R}^2$ by identifying each point $(0,r)$ along its edge with the point $(1,1-r)$ for $r \in I$. This space is known as the *Möbius band*; a topological model of it may easily be constructed in the space $\mathbf{R}^3$. Points of the form $(r,0)$ and of the form $(1-r,1)$ for $r \in I$ form a simple closed curve (that is, a set homeomorphic to the circle) on the Möbius band, since the vertex $(0,0)$ was identified with the vertex $(1,1)$ and the vertex $(0,1)$ with the vertex $(1,0)$; this curve is called the *boundary* of the Möbius band.

The operation of identifying antipodal points along the unit circle bounding the unit disc $\bar{B}^2$ may now be replaced by the operation of pasting together the Möbius band and the disc $\bar{B}^2$ along their homeomorphic boundaries. Instead of identifying antipodal points of S^1 we are now able to join them by a segment lying in the Möbius band, disjoint segments corresponding to distinct pairs. This construction cannot be carried out in the Euclidean space $\mathbf{R}^3$ without creating self-intersections, a consequence of the fact that the projective plane is not homeomorphic with any subset of $\mathbf{R}^3$. The construction can however be carried out in 4-dimensional Euclidean space $\mathbf{R}^4$. ■

We will now be concerned with the establishment of the concept of a limit for sequences of maps. Suppose given a sequence of maps $f_n: X \to Y$ for $n = 1,2,\ldots$ and a map $f_0: X \to Y$. We say that the sequence $\{f_n\}$ is *pointwise convergent* to the map f_0 (which is called its *limit*) if $\lim_n f_n(x) = f_0(x)$ for each $x \in X$.

1.5.14. EXAMPLE. Taking $f_n(x) = x^n$ for $x \in I$, $n = 1,2,\ldots$, where x^n denotes (exceptionally) the n-th power of the number x, we obtain a sequence of maps $f_n: I \to I$ pointwise convergent to the function $f_0: I \to I$ defined by the formula

$$f_0(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

This example demonstrates that the *limit of a pointwise convergent sequence of continuous maps need not itself be continuous.* ■

Such a situation will not however arise when a stronger condition is placed on convergence: We say that the sequence $\{f_n\}$ is *uniformly convergent* to the map f_0 (which is then also called its *limit*) if for each positive real number ϵ there is an index k such that $\rho(f_n(x), f_0(x)) < \epsilon$ for every $n \geq k$ and every point $x \in X$. Of course every uniformly convergent sequence of maps is pointwise convergent and Example 1.5.14 and Theorem 1.5.15, which we are about to prove, together show that the converse is not necessarily true.

1.5.15. THEOREM. *The limit of a uniformly convergent sequence of continuous maps is a continuous map.*

PROOF. Suppose that the sequence of continuous maps $f_n: X \to Y$ for $n = 1,2,\ldots$ is uniformly convergent to the map $f_0: X \to Y$. To prove that the map f_0 is continuous at every point $x \in X$ consider an arbitrary real number $\epsilon > 0$ and choose an index

k such that $\rho(f_n(x'), f_0(x')) < \frac{1}{3}\epsilon$ for each $x' \in X$ and every $n \geq k$. Next using the continuity of the map f_k at the point x choose a real number $\delta > 0$ such that for every point $x' \in X$ satisfying $\rho(x, x') < \delta$ it is the case that $\rho(f_k(x), f_k(x')) < \frac{1}{3}\epsilon$. Then appealing to Theorem 1.1.2 we obtain

$$\begin{aligned}\rho(f_0(x), f_0(x')) &\leq \rho(f_0(x), f_k(x)) + \rho(f_k(x), f_k(x')) + \rho(f_k(x'), f_0(x')) \\ &< \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon,\end{aligned}$$

for each point $x' \in X$ such that $\rho(x, x') < \delta$. Thus f_0 is a continuous map. ■

Exercises

a) Prove that if in a metric space X the only convergent sequences are those that are almost constant, then X is homeomorphic to a space with discrete metric.

b) Prove that the map $f: X \to Y$ is uniformly continuous if and only if for any two sequences $x_n, x'_n \in X$, where $n = 1, 2, \ldots$, the equation $\lim_n \rho(x_n, x'_n) = 0$ implies the equation $\lim_n \rho(f(x_n), f(x'_n)) = 0$.

c) Suppose that $x_n = (x_n^1, x_n^2, \ldots) \in I^\omega$ for $n = 0, 1, 2, \ldots$ Show that $\lim_n x_n = x_0$ in I^ω if and only if $\lim_n x_n^i = x_0^i$ in $\mathbf{R}$ for $i = 1, 2, \ldots$ Give an example to show that the analogous proposition is false for the Hilbert space $\mathbf{R}^\omega$.

1.6. Open and closed sets

Let A be a subset of a fixed metric space X. For each point $a \in A$ one of two possibilities arises: either there is an open ball centred at a entirely contained in A or every open ball centred at a intersects the complement $X \setminus A$. In the former case we say that a is an *interior* point of the set A in the space X, in the latter that a is a *boundary* point of the set A in the space X.

For example, the numbers 0 and 1 are bounday points of the unit interval I in the space of real numbers $\mathbf{R}$, while all the remaining points of the interval are interior points. However, if we view the same unit interval I as a subset of the space of non-negative real numbers $\mathbf{R}_+$, then the number 0 becomes an interior point and only the number 1 is a boundary point of I. We thus see that the classification of points of a set into interior and boundary points is of a relative nature depending not only on the set in question but also on the space in which it is regarded as lying. If the underlying space is beyond doubt, then we shall drop all reference to the space.

The set of interior points of a set A in a space X is called the *interior* of A in the space X and is denoted $\operatorname{int} A$. A set all of whose points are interior points of the set relative to a space X is called an *open* set of the space X; a set all of whose points are boundary points of the set relative to a space X is called a *boundary* set of the space[1]. Open sets A are thus characterized by the equation $\operatorname{int} A = A$ and boundary sets by $\operatorname{int} A = \emptyset$.

[1] Since the term 'boundary set' is not in general use in the English speaking world, the phrase 'set with empty interior', or whatever similar words context permits, will instead be used in the sequel.

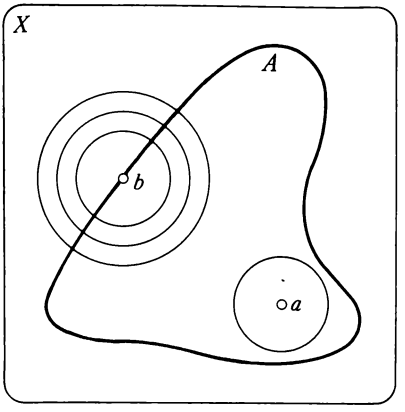


Fig.17. An interior point a and a boundary point b of the set A in the space X.

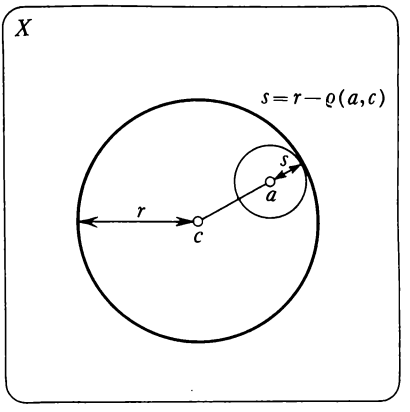


Fig.18. The open ball $B(c;r)$ is an open set of the space X (Lemma 1.6.1).

For instance the empty set $\emptyset$ and the whole space are open in X. Every set in a discrete metric space is open. The set $\{(x^1, x^2) \in \mathbf{R}^2 : x^2 = 0\}$ is a boundary set (i.e. is a set with empty interior) in the Euclidean plane $\mathbf{R}^2$.

1.6.1. LEMMA. *The open ball $B(c;r)$ for $c \in X$ and $r > 0$ is an open set in X.*

PROOF. Let $a \in B(c;r)$ and put $s = r - \rho(a,c) > 0$. Then $B(a;s) \subset B(c;r)$, for if $x \in B(a;s)$ then $\rho(x,c) \leq \rho(x,a) + \rho(a,c) < s + \rho(a,c) = r$ and so $x \in B(c;r)$. ■

A consequence of the lemma is the following.

1.6.2. THEOREM. *The interior of any set is an open set.*

PROOF. If $a \in \operatorname{int} A$ then there is a real number $r > 0$ such that $B(a;r) \subset A$. Since the ball $B(a;r)$ is an open set we have of course $B(a;r) \subset \operatorname{int} A$. ■

We now derive the basic properties of open sets (see also Supplement 1.S.17).

1.6.3. THEOREM. *The union of an arbitrary collection of open sets in a metric space is an open set of the space.*

PROOF. Suppose $a \in A = \bigcup_{t \in T} A_t$ where for each $t \in T$ the set A_t is open in the space X. Say $a \in A_{t_0}$; then there is a positive real number r such that $B(a;r) \subset A_{t_0} \subset A$, so A is open in the space X. ■

1.6.4. THEOREM. *The intersection of a finite number of open sets in a metric space is an open set of the space.*

PROOF. Suppose that $a \in A = \bigcap_{j=1}^{n} A_j$, where the set A_j is open in the space X for $j = 1, 2, \ldots, n$. For $j = 1, 2, \ldots, n$ there thus exists a positive real number r_j such that $B(a;r_j) \subset A_j$. Taking $r = \min\{r_1, r_2, \ldots, r_n\}$ we conclude that $B(a;r) \subset A$ which completes the proof. ■

On the real line $\mathbf{R}$ if we define $A_j = B(0;1/j)$ for $j = 1, 2, \ldots$, we obtain an infinite sequence of open sets whose intersection $\bigcap_{j=1}^{\infty} A_j = \{0\}$ is not an open set in

R. Therefore the theorem above cannot be strengthened to the case of infinitely many sets.

1.6.5. THEOREM. *If X_0 is a metric subspace of the space X, then a set $A_0 \subset X_0$ is open in X_0 if and only if $A_0 = A \cap X_0$ for some set A which is open in X.*

PROOF. If $A_0 = A \cap X_0$ where A is open in X and $a \in A_0 \subset A$, then there is a real number $r_a > 0$ such that $B(a; r_a) \subset A$ where $B(a; r_a)$ denotes the ball in the space X. Denoting by $B_0(a; r_a)$ the ball in the space X_0 we have $B_0(a; r_a) = X_0 \cap B(a; r_a)$ and so $B_0(a; r_a) \subset A \cap X_0 = A_0$.

Conversely, suppose that $A_0 \subset X_0$ is an open set of the space X_0. For each point $a \in A_0$ there is therefore a positive real number r_a with the property that the ball $B_0(a; r_a)$ in the space X_0 is contained in A_0. Take $A = \bigcup_{a \in A_0} B(a; r_a)$, where $B(a; r_a)$ denotes the ball in the space X. Now $B_0(a; r_a) = B(a; r_a) \cap X_0$ so $\bigcup_{a \in A_0} B_0(a; r_a) = A \bigcap X_0$. On the other hand, since $a \in B_0(a; r_a) \subset A_0$ for every point $a \in A_0$ we have $\bigcup_{a \in A_0} B_0(a; r_a) = A_0$. Hence $A_0 = A \cap X_0$ and the set A is open in the space X by Lemma 1.6.1 and Theorem 1.6.3. ■

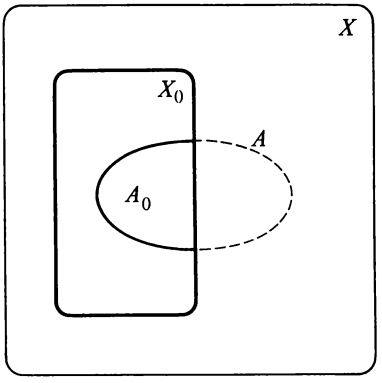


Fig.19. The set A_0 is open (closed) in the subspace X_0 if it is of the form $A \cap X_0$ where the set A is open (closed) in X (Theorems 1.6.5 and 1.6.17).

1.6.6. THEOREM. *If A_i for each $i = 1, 2, \ldots, m$ is an open subset of the space (X_i, ρ_i), then the metric product $\bigtimes_{i=1}^m A_i$ is an open subset of the metric product $\bigtimes_{i=1}^m (X_i, \rho_i)$.*

PROOF. Say $a = (a_1, a_2, \ldots, a_m) \in A = \bigtimes_{i=1}^m A_i \subset X = \bigtimes_{i=1}^m X_i$. For $i = 1, 2, \ldots, m$ there is a positive real number r_i such that $B(a_i; r_i) \subset A_i$. Taking $r = \min\{r_1, r_2, \ldots, r_m\}$ we obtain $B(a; r) \subset A$. For, if $\rho(a, x) < r$ where $x = (x_1, x_2, \ldots, x_m)$ is in X, then by Lemma 1.2.4 we have $\rho_i(a_i, x_i) \leq \rho(a, x) < r \leq r_i$, so $x_i \in B(a_i; r_i) \subset A_i$ for $i = 1, 2, \ldots, m$; hence $x \in A$. ■

We now pass to the definition of certain classes of sets which are in a sense dual to the classes of open sets and boundary sets (i.e. those with empty interior). Suppose X is a fixed metric space and that $A \subset X$. We shall say that a point $x \in X$ is a ***limit point*** of the set A in the space X if there is a sequence of points $a_n \in A$ for $n = 1, 2, \ldots$ such that $\lim_n a_n = x$. Evidently every point of the set A is its limit point but not conversely.

For example if $X = \mathbf{R}$ and A is the interval $(0, 1)$ then not only the points of the set A but also the numbers 0 and 1 are limit points of A in the space of real numbers $\mathbf{R}$.

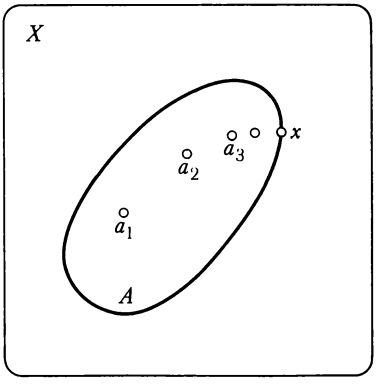


Fig.20. A limit point x of the set A in the space X.

The set of limit points of a set A in a space X is called its *closure* in the space X and denoted $\operatorname{cl} A$. If a set A contains all its limit points in the space X, then we say that the set is *closed* in the space. If every point of the space X is a limit point of the set A, then we say that the set A is *dense* in the space X. Closed sets A are thus characterized by the equation $\operatorname{cl} A = A$ and dense sets A by the equation $\operatorname{cl} A = X$.

For example the empty set $\emptyset$ and the whole space X are closed in X. In any metric space every singleton set is closed. In a discrete metric space every set is closed. The unit interval is closed in the real line $\mathbf{R}$. The set of rational numbers Q is dense in the real line $\mathbf{R}$.

The limit points of a set in a metric space may easily be characterized by their distance to the set. The following in fact obtains.

1.6.7. ASSERTION. *The statements* $x \in \operatorname{cl} A$ *and* $\rho(x, A) = 0$ *are equivalent.* ■

It is also easily observed that closure does not increase the diameter, i.e. the following holds.

1.6.8. ASSERTION. *For each set* $A \subset X$ *we have the equation* $\operatorname{diam} \operatorname{cl} A = \operatorname{diam} A$. ■

We also prove the following analogue of Lemma 1.6.1.

1.6.9. LEMMA. *The closed ball* $\bar{B}(c; r)$ *for* $c \in X$ *and* $r > 0$ *is a closed set of* X.

PROOF. Suppose that $x_n \in \bar{B}(c; r)$ for $n = 1, 2, \ldots$ and that $\lim_n x_n = x$. Then $\rho(x, c) \leq \rho(x, x_n) + \rho(x_n, c)$ for $n = 1, 2, \ldots$ But $\rho(x_n, c) \leq r$ and $\lim_n \rho(x, x_n) = 0$, so $\rho(x, c) \leq r$, which completes the proof. ■

Observe that $\operatorname{cl} B(c; r) \subset \bar{B}(c; r)$, but in general these two sets differ. For instance, in any discrete metric space we have for each point $x \in X$ that $B(x; 1) = \{x\}$ and so $\operatorname{cl} B(x; 1) = \{x\}$, whereas $\bar{B}(x; 1) = X$.

There are sets which are simultaneously closed and open in the space X, for instance the empty set $\emptyset$ and the whole space X. Any such set is called an *open-and-closed* set of X.[2)]

Under a homeomorphism h of a space X onto a space Y the limit points of a set $A \subset X$ are taken to the limit points of its image $h(A) \subset Y$ and conversely. In particular it follows that under a homeomorphism closed set of X are taken to closed sets of Y and dense sets of X to dense sets of Y.

We now prove a basic result.

1.6.10. THEOREM. *A point $x \in X$ is a limit point of a set A in the space X if and only if it is not an interior point of the complement $X\backslash A$ in the space X.*

PROOF. If $x \in \operatorname{cl} A$ then there is a sequence of points $a_n \in A$ where $n = 1, 2, \ldots$ such that $\lim_n a_n = x$. Then any open ball $B(x, \epsilon)$ contains points of the set A namely the terms of the sequence $\{a_n\}$ with sufficiently high index. Thus $x \notin \operatorname{int}(X\backslash A)$.

Conversely, if $x \notin \operatorname{int}(X\backslash A)$, every open ball centred at x meets the set A. By taking in turn the balls of radius $1/n$ for $n = 1, 2, \ldots$ we may choose points $a_n \in B(x; \frac{1}{n}) \cap A$ for $n = 1, 2, \ldots$ Then $\rho(x, a_n) < 1/n$ and so $\lim_n a_n = x$, hence we deduce that $x \in \operatorname{cl} A$. ■

1.6.11. COROLLARY. *For any set $A \subset X$ the equations $X\backslash \operatorname{cl} A = \operatorname{int}(X\backslash A)$ and $X\backslash \operatorname{int} A = \operatorname{cl}(X\backslash A)$ hold.* ■

1.6.12. COROLLARY. *A set A is closed (respectively, dense) in the space X if and only if the complement $X\backslash A$ is open (respectively, has empty interior) in the space X. A set A is open (respectively, has empty interior) if and only if the complement $X\backslash A$ is closed (respectively dense) in the space X.* ■

The definition of open sets and boundary sets (i.e. those that have empty interior) and Corollary 1.6.11 imply the following

1.6.13. COROLLARY. *A set is a boundary set in the space X if and only if its only subset open in X is the empty set. A set is dense in the space X if and only if it has non-empty intersection with every non-empty open set in the space X.* ■

In particular it follows from Corollary 1.6.11 that under a homeomorphism h of the space X onto a space Y interior points of a set $A \subset X$ are taken to interior points of its image $h(A) \subset Y$ and conversely. It follows that under a homeomorphism open sets of the space are taken to open sets of the space Y and boundary sets of the space X are taken to boundary sets of Y.

1.6.14. EXAMPLE. *The set of proper points of the projective space P^m is open and dense in P^m.* For, by Theorem 1.5.10 it follows that no proper point is a limit point of the set of improper points. On the other hand, it follows from Corollary 1.5.12 that every improper point is a limit point of the set of proper points. ■

1.6.15. EXAMPLE. *Every n-dimensional affine subspace of the space $\mathbf{R}^m$ is a closed set, and when $n < m$ has empty interior.* It follows from Theorem 0.4.20 that for every n-dimensional affine subspace H of $\mathbf{R}^m$ there is an affine isomorphism $f: \mathbf{R}^m \to \mathbf{R}^m$ such

[2)] Colloquially they are called *clopen sets* - though this is a linguistic malpractice.

that $f(H)$ is the affine subspace $H' = \{(x^1, x^2, \ldots, x^m) \in \mathbf{R}^m,\ x^{n+1} = \ldots = x^m = 0\}$. By Corollary 1.3.27 it thus suffices to check that H' has the required properties. That H' is closed in $\mathbf{R}^m$ follows easily from Theorem 1.5.9. To verify that H' has empty interior in $\mathbf{R}^m$ for $n < m$ consider $a = (a^1, a^2, \ldots, a^n, 0, \ldots, 0) \in H'$ and let $r > 0$. Then $(a^1, a^2, \ldots, a^n, \frac{1}{2}r, 0, \ldots, 0) \in B(a; r) \backslash H'$. ■

The following Theorem is based on the result just proved and will be of use to us in Section 6.8. We prove it here rather than later as an illustration of the concepts of closed set and set with empty interior.

1.6.16. THEOREM. *For every set of points $a_1, a_2, \ldots, a_n \in \mathbf{R}^m$ of which $a_1, a_2, \ldots, a_k$ are in general position and for every positive real number ϵ there is a set of points $b_1, b_2, \ldots, b_n \in \mathbf{R}^m$ in general position such that $a_j = b_j$ for $j = 1, 2, \ldots, k$ and $\rho(a_j, b_j) < \epsilon$ for $j = k+1, k+2, \ldots, n$.*

PROOF. We proceed by induction on the number n of points $a_1, a_2, \ldots, a_n$. If $n = 1$, then it suffices to take $b_1 = a_1$. Suppose that of the points $a_1, a_2, \ldots, a_n$ the points $a_1, a_2, \ldots, a_k$ are in general position, that the points $b_1, b_2, \ldots, b_n$ are in general position with $a_j = b_j$ for $j = 1, 2, \ldots, k$ and $\rho(a_j, b_j) < \epsilon$ for $j = k+1, k+2, \ldots, n$. Consider the point $a_{n+1} \in \mathbf{R}^m$.

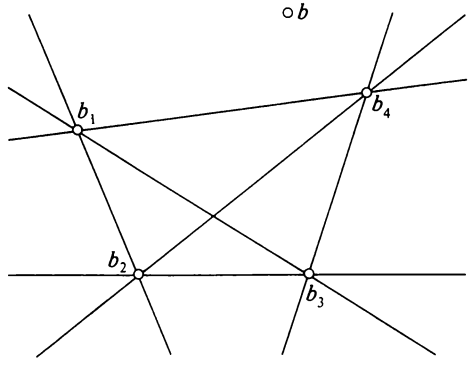


Fig.21. Illustration of the proof of Theorem 1.6.16. The point b which together with the points b_1, b_2, b_3, b_4 forms a set in general position lies outside the union of six lines and thus outside a set with empty interior in the plane.

If the points $b_1, b_2, \ldots, b_n, a_{n+1}$ are in general position, we take $b_{n+1} = a_{n+1}$. Otherwise denote by B the set of points $b \in \mathbf{R}^m$ such that $b_1, b_2, \ldots, b_n, b$ is not in general position. Then $b \in B$ if and only if the set $b_1, b_2, \ldots, b_n, b$ contains an affinely dependent subset consisting of at most $m+1$ points; the subset must of course contain b. From Theorem 0.4.9 it follows that this happens if and only if b together with at most m points of $b_1, b_2, \ldots, b_n$ lie in an affine subspace of dimension less than m.

By Example 1.6.15 every affine hull of a subset of $b_1, b_2, \ldots, b_n$ which is of dimension less than m has empty interior in $\mathbf{R}^m$. Since the number of such affine hulls is finite and a finite union of closed sets with empty interior obviously has empty interior it follows that B has empty interior. The complement $\mathbf{R}^m \backslash B$ is thus a dense set and we may select a point $b_{n+1} \in \mathbf{R}^m \backslash B$ such that $\rho(a_{n+1}, b_{n+1}) < \epsilon$ which completes the construction. ■

Appealing to Lemmas 1.6.11 and 1.6.12 we may also prove the following analogues of Theorems 1.6.2–1.6.5.

1.6.17. THEOREM. *The closure of any set is a closed set.*

PROOF. If $A \subset X$, then by Corollary 1.6.11 we have have $\operatorname{cl} A = X \setminus \operatorname{int}(X \setminus A)$. Using Theorem 1.6.2 we infer that the set $\operatorname{int}(X \setminus A)$ is open in X. By Corollary 1.6.11 we conclude that $\operatorname{cl} A$ is closed in X. ■

1.6.18. THEOREM. *The intersection of an arbitrary collection of closed sets in a metric space is a closed set of the space.*

PROOF. If for each $t \in T$ the set $A_t \subset X$ is closed in X, then its complement $X \setminus A_t$ is an open set in X for each $t \in T$. Hence the union $\bigcup_{t \in T}(X \setminus A_t)$ is an open set in X and so its complement $X \setminus \bigcup_{t \in T}(X \setminus A_t)$ is closed in X. But by De Morgan's Laws we have $X \setminus \bigcup_{t \in T}(X \setminus A_t) = \bigcap_{t \in T} A_t$ which completes the proof. ■

1.6.19. THEOREM. *The union of a finite number of closed sets in a metric space is a closed set of the space.*

PROOF. If for $j = 1, 2, \ldots, n$ each set $A_j \subset X$ is closed in the space X, then each of the complements $X \setminus A_j$ is open for $j = 1, 2, \ldots, n$. Hence the intersection $\bigcap_{j=1}^{n}(X \setminus A_j)$ is an open subset of X and so its complement $X \setminus \bigcap_{j=1}^{n}(X \setminus A_j)$ is closed. By De Morgan's Laws we have $X \setminus \bigcap_{j=1}^{n}(X \setminus A_j) = \bigcup_{j=1}^{n} A_j$, which completes the proof. ■

1.6.20. THEOREM. *If X_0 is a metric subspace of the space X, then a set $A_0 \subset X_0$ is closed in X_0 if and only if $A_0 = A \cap X_0$ for some set A which is closed in X.*

PROOF. The set A_0 is closed in X_0 if and only if the set $X_0 \setminus A_0$ is open in X_0 and by Theorem 1.6.5 this is so if and only if $X_0 \setminus A_0 = B \cap X_0$ for some open set B of X. Taking $A = X \setminus B$ we obtain a closed set of X for which $X_0 \setminus A_0 = (X \setminus A) \cap X_0$, that is $A_0 = A \cap X_0$. ■

Likewise using Theorem 1.5.9 we prove the following.

1.6.21. THEOREM. *If A_i for each $i = 1, 2, \ldots, m$ is a closed subset of the metric space (X_i, ρ_i), then the metric product $\bigtimes_{i=1}^{m} A_i$ is a closed subset of the metric product $\bigtimes_{i=1}^{m}(X_i, \rho_i)$.*

PROOF. If for $n = 1, 2, \ldots$, $a_n = (a_n^1, a_n^2, \ldots, a_n^m) \in A = \bigtimes_{i=1}^{m} A_i \subset X = \bigtimes_{i=1}^{m} X_i$ and $\lim_n a_n = x = (x^1, x^2, \ldots, x^m) \in X$, then $\lim_n a_n^i = x^i$ for $i = 1, 2, \ldots, m$. Since A_i is closed in X_i we conclude that $x^i \in A_i$ for $i = 1, 2, \ldots, m$, so $x \in A$. ■

Before going on we verify a useful property of closed subsets on the real line $\mathbf{R}$.

1.6.22. LEMMA. *If a non-empty set $A \subset \mathbf{R}$ is closed in $\mathbf{R}$ and $\sup A$ (respectively, $\inf A$) is finite then $\sup A \in A$ (respectively, $\inf A \in A$).*

PROOF. From the definition of the least upper bound there is for each $n = 1, 2, \ldots$ a real number $a_n \in A$ satisfying the inequality $(\sup A) - 1/n \leq a_n \leq \sup A$. Hence

$\lim_n a_n = \sup A$, so $\sup A$ is a limit point of the set A and, since A is closed in $\mathbf{R}$, $\sup A \in A$. For the greatest lower bound the proof runs analogously. ■

Let X be a metric space and say $x \in X$. Every open set of X which contains the point x is called a *neighbourhood*[3] of the point x in the space X.

For example in any metric space the open ball $B(c;r)$ is a neighbourhood of the point c. Using the concept of neighbourhood we can give the following characterization of continuous maps.

1.6.23. THEOREM. *Let $f: X \to Y$ be a map between metric spaces and let $y_0 = f(x_0)$ where $x_0 \in X$. For the map f to be continuous at the point x_0 it is necessary and sufficient that for every neighbourhood V of the point y_0 there is a neighbourhood U of the point x_0 such that $f(U) \subset V$.*

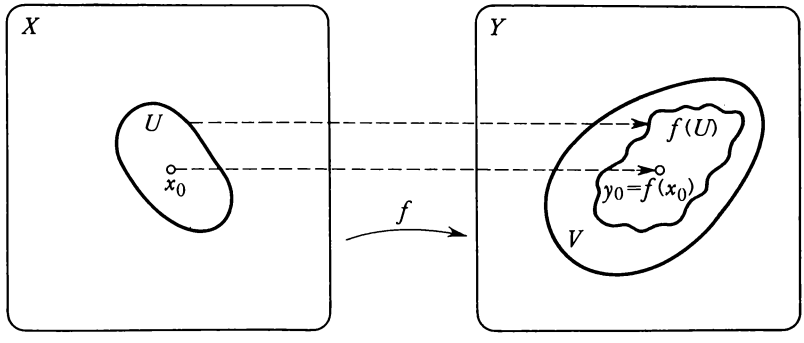


Fig.22. The map f is continuous at x_0 if for every neighbourhood V of the point $y_0 = f(x_0)$ there is a neighbourhood U of the point x_0 such that $f(U) \subset V$ (Theorem 1.6.23).

PROOF. Suppose that the map f is continuous at x_0 and consider any neighbourhood V of the point y_0. Then there exists a positive real number ϵ such that $B(y_0;\epsilon) \subset V$. Choose a positive real number δ such that for every $x \in X$ satisfying $\rho(x_0,x) < \delta$ we have $\rho(f(x_0),f(x)) < \epsilon$. Then taking $U = B(x_0;\delta)$ we have $f(U) \subset B(y_0;\epsilon) \subset V$.

Conversely suppose that for every neighbourhood V of the point y_0 there is a neighbourhood U of the point x_0 such that $f(U) \subset V$. Now consider an arbitrary positive real number ϵ. There then exists a neighbourhood U of x_0 such that $f(U) \subset B(y_0;\epsilon)$. Since U is an open set there exists a positive real number δ such that $B(x_0;\delta) \subset U$. Thus $f(B(x_0;\delta)) \subset B(y_0;\epsilon)$ which proves that the map f is continuous at the point x_0. ■

We derive another condition for continuity from the Theorem above.

1.6.24. THEOREM. *For a map $f: X \to Y$ between metric spaces to be continuous it is necessary and sufficient that for every open (respectively, closed) set B of the space Y the inverse image $f^{-1}(B)$ is open (respectively, closed) in X.*

[3] Some authors prefer to say 'open neighbourhood' reserving the term neighbourhood to mean any set U of which x is an interior point.

PROOF. We first carry out the proof in the open set formulation.

Suppose that the map f is continuous and consider any open set B of the space Y. It is a neighbourhood of each of its members, so if $x \in f^{-1}(B)$ there exists a neighbourhood U_x of the point x in the space X such that $f(U_x) \subset B$. The inverse image $f^{-1}(B)$ is then the union $\bigcup_{x \in f^{-1}(B)} U_x$ of open sets, hence is itself an open set of X by Theorem 1.6.3.

Conversely, if the inverse image of every open set is open and if $x \in X$, then taking any neighbourhood V of the point $y = f(x)$ in the space Y we can find a neighbourhood U of the point x in the space X such that $f(U) \subset V$ — in fact $U = f^{-1}(V)$ is open and has this property. Thus the map f is continuous at the point x.

To prove the Theorem in the closed sets formulation it is enough to observe that the sets B and $f^{-1}(B)$ are closed in the respective spaces X and Y if and only if the sets $Y \setminus B$ and $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$ are open. ■

1.6.25. COROLLARY. *Every closed (respectively, open) half-space of Euclidean space* $\mathbf{R}^m$ *is a closed (respectively, open) set in* $\mathbf{R}^m$.

PROOF. Consider a hyperplane H in $\mathbf{R}^m$ given by the equation $\alpha_0 + \sum_{i=1}^m \alpha_i x^i = 0$. The inverse images under the continuous map f defined by $f(x^1, x^2, \ldots, x^m) = \alpha_0 + \sum_{i=1}^m \alpha_i x^i$ of the half-lines $[0, \infty)$ and $(-\infty, 0]$ are then the closed half-spaces and those of the half-lines $(0, \infty)$, $(-\infty, 0)$ are the open half-spaces determined by H. ■

Using Theorem 1.6.24 we shall derive an important property of metric spaces. First we show that the following holds.

1.6.26. LEMMA. *For any two closed disjoint sets* $A, B \subset X$ *there is a continuous function* $f: X \to I$ *such that* $f(x) = 0$ *for* $x \in A$ *and* $f(x) = 1$ *for* $x \in B$.

PROOF. Since the sets A and B are closed and disjoint it follows from Assertion 1.6.7 that $\rho(x, A) + \rho(x, B) > 0$ for all $x \in X$. Put

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)} \quad \text{for} \quad x \in X.$$

By Theorem 1.4.10 the function $f: X \to I$ is continuous and if $x \in A$ then $f(x) = 0$ while if $x \in B$ then $f(x) = 1$. ■

1.6.27. THEOREM. *For any two disjoint closed sets* $A, B \subset X$ *there are disjoint open sets* $U, V \subset X$ *satisfying* $A \subset U$, $B \subset V$.

PROOF. By Lemma 1.6.26 there is a continuous function $f: X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. Taking $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$ we obtain disjoint sets satisfying $A \subset U$, $B \subset V$. By Theorem 1.6.24 these are open sets. ■

We prove one more result from Theorem 1.6.24.

1.6.28. THEOREM. *Let* A_1 *and* A_2 *be closed subsets of a metric space* X *and suppose* $A_1 \cup A_2 = X$. *If* $f: X \to Y$ *and* $f|A_1$ *and* $f|A_2$ *are continuous, then the map* f *is continuous.*

PROOF. Let F be a closed set in the space Y. By Theorem 1.6.24 the sets $(f|A_1)^{-1}(F)$ and $(f|A_2)^{-1}(F)$ are closed respectively in A_1 and A_2 and so are closed in X. Hence the set $f^{-1}(F) = (f|A_1)^{-1}(F) \cup (f|A_2)^{-1}(F)$ is closed in the space X. Using Theorem 1.6.24 a second time we deduce that f is continuous. ■

A consequence of the theorem above is the next corollary which is useful in the construction of various continuous maps.

1.6.29. COROLLARY. *Let A_1, A_2 be closed subsets of a metric space X and suppose $A_1 \cup A_2 = X$. If the continuous maps $f_1: A_1 \to Y$ and $f_2: A_2 \to Y$ satisfy the condition $f_1|A_1 \cap A_2 = f_2|A_1 \cap A_2$, then the map $f: X \to Y$ defined by the formula:*

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in A_1, \\ f_2(x), & \text{if } x \in A_2, \end{cases}$$

is continuous. ■

By Theorem 1.6.24 and in view of Theorem 1.4.10 we have the following generalization of Lemma 1.6.1.

1.6.30. ASSERTION. *For every $A \subset X$ and $r > 0$ the generalized open ball $B(A; r)$ is an open set of X.* ■

Let $A \subset X$. We say that the point $x \in X$ is an *accumulation* point of the set A if $x \in \mathrm{cl}(A \backslash \{x\})$; in other words, x is an accumulation point of a set A if there is a sequence of points $a_n \in A$ such that $a_n \neq x$ for $n = 1, 2, \ldots$ and $\lim_n a_n = x$. A point of a set A which is not an accumulation point of A is called an *isolated* point of the set.

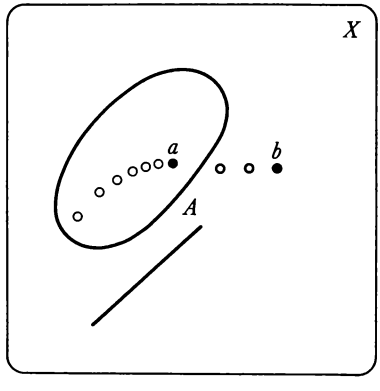


Fig.23. An accumulation point a and an isolated point b of the set A in X.

Thus for a point x to be isolated in the space X it is necessary and sufficient that $X \backslash \{x\}$ be closed in X; by Corollary 1.6.12 this is equivalent to saying that the singleton set $\{x\}$ is open in the space. For instance, in a discrete metric space every point is isolated, whereas every point of the real line $\mathbf{R}$ is an accumulation point.

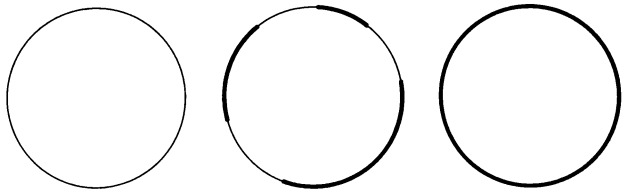


Fig.24. The open disc, the disc with some of its boundary and with all of its boundary. The boundaries of all three sets in the plane are identical.

The intersection of the closure of a set A in a space X with the closure of its complement $X\setminus A$ is called the *boundary* of the set A in the space X and is denoted by the symbol $\operatorname{bd} A$. Now, by Corollary 1.6.11 the equation $\operatorname{cl} A \cap \operatorname{cl}(X\setminus A) = X\setminus(\operatorname{int} A \cup \operatorname{int}(X\setminus A))$ holds whilst $X\setminus(\operatorname{int} A \cup \operatorname{int}(X\setminus A)) = (A\setminus \operatorname{int} A) \cup ((X\setminus A)\setminus \operatorname{int}(X\setminus A))$ so the boundary $\operatorname{bd} A$ of a set A consists of boundary points of the set A and boundary points of its complement. We have the following conclusion.

1.6.31. COROLLARY. *A set A is open-and-closed in X if and only if* $\operatorname{bd} A = \emptyset$. ■

Exercises

a) Show that in the Euclidean space $\mathbf{R}^m$ the closure of the open ball $B(c;r)$ is identical with the closed ball $\bar{B}(c;r)$ and its boundary is the sphere $S(c;r)$.

b) Give a proof of the equation $\operatorname{cl} A = \{x \in X : \rho(x,A) = 0\}$ for every $A \subset X$ (Assertion 1.6.7). Express $\operatorname{int} A$ by means of the distance to the set $X\setminus A$.

c) Carry out the proof of the equation $\operatorname{diam}\operatorname{cl} A = \operatorname{diam} A$ for any $A \subset X$ (Assertion 1.6.8). Is it always true that $\operatorname{diam}\operatorname{int} A = \operatorname{diam} A$?

d) Generalize Theorem 1.6.28 and Corollary 1.6.29 to the case of a finite number of closed sets. State and prove the analogues of these results for open sets.

e) Prove that for every set $A \subset X$ the generalized open ball $B(A;r)$ is a union of balls $B(a;r)$ for $a \in A$. Hence deduce Assertion 1.6.30.

f) Give an example of a continuous map $f: X \to Y$ and of an open (respectively, closed) set A in X such that the image $f(A)$ is not open (respectively, closed) in Y.

g) Give an example of a metric space X and of two homeomorphic subspaces $A, B \subset X$ one of which is open and the other not.

h) Let X_1, X_2 be arbitrary metric spaces and let $A_1 \subset X_1$ and $A_2 \subset X_2$. Show that $\operatorname{bd}(A_1 \times A_2) = (\operatorname{bd} A_1 \times \operatorname{cl} A_2) \cup (\operatorname{cl} A_1 \times \operatorname{bd} A_2)$.

i) Show that if $f, g: X \to Y$ are continuous maps and there is a dense set A of X such that $f|A = g|A$, then $f = g$ on X.

1.7. Connected spaces

A metric space which cannot be expressed as the union of two disjoint, non-empty closed subsets is said to be *connected*. Thus a space X is disconnected if there

is a decomposition of X into two subsets $X = A \cup B$ with the sets A, B non-empty, disjoint and closed in X. For example the metric subspace $\mathbf{R}\backslash\{0\}$ of the real line $\mathbf{R}$ is disconnected since it can be written as a union $\mathbf{R}\backslash\{0\} = A \cup B$ where $A = \{r \in \mathbf{R} : r > 0\}$ and $B = \{r \in \mathbf{R} : r < 0\}$. Another example of a disconnected space is a discrete metric space with at least two points; since every subset of such a space is closed the required decomposition may be obtained by taking an arbitrary point and its complement. Also we have the following theorem.

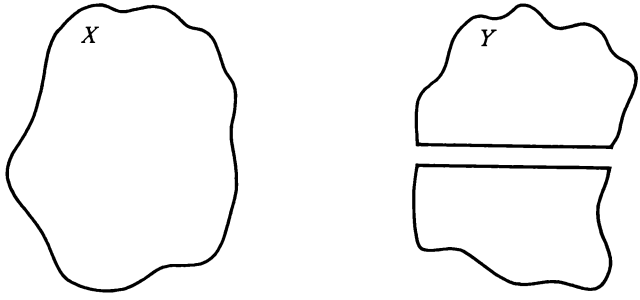


Fig.25. A connected space X and a disconnected space Y.

1.7.1. THEOREM. *For a metric space to be disconnected it is necessary and sufficient that there exists a continuous map of the space onto the two-point discrete metric space* D.

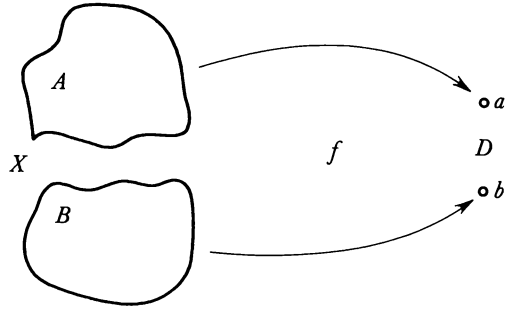


Fig.26. The space X is disconnected when there is a continuous map onto the two-point discrete space D (Theorem 1.7.1).

PROOF. Suppose that $X = A \cup B$ where $\operatorname{cl} A = A \neq \emptyset \neq B = \operatorname{cl} B$ and $A \cap B = \emptyset$. Let $D = \{a, b\}$. The map g of the space X onto the space D defined by the formula:

$$g(x) = \begin{cases} a, & \text{if } x \in A, \\ b, & \text{if } x \in B, \end{cases}$$

is continuous by Theorem 1.6.4 since the inverse images of closed subsets of the space D are closed in X.

Conversely, if there is a continuous map g of the space X onto the space D, then taking $A = g^{-1}(a)$ and $B = g^{-1}(b)$ we obtain a decomposition $X = A \cup B$ where $\operatorname{cl} A = A \neq \emptyset \neq B = \operatorname{cl} B$ and $A \cap B = \emptyset$. ■

Observe also that if $X = A \cup B$ with $A \cap B = \emptyset$, then the sets A and B are mutually complementary and so are both closed if and only if they are both open. We may thus replace the word closed in the definition of connectedness by the word open. For the same reason a space X is disconnected if and only if it contains a proper non-empty set which is open-and-closed.

Observe that in defining a connected space we required the absence of a decomposition $X = A \cup B$ where the sets A and B satisfy three conditions:

1) $A \neq \emptyset \neq B$,
2) $A = \operatorname{cl} A, \quad B = \operatorname{cl} B$,
3) $A \cap B = \emptyset$.

In proving that a space X is connected we often take the contrapositive and assume given a decomposition $X = A \cup B$, where the sets A and B satisfy two (arbitrarily chosen) conditions out of the three, and show that this leads to a contradiction of the remaining condition.

1.7.2. EXAMPLE. *The unit interval is a connected space.* For, suppose that we have a decomposition $I = A \cup B$ where $\operatorname{cl} A = A \neq \emptyset \neq B = \operatorname{cl} B$. We may of course suppose that $1 \in B$. Let $a = \sup A$; this is some real number. Since the set A is closed in the interval I and the interval is in turn closed in the reals $\mathbf{R}$, it follows that A is a closed subset of the real line $\mathbf{R}$. By Lemma 1.6.22 we infer $a \in A$. If $a = 1$, then $A \cap B \neq \emptyset$ and the argument is complete. So suppose $a < 1$. Then the set $\{x \in B : a < x\}$ is non-empty; it is also a closed set of the real line $\mathbf{R}$ since if $x_n \in B$ and $a < x_n$ for $n = 1, 2, \ldots$ and $\lim_n x_n = x_0 \in \mathbf{R}$, then $x_0 \in B$ and $a \leq x_0$ and so $a < x_0$. Put $b = \inf\{x \in B : a < x\}$. Thus Lemma 1.6.22 implies that $b \in B$. Evidently $0 \leq a < b \leq 1$ and moreover no real number in the interval (a, b) can belong either to the set A or the set B which contradicts the hypothesis that $I = A \cup B$. ■

The concept of a connected space belongs not only to metric geometry but also to topology. More in fact is true, as the next theorem shows.

1.7.3. THEOREM. *If f is a continuous map of a connected space X onto a space Y, then Y is also connected.*

PROOF. Suppose that Y is disconnected. By Theorem 1.7.1 there exists a continuous map g of the space Y onto the two-point discrete space D. The composition gf is then a continuous map of the space X onto the space D which is impossible as X was assumed connected. ■

1.7.4. COROLLARY. *Any line segment in Euclidean space $\mathbf{R}^m$ is a connected space.*

PROOF. If $a, b \in \mathbf{R}^m$, then the continuous map f defined by the formula $f(r) = (1 - r)a + rb$ for $r \in I$ maps the interval I onto the segment $\overline{ab}$. ■

The union of connected subspaces of a fixed metric space need not of course be connected. However the following does hold.

1.7.5. THEOREM. *If $X = \bigcup_{t \in T} X_t$, where each of the subspaces X_t is connected for $t \in T$, and $\bigcap_{t \in T} X_t \neq \emptyset$, then the space X is connected.*

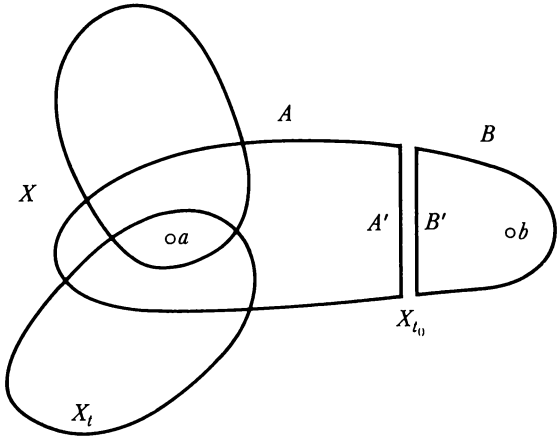


Fig.27. Illustration for the proof of Theorem 1.7.5.

PROOF. Suppose that X is disconnected and $X = A \cup B$ where $\operatorname{cl} A = A \neq \emptyset \neq B = \operatorname{cl} B$ and $A \cap B = \emptyset$. Let $a \in \bigcap_{t \in T} X_t$; we may of course suppose that $a \in A$. Consider any point $b \in B$ and suppose $b \in X_{t_0}$. Take $A' = A \cap X_{t_0}$ and $B' = B \cap X_{t_0}$. Since $a \in A'$ and $b \in B'$ we see that these two sets are non-empty. Moreover $A' \cup B' = (A \cup B) \cap X_{t_0} = X_{t_0}$ and $A' \cap B' = A \cap B \cap X_{t_0} = \emptyset$. By Theorem 1.6.20 the sets A' an B' are closed in X_{t_0}. The subspace X_{t_0} would thus have to be disconnected and this contradiction completes the proof. ■

1.7.6. THEOREM. *If a metric space X for each pair of points there exists a connected subspace containing the pair of points, then the space X is connected.*

PROOF. Consider a fixed point $a \in X$ and for each point $b \in X$ let $C(b)$ denote a connected subspace of X containing both points a and b. Then $X = \bigcup_{b \in X} C(b)$ and our proposition follows from the last theorem since $a \in \bigcap_{b \in X} C(b) \neq 0$. ■

Corollary 1.7.4 and Theorem 1.7.6 together imply the following.

1.7.7. COROLLARY. *Every convex subset of Euclidean space $\mathbf{R}^m$ is a connected space.* ■

In particular we have from Example 0.4.14 as follows.

1.7.8. EXAMPLE. The following are connected: the space $\mathbf{R}^m$, any (closed or open) half-space of the space $\mathbf{R}^m$, every affine subspace of $\mathbf{R}^m$, every m-dimensional cube, every m-dimensional ball (whether open or closed). In particular the real line $\mathbf{R}$ and the half-line $\mathbf{R}_+$ are connected. ■

Using Theorem 1.7.6 we may prove the following.

1.7.9. THEOREM. *The metric product of a finite number of connected spaces is a connected space.*

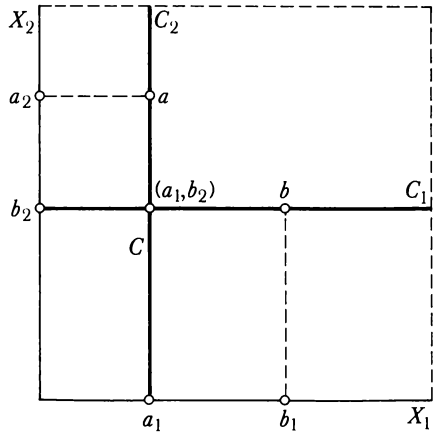


Fig.28. Illustration for the proof of Theorem 1.7.9.

PROOF. By Corollary 1.3.21 it is enough to prove the case of the metric product of two connected spaces. So suppose that $X = X_1 \times X_2$ where X_1 and X_2 are connected spaces. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$ be any two points of the space X; take

$$C_1 = \{(x_1, b_2) \in X : x_1 \in X_1\} \quad \text{and} \quad C_2 = \{(a_1, x_2) \in X : x_2 \in X_2\}.$$

The map taking a point $x_1 \in X_1$ to the point $(x_1, b_2) \in C_1$ is an isometry, hence we deduce that C_1 is connected. We similarly see that C_2 is connected. The two subspaces have the point (a_1, b_2) in common, hence by Theorem 1.7.5 the union $C = C_1 \cup C_2$ is a connected space. But $a \in C_2$ and $b \in C_1$ so $a, b \in C$. Theorem 1.7.6 now implies that X is connected. ■

Theorem 1.7.5 implies also the following.

1.7.10. COROLLARY. *The $(m-1)$-dimensional unit sphere S^{m-1} is a connected space for each $m > 1$.*

PROOF. Let $a = (0, \ldots, 0, 1)$, $b = (0, \ldots, 0, -1) \in S^{m-1}$. By Corollary 1.3.30 each of the sets $S^{m-1} \backslash \{a\}$ and $S^{m-1} \backslash \{b\}$ is homeomorphic to $\mathbf{R}^{m-1}$ and so is connected. If $m > 1$, then $(S^{m-1} \backslash \{a\}) \cap (S^{m-1} \backslash \{b\}) \neq \emptyset$, hence $S^{m-1} = (S^{m-1} \backslash \{a\}) \cup (S^{m-1} \backslash \{b\})$ is connected. The 0-dimensional sphere S^0 is of course disconnected. ■

1.7.11. COROLLARY. *The m-dimensional projective space P^m is connected for each m.*

PROOF. By Example 1.3.5 the projective space P^m is a continuous image of the sphere S^m; if $m > 0$ we make use of Theorem 1.7.3. The space P^0 consists of just one point. ■

If $A \subset X$ where X is some fixed metric space, then we sometimes say that the *set* A is *connected* or *disconnected* tacitly treating the set as a metric subspace of the space X. An open connected subset of a metric space is called a *region* in the space. For example, regions on the real line $\mathbf{R}$ take one of the forms: the whole line, open half-line,

open interval, empty set. In a discrete metric space the only regions are singleton sets and the empty set.

We now study the behaviour of connectedness under the closure operation. We prove the following.

1.7.12. THEOREM. *If A is a connected subset of a metric space X, then the closure* $\operatorname{cl} A$ *of the set A in the space X is also a connected set.*

PROOF. If the subspace $\operatorname{cl} A$ were not connected, then by Theorem 1.7.1 there would exist a continuous map f of the subspace onto the discrete two-point space $D = \{a, b\}$. Since $f|A$ is continuous and the subspace A is connected, we have by the same theorem that $f(A)$ is a proper subset of D. Suppose for the sake of argument that $f(A) = \{a\}$. It follows from the definition of the closure $\operatorname{cl} A$ of the set A and from the continuity of the map f that $f(\operatorname{cl} A) = \{a\}$. The contradiction just reached completes the proof. ■

The next example shows that the interior of a connected space need not be connected.

1.7.13. EXAMPLE. Let $A = \{(x^1, x^2) \in \mathbf{R}^2 : x^1 x^2 \geq 0\}$. Since every point of A can be joined to the origin by a line segment in A, we have that A is connected. The interior of the set relative to the plane $\mathbf{R}^2$ is the set $\{(x^1, x^2) \in \mathbf{R}^2 : x^1 x^2 > 0\}$ and is easily seen to be disconnected. ■

Let X be any metric space. A subset of the space X which is connected and inclusion-maximal with respect to this property is called a *component* of the space X. In other words a set $S \subset X$ is a component of the space if it is connected and for every connected set $C \subset X$ satisfying $S \subset C$ we have the equality $S = C$.

1.7.14. EXAMPLE. Every isolated point of a metric space is a component. In particular every point of a discrete space is its component. ■

1.7.15. THEOREM. *The components of a space are pairwise disjoint.*

PROOF. If S' and S'' are components of the space X and $p \in S' \cap S''$, then by Theorem 1.7.5 the union $S = S' \cup S''$ is a connected space. By maximality of S' and S'' we infer that $S' = S = S''$. ■

1.7.16. THEOREM. *Every metric space is the union of its components.*

PROOF. Say $p \in X$ and let S_p denote the union of all the connected subspaces of the space X that contain the point p. By Theorem 1.7.5 the subspace S_p is connected and is of course maximal with respect to this property; it is thus the component of X containing the point p. ■

1.7.17. THEOREM. *For two points of a metric space to belong to the same component, it is necessary and sufficient that there is a connected subspace containing the two points.*

PROOF. Necessity of the condition is obvious. To prove that the condition is sufficient observe that if S_a denotes the component of the space X containing the point

a and C is a connected subspace containing the points a and b, then $C \subset S_a$ from which it follows that $b \in S_a$. ■

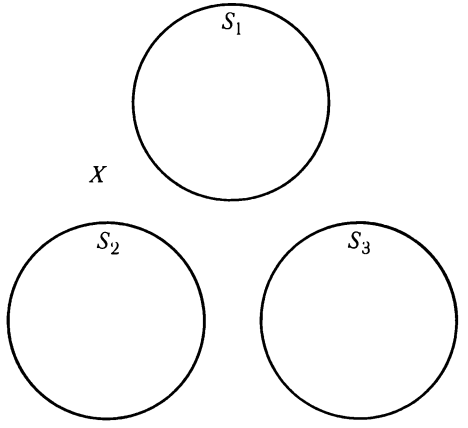


Fig.29. The components S_1, S_2 and S_3 of the space X.

A consequence of the theorem above and Theorem 1.7.5 is the following.

1.7.18. COROLLARY. *A metric space is connected if and only if it has exactly one component.* ■

An immediate conclusion of Theorem 1.7.12 is the following.

1.7.19. COROLLARY. *Every component of a space X is closed in X.*

PROOF. If A is a component of the space X, then its closure $\operatorname{cl} A$, being a connected set containing A, satisfies $A = \operatorname{cl} A$. ■

The next example shows that the components of a space need not be open sets.

1.7.20. EXAMPLE. Let $X = \bigcup_{n=1}^{\infty} \{1/n\} \cup \{0\}$ be the metric subspace of the real line $\mathbf{R}$. The set $\{0\}$ is a component of the space X but is not open in X. ■

We conclude this section by discussing real-valued functions defined on connected spaces. The following turns out to be true.

1.7.21. THEOREM (Darboux). *If f is a continuous real-valued function defined on a connected space X, then for any two points $a, b \in X$ and any real number $r \in \mathbf{R}$ satisfying $f(a) \leq r \leq f(b)$ there is a point $c \in X$ such that $f(c) = r$.*

PROOF. By Theorem 1.7.3 the subspace $f(X) \subset \mathbf{R}$ is connected. If there was a number $r \notin f(X)$ satisfying $f(a) < r < f(b)$, then taking $A = \{y \in f(X) : y \leq r\}$ and $B = \{y \in f(X) : y \geq r\}$ we would obtain two disjoint sets which are closed in $f(X)$ and non-empty because $f(a) \in A$ and $f(b) \in B$. Since their union is $f(X)$ this contradicts connectedness. It follows that every real number r satisfying $f(a) \leq r \leq f(b)$ belongs to $f(X)$; which completes the proof. ■

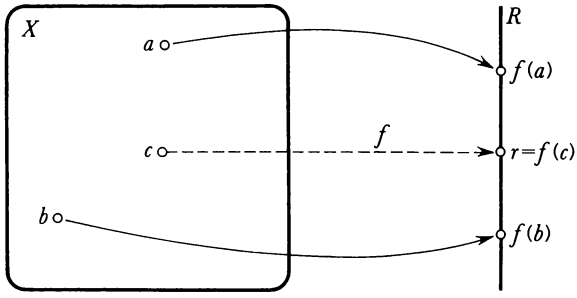


Fig.30. A continuous function defined on a connected space assumes all intermediate values between any two values in its range (Darboux's Theorem – 1.7.21).

We now use the concepts so far introduced to examine in greater detail the half-spaces of $\mathbf{R}^m$.

1.7.22. THEOREM. *The open half-spaces determined by a hyperplane H in $\mathbf{R}^m$ are components of the complement $\mathbf{R}^m \backslash H$ and H is the boundary of each.*

PROOF. Let H be the hyperplane with equation $\alpha_0 + \sum_{i=1}^m \alpha_i x^i = 0$. Each of the two subspaces given by the inequalities $\alpha_0 + \sum_{i=1}^m \alpha_i x^i > 0$, $\alpha_0 + \sum_{i=1}^m \alpha_i x^i < 0$ is connected by Example 1.7.8. Their union $\mathbf{R}^m \backslash H$ is not connected by Theorem 1.7.21 since the continuous function $f(x^1, x^2, \ldots, x^m) = \alpha_0 + \sum_{i=1}^m \alpha_i x^i$ does not assume the value 0 on the set $\mathbf{R}^m \backslash H$. Hence indeed the open half-spaces are the components of $\mathbf{R}^m \backslash H$.

By Theorem 0.4.20 there is an affine isomorphism $f: \mathbf{R}^m \to \mathbf{R}^m$ which takes the hyperplane H to the hyperplane H' defined by the equation $x^m = 0$. This isomorphism takes the components of $\mathbf{R}^m \backslash H$ onto components of $\mathbf{R}^m \backslash H'$ (though the normals to the inequalities describing the components may be reversed). It is thus enough to check that the hyperplane H' is the boundary of the half-space $x^m > 0$ and of the half-space $x^m < 0$, but that is obvious. ■

Some further remarks on connectedness are contained in Supplement 1.S.19.

Exercises

a) Show that the Hilbert space $\mathbf{R}^\omega$ and the Hilbert cube I^ω are connected.

b) Show that if a metric product is a non-empty connected space, then all the factors are connected.

c) Prove that if $X = \bigcup_{t \in T} X_t$ where for each $t \in T$ the subspace X_t is connected and there is an index t_0 such that for all $t \in T$, $X_t \cap X_{t_0} \neq \emptyset$, then the space X is connected.

d) Prove that if A is a connected subspace of the metric space X and $A \subset B \subset \mathrm{cl}\, A$, then B is a connected subspace.

e) Give an example of a map $f: \mathbf{R} \to \mathbf{R}$ which takes connected subspaces to connected subspaces but is not continuous.

1.8. Compact spaces

We begin with a classical result concerning numerical sequences with terms in the unit interval.

1.8.1. THEOREM (Bolzano, Weierstrass). *For every sequence of numbers $x_n \in I$ where $n = 1, 2, \ldots$ there is a subsequence convergent in I.*

PROOF. We begin by defining inductively two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ for $n = 1, 2, \ldots$ such that for each index n infinitely many terms of the sequence $\{x_k\}$ satisfy $a_n \leq x_k \leq b_n$. We start off with $a_1 = 0$ and $b_1 = 1$ and, assuming that we have already defined the terms a_n and b_n, we take either

$$a_{n+1} = a_n \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2},$$

or

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = b_n,$$

choosing in such a way that there are infinitely many terms of the sequence $\{x_k\}$ satisfying $a_{n+1} \leq x_k \leq b_{n+1}$.

It follows from the construction that the sequence $\{a_n\}$ is non-decreasing, the sequence $\{b_n\}$ is non-increasing and $b_n - a_n = 2^{1-n}$ for $n = 1, 2, \ldots$ Set $x_0 = \sup\{a_n : n = 1, 2, \ldots\}$. It follows from the properties of the sequences $\{a_n\}$ and $\{b_n\}$ and of the least upper bound that $a_n \leq x_0 \leq b_n$ for $n = 1, 2, \ldots$

We now define inductively a subsequence $\{k_n\}$ of the natural numbers so that $k_1 = 1$ and k_{n+1} is the smallest natural number greater than k_n such that $a_{n+1} \leq x_{k_{n+1}} \leq b_{n+1}$. The properties of the sequences $\{a_n\}$ and $\{b_n\}$ guarantees the feasibility of the construction. Thus $\rho(x_{k_n}, x_0) = |x_{k_n} - x_0| \leq b_n - a_n = 2^{1-n}$ for $n = 1, 2, \ldots$ so $\lim_n x_{k_n} = x_0$. The subsequence $\{x_{k_n}\}$ of the sequence $\{x_n\}$ is therefore convergent, which completes the proof. ■

The property of the unit interval I stated in the theorem above holds also of any closed bounded interval, being a set similar to the unit interval. It is not however a property of all metric spaces not even of subspaces of the real line $\mathbf{R}$, for instance on the real line $\mathbf{R}$ the sequence $x_n = n$ for $n = 1, 2, \ldots$ does not have a convergent subsequence in view of Lemma 1.5.5.

A metric space X is said to be a *compact* space, if every sequence of points of the space has a convergent subsequence. Thus the unit interval is a compact space while the real line $\mathbf{R}$ is not. A discrete metric space is compact if and only if it has finitely many points. Compactness of a finite space is obvious whereas the non-compactness of an infinite discrete metric space follows immediately from Lemma 1.5.5. (See also Supplement 1.S.20).

We now show that the concept of a compact space is topological. We do in fact prove more, as follows.

1.8.2. THEOREM. *If f is a continuous map of a compact space X onto a space Y, then Y is also compact.*

PROOF. Suppose given a sequence of points $y_n \in Y$ for $n = 1, 2, \ldots$ For each index n choose a point x_n from the non-empty set $f^{-1}(y_n)$. Using the compactness of X we may select a subsequence $\{x_{k_n}\}$ of the sequence $\{x_n\}$ and a point $x_0 \in X$ such that $\lim_n x_{k_n} = x_0$. By Theorem 1.5.6 we then have $\lim_n f(x_{k_n}) = f(x_0)$. Since $f(x_{k_n}) = y_{k_n}$, the subsequence $\{y_{k_n}\}$ of the sequence $\{y_n\}$ is convergent, which completes the proof. ■

A metric subspace of a compact space need not of course be compact. The following however does hold.

1.8.3. THEOREM. *If X is a compact space and A is a closed subset of X, then the subspace A is also compact.*

PROOF. Consider an arbitrary sequence of points $a_n \in A$ for $n = 1, 2, \ldots$ and using the compactness of X select a subsequence $\{a_{k_n}\}$ convergent to some point $x \in X$. Then x is a limit point of the set A in the space X, and since A is closed in X we infer that $x \in A$ which completes the proof. ■

A converse of a kind to this theorem also holds. In fact we have the following.

1.8.4. THEOREM. *If $A \subset X$ and A is a compact subspace, then A is a closed set of X.*

PROOF. Consider a sequence of points $a_n \in A$ for $n = 1, 2, \ldots$ such that $\lim_n a_n = x \in X$. Using the compactness of A take a subsequence $\{a_{k_n}\}$ and a point $a \in A$ such that $\lim_n a_{k_n} = a$. But $\lim_n a_{k_n} = x$ in the space X and since a sequence can converge to only one limit at most, we have $x = a$, that is $x \in A$.

We have thus shown that every limit point of the set A in the space X belongs to A and this completes the proof. ■

Passing to the operation of metric product we prove the following.

1.8.5. THEOREM. *The metric product of a finite number of compact spaces is a compact space.*

PROOF. By Corollary 1.3.21 it is enough to prove the case of a product of two spaces. So suppose that X and Y are compact spaces and consider a sequence of points $p_n = (x_n, y_n) \in X \times Y$ for $n = 1, 2, \ldots$ Using the compactness of X select a subsequence $\{x_{k_n}\}$ converging to some point $x_0 \in X$. Now consider the subsequence $\{y_{k_n}\}$ of the sequence $\{y_n\}$ and using the compactness of Y select a further subsequence $\{y_{k_{l_n}}\}$ convergent to some point $y_0 \in Y$. The sequence $\{x_{k_{l_n}}\}$ converges to x_0 so by Theorem 1.5.9 we conclude that $\lim_n p_{k_{l_n}} = (x_0, y_0)$. Thus the space $X \times Y$ is compact. ■

1.8.6. COROLLARY. *The unit cube I^m is a compact space for each m.* ■

1.8.7. COROLLARY. *The closed unit ball $\bar{B}^m$ and the unit sphere S^{m-1} are compact for each m.*

PROOF. Let $X = \{(x^1, x^2, \ldots, x^m) \in \mathbf{R}^m : -1 \leq x^i \leq 1 \text{ for } i = 1, 2, \ldots, m\}$. Since the interval $[-1, 1]$ is compact we deduce from Theorem 1.8.5 that X is also a compact

space. By Lemmas 1.6.9, 1.6.1 and Theorem 1.6.20 it follows that the set $\bar{B}^m$ and S^{m-1} are closed in X and so by Theorem 1.8.3 are compact. ■

From Example 1.3.5, Theorem 1.8.2 and that part of Corollary 1.8.7 which concerns spheres, we obtain:

1.8.8. COROLLARY. *The m-dimensional projective space P^m is compact for each m.* ■

The following theorem on decreasing sequences of non-empty closed sets in a compact space is useful in a variety of constructions.

1.8.9. THEOREM (Cantor). *If X is a compact space and $X \supset F_1 \supset F_2 \supset \ldots$ where $\emptyset \neq F_n = \operatorname{cl} F_n$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

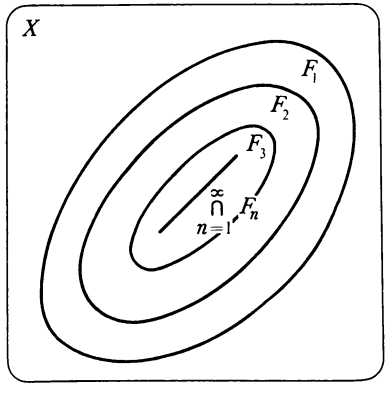


Fig.31. The intersection of a decreasing sequence of non-empty closed sets in a compact space is non-empty (Cantor's Theorem – 1.8.9).

PROOF. Choose an arbitrary point $x_n \in F_n$ for $n = 1, 2, \ldots$ and then extract from the sequence $\{x_n\}$ a subsequence $\{x_{k_n}\}$ converging to some point $x \in X$. This point is a limit point of the set F_j for $j = 1, 2, \ldots$ since $x_{k_n} \in F_{k_n} \subset F_j$ for $k_n \geq j$. Thus $x \in F_j$ for $j = 1, 2, \ldots$ whence the proposition. ■

We pass now to some theorems about coverings of compact spaces. A family $\{U_t\}_{t \in T}$ of subsets of a set X is called a *covering* of the set, if $X = \bigcup_{t \in T} U_t$. If X is a metric space and the sets U_t for $t \in T$ are open in X, then the covering $\{U_t\}_{t \in T}$ is said to be *open*.

1.8.10. LEMMA. *If X is a compact space, then for every positive real number ϵ there is a finite covering of the space X by open balls of radius ϵ.*

PROOF. Let $\epsilon > 0$ be given. Choose an arbitrary point $x_1 \in X$. Suppose that we have defined a finite sequence of points $x_1, x_2, \ldots, x_n \in X$ with the property that $\rho(x_j, x_k) \geq \epsilon$ for $1 \leq j \neq k \leq n$. If the set $X \setminus \bigcup_{j=1}^{n} B(x_j, \epsilon)$ is non-empty, then we may choose from it a point x_{n+1}. We thus obtain a sequence $x_1, x_2, \ldots, x_{n+1} \in X$ with the property that $\rho(x_j, x_k) \geq \epsilon$ for $1 \leq j \neq k \leq n+1$. By Lemma 1.5.5 and by the compactness of the space X this construction cannot be continued indefinitely.

Thus there exists an index n such that the complement $X \setminus \bigcup_{j=1}^{n} B(x_j; \epsilon)$ is empty, and therefore $X = \bigcup_{j=1}^{n} B(x_j; \epsilon)$. ■

1.8.11. COROLLARY. *Every compact space is bounded.*

PROOF. For, by Lemma 1.8.10, every compact space is a finite union of open unit balls that is to say bounded sets. It remains to appeal to Lemma 1.4.9. ■

We pass now to the following important property of compact spaces:

1.8.12. THEOREM (Borel, Lebesgue). *Any open covering of a compact space contains a finite covering.*

PROOF. Suppose that X is a compact space and $X = \bigcup_{t \in T} U_t$, where, for each $t \in T$, U_t is an open subset of X. The proof proceeds in two stages; first, we extract a countable covering from the covering $\{U_t\}_{t \in T}$.

For each natural number ℓ there is by Lemma 1.8.10 a finite sequence of points $x_1^\ell, x_2^\ell, \ldots, x_{n_\ell}^\ell \in X$ with the property that $X = \bigcup_{j=1}^{n_\ell} B(x_j^\ell; 1/\ell)$. If the ball $B(x_j^\ell, 1/\ell)$ for $j = 1, 2, \ldots, n_\ell$ and $\ell = 1, 2, \ldots$ is contained in at least one of the sets of the family $\{U_t\}_{t \in T}$, choose exactly one of the sets that contains it. The sets thus selected form a countable subfamily since their number does not exceed the number of pairs (j, ℓ) where $j = 1, 2, \ldots, n_\ell$ and $\ell = 1, 2, \ldots$ Arrange the sets of this countable subfamily into a sequence $\{U_{t_n}\}$ where $n = 1, 2, \ldots$ We shall show that $X = \bigcup_{n=1}^{\infty} U_{t_n}$.

For, if $x \in X$, then there is an index $t_0 \in T$ with $x \in U_{t_0}$. Since U_{t_0} is open there is a number $r > 0$ such that $B(x; r) \subset U_{t_0}$. Choose a natural number ℓ so that $1/\ell < r/2$. There is an index j with $1 \leq j \leq n_\ell$ such that $x \in B(x_j^\ell; 1/\ell)$; then $B(x_j^\ell; 1/\ell) \subset B(x; r) \subset U_{t_0}$. Hence there is a set in the sequence $\{U_{t_n}\}$ (not necessarily equal to U_{t_0}) which contains the point x.

Thus we have shown that a countable covering $\{U_{t_n}\}$ for $n = 1, 2 \ldots$ can be extracted from the covering $\{U_t\}_{t \in T}$. Assume now that a finite covering cannot be extracted from the covering $\{U_{t_n}\}$. Then each of the closed sets $F_n = X \setminus \bigcup_{k=1}^{n} U_{t_k}$ is non-empty and the inclusions $F_{n+1} \subset F_n$ hold for $n = 1, 2, \ldots$, but on the other hand $\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (X \setminus \bigcup_{k=1}^{n} U_{t_k}) = X \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} U_{t_k} = \emptyset$, contrary to Cantor's Theorem (1.8.9). The contradiction completes the proof. ■

The next result is also concerned with open coverings of a compact metric space.

1.8.13. LEMMA (Lebesgue). *For any open covering $\mathcal{U}$ of a compact space X there is a positive real number λ with the property that any subset of the space X with diameter less than λ is contained in a member of $\mathcal{U}$.*

PROOF. For each point $x \in X$ there is an element $U_x \in \mathcal{U}$ such that $x \in U_x$. As this is an open set there is a positive real number λ_x such that $B(x; 2\lambda_x) \subset U_x$. The balls $B(x; \lambda_x)$ for $x \in X$ evidently form an open covering of the space X. By the Borel-Lebesgue Theorem (1.8.12) there is a finite number of points $x_1, x_2, \ldots, x_n \in X$ with the property that the balls $B(x_j; \lambda_{x_j})$ for $j = 1, 2, \ldots, n$ form a covering of the space. The number $\lambda = \min(\lambda_{x_1}, \lambda_{x_2}, \ldots, \lambda_{x_n})$ has the required property.

For, if $A \subset X$, $\operatorname{diam} A < \lambda$ and $a \in A$, then there is an index j with $1 \leq j \leq n$ such that $a \in B(x_j; \lambda_{x_j})$. We then have $A \subset B(a; \lambda) \subset B(x_j; 2\lambda_{x_j}) \subset U_{x_j}$, which completes the proof. ■

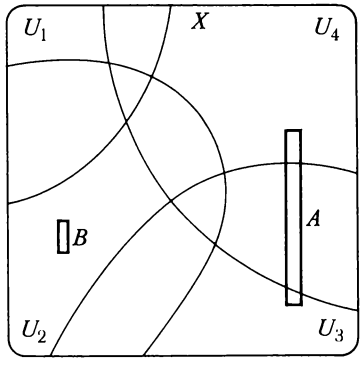


Fig.32. The set A is not contained in any member of the covering of the space X, it is of diameter greater than the Lebesgue number of the covering. The set B has diameter less than this number and is contained in U_2.

Any number λ which has the property asserted in Lebesgue's Lemma is called a *Lebesgue number* of the covering $\mathcal{U}$. It is evidently not determined uniquely by the covering.

1.8.14. THEOREM (Heine). *Every continuous map defined on a compact space is uniformly continuous.*

PROOF. Let $f: X \to Y$ be a continuous map. Let ϵ be any positive real number and consider the open covering of the space Y by the balls $B(y; \frac{1}{2}\epsilon)$ for $y \in Y$. By Theorem 1.6.24 the sets $f^{-1}(B(y; \frac{1}{2}\epsilon))$ for $y \in Y$ form an open covering of the space X. Let δ be a Lebesgue number of the covering. Thus if $\rho(x, x') < \delta$, then $\operatorname{diam}\{x, x'\} < \delta$ and so there is $y \in Y$ such that $x, x' \in f^{-1}(B(y, \frac{1}{2}\epsilon))$, that is $f(x), f(x') \in B(y; \frac{1}{2}\epsilon)$. Hence we obtain $\rho(f(x), f(x')) \leq \rho(f(x), y) + \rho(y, f(x')) \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$. ■

In checking that a bijective map between two compact spaces is a homeomorphism it turns out to be unnecessary to check the continuity of the inverse map. This useful remark is a consequence of the following theorem.

1.8.15. THEOREM. *If f is a continuous bijective map of a compact space X onto a space Y, then the inverse map f^{-1} is continuous.*

PROOF. By Theorem 1.6.4 it is enough to prove that for each closed set A in the space X the set $f(A)$ is closed in the space Y. To this aim observe that from Theorem 1.8.3 it follows that A is a compact space and so by Theorem 1.8.2 we infer that $f(A)$ is a compact space, but by Theorem 1.8.4 that has to be a closed subset of the space Y. ■

In the closing parts of this section we consider real-valued functions defined on compact spaces. The following is the case.

1.8.16. THEOREM (Weierstrass). *Every real-valued continuous function defined on a compact space is bounded and achieves its infimum and supremum.*

PROOF. Let $f: X \to \mathbf{R}$ where X is a compact space. The function is bounded, since by Theorem 1.8.2 the image $f(X) \subset \mathbf{R}$ is a compact space and so by Corollary 1.8.11 is a bounded set. To prove that the infimum and supremum belong to the set of values it is enough to invoke Theorem 1.8.4 to see that $f(X)$ is a closed subset of $\mathbf{R}$ and then to observe that by Lemma 1.6.22 the latter contains its infimum and supremum. ■

Sometimes it is possible to apply Weierstrass' Theorem to functions defined on certain non-compact spaces. We explain the method through the following Example.

1.8.17. EXAMPLE. *If H is an affine subspace in the Euclidean space $\mathbf{R}^m$ and $x \in \mathbf{R}^m$, then the function f defined by the formula $f(z) = \rho(x, z)$ for $z \in H$ achieves its infimum.* For, applying if necessary a translation and a homotheticity we may straight away suppose that $x = 0$ and that there exists a point $z \in H$ with $\|z\| \leq 1$. Thus the intersection $H \cap \bar{B}^m$ is non-empty. By Example 1.6.15, Theorem 1.8.3 and Corollary 1.8.7 the intersection is compact. By Example 1.3.4 the function f is continuous, it thus achieves its infimum c in $H \cap \bar{B}^m$. This infimum is also an infimum over the whole subspace H since $c \leq 1 < \|z\|$ for $z \in H \setminus \bar{B}^m$.

Observe that the point $y \in H$ for which $\rho(x, y) = \inf\{\rho(x, z) : z \in H\}$ is the orthogonal projection of the point x onto H (cf. Example 1.3.3). To this end proof is needed that if $p, q \in H$ and $r = (x - y) \cdot (p - q)$, then $r = 0$. We may of course assume that $\|p - q\| = 1$. It is easy to see that $z = y + r(p - q) \in H$ and so $\|x - y\| \leq \|x - z\| = \|x - y - r(p - q)\|$. Thus $0 \leq \|(x - y) - r(p - q)\|^2 - \|x - y\|^2 = \|x - y\|^2 - 2r(x - y) \cdot (p - q) + r^2\|p - q\|^2 - \|x - y\|^2 = -2r^2 + r^2 = -r^2$, hence certainly $r = 0$. ■

A metric space that is both compact and connected is called a *continuum*. From Theorems 1.7.3 and 1.8.2 we obtain the following.

1.8.18. COROLLARY. *If f is a continuous map of the continuum X onto the space Y, then the space Y is also a continuum.* ■

Similarly Theorems 1.7.9 and 1.8.5 imply:

1.8.19. COROLLARY. *The metric product of a finite number of continua is a continuum.* ■

We now prove the analogue of Cantor's Theorem (1.8.9) for continua.

1.8.20. THEOREM. *If X_n is a non-empty continuum and X_{n+1} is a metric subspace of the space X_n for $n = 1, 2, \ldots$, then the intersection $\bigcap_{n=1}^{\infty} X_n$ is a non-empty continuum.*

PROOF. The compactness of the intersection $X = \bigcap_{n=1}^{\infty} X_n$ is obvious and it is non-empty by Cantor's Theorem (1.8.9). To prove that X is a connected space suppose that $X = A \cup B$ where the sets A and B are closed and disjoint. By Theorem 1.6.27 there are sets U, V open in X_1 with $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The compact sets $X_n \setminus (U \cup V)$ for $n = 1, 2, \ldots$ form a decreasing sequence and since their intersection is empty, there is an n with $X_n \subset U \cup V$ whence $X_n = (X_n \cap U) \cup (X_n \cap V)$. The

connectedness of X_n implies that either $X_n \cap U = \emptyset$ or $X_n \cap V = \emptyset$. Since $A \subset X_n \cap U$ and $B \subset X_n \cap V$, we have either $A = \emptyset$ or $B = \emptyset$ which completes the proof. ■

We add the remark that the intersection of a decreasing sequence of connected sets need not be connected. If A is an arbitrary line segment with distinct endpoints a_1, a_2 in the plane $\mathbf{R}^2$, then taking $X_n = B(A; \frac{1}{n}) \backslash (A \backslash \{a_1, a_2\})$ for $n = 1, 2, \ldots$ we obtain a decreasing sequence of connected sets whose intersection is the set $\{a_1, a_2\}$.

Exercises

a) Show that the Hilbert cube I^ω is a compact space.

b) Show that the closed ball $\bar{B}(0; 1)$ in Hilbert space $\mathbf{R}^\omega$ where $0 = (0, 0, \ldots)$ is not compact. Give an example of a continuous real-valued function defined on $\mathbf{R}^\omega$ which is not bounded on the ball $\bar{B}(0; 1)$.

c) Give an example of a decreasing sequence of non-empty closed subsets of the real line $\mathbf{R}$ with empty intersection.

d) Let $a_1 = (-1, 0)$, $a_2 = (1, 0) \in \mathbf{R}^2$, $A = \overline{a_1 a_2}$ and $X_n = B(A; \frac{1}{n}) \backslash (A \backslash \{a_1, a_2\})$ for $n = 1, 2, \ldots$ Show that the set X_n is connected for $n = 1, 2, \ldots$, $X_1 \supset X_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} X_n = \{a_1, a_2\}$ (cf. Theorem 1.8.20).

1.9. Complete spaces

We begin by distinguishing in the context of metric spaces a class of sequences which is more general than the class of convergent sequences. We say that the sequence $\{x_n\}$ where $x_n \in X$ for $n = 1, 2, \ldots$ satisfies the *Cauchy condition* (or more briefly, is a *Cauchy sequence*) if, for every positive real number ϵ, there is an index k such that $\rho(x_n, x_{n'}) < \epsilon$ for all $n, n' \geq k$. For instance in a discrete metric space the only sequences satisfying the Cauchy condition are those that are almost constant. We now give the basic properties of Cauchy sequences.

1.9.1. THEOREM. *The terms of a Cauchy sequence form a bounded set.*

PROOF. Appealing to the definition we infer that if $\{x_n\}$ is a Cauchy sequence, then there is an index k such that $\rho(x_n, x_k) < 1$ for all $n \geq k$. Hence the set $\{x_n : n = 1, 2, \ldots\}$ is the union of the $k - 1$ singletons $\{x_1\}, \{x_2\}, \ldots, \{x_{k-1}\}$ together with a bounded set which by Lemma 1.4.9 completes the proof. ■

1.9.2. THEOREM. *Every convergent sequence satisfies the Cauchy condition.*

PROOF. If $\lim_n x_n = x_0$, then for every positive real number ϵ there is an index k such that $\rho(x_n, x_0) < \frac{1}{2}\epsilon$ for all $n \geq k$. Then for $n, n' \geq k$ we have

$$\rho(x_n, x_{n'}) \leq \rho(x_n, x_0) + \rho(x_{n'}, x_0) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \blacksquare$$

The converse assertion is not in general true; the sequence $x_n = 1/n$ for $n = 1, 2, \ldots$ of points of the interval $(0, 1)$ satisfies the Cauchy condition but is not convergent over that interval. However a partial converse of Theorem 1.9.2 is available as follows.

1.9.3. THEOREM. *If a Cauchy sequence contains a convergent subsequence, then the sequence itself is convergent.*

PROOF. Suppose the sequence $\{x_n\}$ satisfies the Cauchy condition and $\lim_n x_{k_n} = x_0$. We show that for every positive real number ϵ there is an index k such that $\rho(x_n, x_0) < \epsilon$ for every $n \geq k$. To this end note the triangle inequality $\rho(x_n, x_0) \leq \rho(x_n, x_{k_n}) + \rho(x_{k_n}, x_0)$. Since the sequence $\{x_n\}$ satisfies the Cauchy condition there is an index k' such that $\rho(x_n, x_{k_n}) < \frac{1}{2}\epsilon$ for all $n \geq k'$. Since the sequence $\{x_{k_n}\}$ converges to the point x_0 there is an index k'' such that $\rho(x_{k_n}, x_0) < \frac{1}{2}\epsilon$ for all $n \geq k''$. Taking $k = \max(k', k'')$ we obtain $\rho(x_n, x_{n'}) < \epsilon$ for all $n, n' \geq k$ which completes the proof. ■

By Theorems 1.9.1 and 1.9.3 the following is now immediate:

1.9.4. THEOREM. *If a metric space X has the property that every bounded subset is contained in a compact subspace, then every Cauchy sequence in X is convergent.* ■

Now it follows from Lemma 1.4.8 that every bounded set on the real line $\mathbf{R}$ is contained in a closed bounded interval, hence in a compact subspace, thus Theorems 1.9.1 and 1.9.4 imply the following.

1.9.5. THEOREM (Cauchy). *On the real line $\mathbf{R}$ every Cauchy sequence is convergent.* ■

The concept of a Cauchy sequence belongs not only to metric geometry but also to uniform topology. We shall prove the following stronger result.

1.9.6. THEOREM. *If $f: X \to Y$ is a uniformly continuous map and the sequence $\{x_n\}$, where $x_n \in X$ for $n = 1, 2 \ldots$, satisfies the Cauchy condition in the space X, then the sequence $\{f(x_n)\}$ satisfies the Cauchy condition in the space Y.*

PROOF. Take any positive real number ϵ and choose the appropriate real number δ so that $\rho(x, x') < \delta$ for the points $x, x' \in X$ implies $\rho(f(x), f(x')) < \epsilon$. Next choose the index k so that $\rho(x_n, x_{n'}) < \delta$ for all $n, n' \geq k$. Then $\rho(f(x_n), f(x_{n'})) < \epsilon$ for all $n, n' \geq k$ and that completes the proof. ■

Observe however that there are homeomorphisms between metric spaces which take Cauchy sequences onto sequences which do not satisfy the Cauchy condition.

1.9.7. EXAMPLE. The homeomorphism h defined by the formula $h(1/n) = n$ for $n = 1, 2, \ldots$ takes the space $X = \{1/n : n = 1, 2, \ldots\}$ with the subspace metric of the real line $\mathbf{R}$ onto the space $Y = \{n : n = 1, 2, \ldots\}$ with the analogous metric. Under this homeomorphism the Cauchy sequence $x_n = 1/n \in X$ for $n = 1, 2, \ldots$ is mapped to the sequence $h(x_n) = n \in Y$ which does not satisfy the Cauchy condition. ■

The example above shows that the concept of a Cauchy sequence does not belong to topology.

The following is a direct consequence of Lemma 1.2.4.

1.9.8. THEOREM. *If $(X, \rho) = \bigtimes_{i=1}^{m}(X_i, \rho_i)$, then the sequence $x_n = (x_n^1, x_n^2, \ldots, x_n^m) \in X$ for $n = 1, 2, \ldots$ satisfies the Cauchy condition in the space X if and only if each of the sequences $\{x_n^i\}$ satisfies the Cauchy condition in the space X_i for $i = 1, 2, \ldots, m$.* ■

Not every metric space enjoys the property which the real line $\mathbf{R}$ has by Cauchy's Theorem (1.9.5). For instance the space of all rational numbers Q (viewed as a metric subspace of the real line $\mathbf{R}$) contains Cauchy sequences that do not converge; every sequence of decimal approximations of any irrational number has this property.

We say that a metric space is *complete* if every Cauchy sequence in the space is convergent. Thus for example the real line $\mathbf{R}$ is complete whereas the space of rationals Q is not complete. A discrete metric space of any cardinality is complete. (See also Supplement 1.S.21).

An immediate corollary of Theorems 1.9.6 and 1.5.6 is:

1.9.9. THEOREM. *If f is a uniform homeomorphism of a complete space X onto a space Y, then the space Y is also complete.* ■

1.9.10. COROLLARY. *The real line $\mathbf{R}$ and an open interval are never uniformly homeomorphic.* ■

It is easy to see that the inverse of the homeomorphism h described in Example 1.9.7 takes a complete space Y onto an incomplete space X. Thus the notion of a complete space does not belong to topology. It is also worth noting that the homeomorphism just mentioned is a uniformly continuous map of a complete space onto an incomplete space so Theorem 1.9.9 cannot be strengthened in that direction.

From Theorem 1.9.4 we immediately obtain the following.

1.9.11. THEOREM. *Every compact space is complete.* ■

The example of the real line $\mathbf{R}$ shows that the implication in this theorem cannot be reversed.

We now prove two theorems which are in a sense the analogues of 1.8.3 and 1.8.4.

1.9.12. THEOREM. *If X is a complete space and A is a closed subset of X, then the subspace A is complete.*

PROOF. Consider a sequence of points $a_n \in A$ for $n = 1, 2, \ldots$ satisfying the Cauchy condition in the subspace A. Hence it satisfies the Cauchy condition also in the space X and since X is a complete space, there is a point $x \in X$ with $\lim_n a_n = x$. It follows that x is a limit point of the set A and since A is closed in X, $x \in A$. Thus the sequence $\{a_n\}$ is convergent in A, which completes the proof. ■

1.9.13. THEOREM. *If $A \subset X$ and A is a complete subspace, then A is a closed subset of X.*

PROOF. Consider a sequence of points $a_n \in A$ for $n = 1, 2, \ldots$ such that $\lim_n a_n = x \in X$. As a convergent sequence of X the sequence satisfies the Cauchy condition in X; it thereby also satisfies the Cauchy condition in the subspace A. From the completeness of the subspace it follows that there is a point $a \in A$ with $\lim_n a_n = a$. Since a sequence in the space X may converge to only one limit, we have $a = x$, so $x \in A$. We have thus shown that every limit point of the set A in the space X belongs to A and this completes the proof. ■

From Theorem 1.9.8 we immediately have:

1.9.14. THEOREM. *The metric product of a finite number of complete spaces is a complete space.* ■

1.9.15. COROLLARY. *The Euclidean m-dimensional space* $\mathbf{R}^m$ *is complete for each* m. ■

Exercises

a) Show that a sequence $\{x_n\}$ satisfies the Cauchy condition if and only if for each positive number ϵ there is an index k such that $\rho(x_k, x_{k+m}) < \epsilon$ for $m = 1, 2, \ldots$

b) Prove that the Hilbert space $\mathbf{R}^\omega$ is complete.

c) Show that if in a metric space every bounded subspace is compact, then the space X is complete. Give an example indicating that the reverse implication does not hold.

1.10. Metric and topological concepts in Euclidean spaces

We will now apply the concepts studied in the previous sections to the special case of Euclidean spaces and their subsets. We first prove an analogue of Lemma 1.4.8.

1.10.1. LEMMA. *For a set* $A \subset \mathbf{R}^m$ *to be bounded it is necessary and sufficient that it be contained in some m-dimensional cube.*

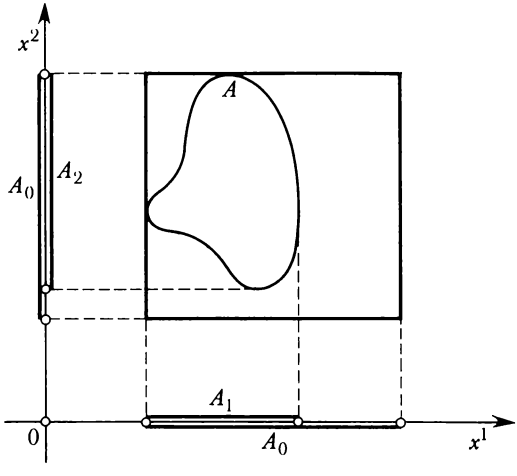


Fig.33. The set $A \subset \mathbf{R}^2$ is bounded if it is contained in some square (Lemma 1.10.1 in the case $m = 2$).

PROOF. Sufficiency of the condition follows immediately from Corollary 1.4.2 in view of Example 1.4.5. To prove that the condition is necessary denote by $p_i: \mathbf{R}^m \to \mathbf{R}$ the map defined by $p_i(x^1, x^2, \ldots, x^m) = x^i$ for $(x^1, x^2, \ldots, x^m) \in \mathbf{R}^m$ and for $i = 1, 2, \ldots, m$. Thus p_i is the projection of the metric product $\mathbf{R}^m$ onto its i^{th}-factor $\mathbf{R}$. Denoting $A_i = p_i(A)$ for $i = 1, 2, \ldots, m$ we infer on the basis of the arguments given in

Example 1.3.2 that $\operatorname{diam} A_i \leq \operatorname{diam} A < \infty$. From Lemma 1.4.9 it follows that the set $A_0 = \bigcup_{i=1}^{m} A_i$ is bounded on the real line $\mathbf{R}$. Taking $a_0 = \inf A_0$ and $b_0 = \sup A_0$ we have $A_0 \subset [a_0, b_0]$ and hence that $A \subset [a_0, b_0]^m$ which completes the proof. ■

The lemma above allows us to embark on the study of compact subsets of Euclidean spaces. We prove the following.

1.10.2. THEOREM. *For a metric subspace $A \subset \mathbf{R}^m$ to be compact it is necessary and sufficient that the set A be bounded and closed in the space $\mathbf{R}^m$.*

PROOF. Necessity of the condition follows from Theorem 1.8.4 and Corollary 1.8.11. To prove sufficiency suppose that $A \subset \mathbf{R}^m$ is a set bounded and closed in the space $\mathbf{R}^m$. By Lemma 1.10.1 there is an m-dimensional cube of the space $\mathbf{R}^m$ containing the set A. Since the set A is closed in the space $\mathbf{R}^m$ it is by Theorem 1.6.20 also closed in the cube. By Corollary 1.8.6 it follows that the cube is a compact space. By Theorem 1.8.3 the set A being a closed subset of a compact space is itself a compact space. ■

We now prove the analogue of Theorem 1.10.2 for complete spaces.

1.10.3. THEOREM. *For a metric subspace $A \subset \mathbf{R}^m$ to be complete it is necessary and sufficient that the set A be closed in the space $\mathbf{R}^m$.*

PROOF. Necessity of the condition follows from Theorem 1.9.13. To see that the condition is sufficient it is enough to invoke Theorem 1.9.12 and Corollary 1.9.15. ■

We now move on to consider the connected subspaces of Euclidean spaces. In Corollary 1.7.7 we observed that every convex set in the space $\mathbf{R}^m$ is connected. The example of the sphere (cf. Example 0.4.14 and Corollary 1.7.10) shows that the converse of the theorem cannot hold. We examine however some circumstances in which at least a partial converse is possible.

1.10.4. THEOREM. *Every connected subset of the real line $\mathbf{R}$ is convex.*

PROOF. Suppose $C \subset \mathbf{R}$ is not convex and consider points $a, b \in C$ with $a < b$ for which there is a point $c \in [a, b] \setminus C$; then of course $c \in (a, b) \setminus C$. Taking $A = \{x \in \mathbf{R} : x < c\} \cap C$ and $B = \{x \in \mathbf{R} : x > c\} \cap C$ we deduce that $C = A \cup B$ and $A \cap B = \emptyset$ while $a \in A$ and $b \in B$ so that $A \neq \emptyset \neq B$. Since $A = (-\infty, c) \cap C$ and $B = (c, +\infty) \cap C$, by Theorem 1.6.5 the sets A and B are open in C. The set C is thus disconnected. ■

Corollary 1.7.7 may be strengthened by the use of the notion of a broken line in the space $\mathbf{R}^m$. Any set in $\mathbf{R}^m$ which is the union of a finite number of line segments $\overline{a_j b_j}$ for $j = 1, 2, \ldots, n$ where $a_1 = a$, $b_n = b$ and $a_{j+1} = b_j$ for $j = 1, 2, \ldots, n-1$, is called a *broken line* joining a to b. Observe that in contrast to a line segment a broken line does not uniquely determine the points which it joins.

The following easily proved generalization of the property described in Example 1.7.2 holds.

1.10.5. THEOREM. *Every broken line is a connected space.*

PROOF. We use the notation of the definition. In the case $n = 1$ the proposition follows from Example 1.7.2. Assuming the theorem is true for every broken line con-

sisting of $n-1$ segments for $n > 1$ consider a broken line A which is the union of n line segments $\overline{a_j b_j}$ for $j = 1, 2, \ldots n$. Then $A = (\bigcup_{j=1}^{n-1} \overline{a_j b_j}) \cup \overline{a_n b_n}$ where $\bigcup_{j=1}^{n-1} \overline{a_j b_j}$ and $\overline{a_n b_n}$ are connected subspaces and $a_n = b_{n-1} \in (\bigcup_{j=1}^{n-1} \overline{a_j b_j}) \cap \overline{a_n b_n} \neq \emptyset$. From Theorem 1.7.5 we infer that the broken line A is a connected space which completes the proof. ■

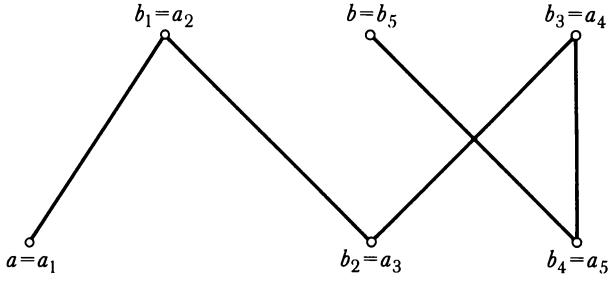


Fig.34. A broken line joining a to b.

1.10.6. COROLLARY. *If every two points of a set $A \subset \mathbf{R}^m$ may be joined by a broken line lying in A, then A is a connected set.* ■

Again the example of the $(m-1)$-dimensional unit sphere $S^{m-1} \subset \mathbf{R}^m$ for $m > 1$ shows that the converse of the Corollary does not hold. However the following theorem characterizing regions in Euclidean spaces is available.

1.10.7. THEOREM. *An open set A in the Euclidean space $\mathbf{R}^m$ is connected if and only if every two points of A may be joined by a broken line lying in A.*

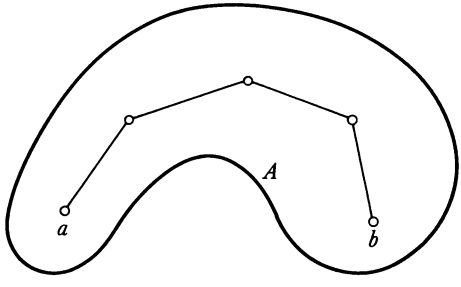


Fig.35. An open set in the space $\mathbf{R}^m$ is connected if every pair of points in it may be joined by a broken line lying in the set (Theorem 1.10.7).

PROOF. Sufficiency follows from Corollary 1.10.6. To prove necessity consider a fixed, though arbitrary, point $a \in A$ and denote by B the set of points b in A for which there exists a broken line lying in A joining a to b. Our aim is to show that $B = A$. To achieve this aim we show that the sets B and $A \setminus B$ are open in A; since $a \in B$ we have $B \neq \emptyset$; from the connectedness of the subspace A it will then follow that $A \setminus B = \emptyset$, that is that $B = A$.

To show that B is open in the subspace A suppose $b \in B$ and using the fact that A is open in $\mathbf{R}^n$ choose a positive real number r so that $B(b; r) \subset A$. We shall show

that $B(b;r) \subset B$. To this end suppose that $x \in B(b;r)$. Since $b \in B$ there is a broken line L lying in A joining the point a to the point b. Using the convexity of the ball $B(b;r)$ verified in Example 0.4.14 we infer that the line segment $\overline{bx}$ lies in $B(b;r)$ and hence also in A. The union $L \cup \overline{bx}$ is thus a broken line in A joining the point a to the point x, that is $x \in B$.

To show that $A \backslash B$ is open in A suppose that $c \subset A \backslash B$ and using the fact that A is open in the space $\mathbf{R}^m$ choose a positive real number r so that $B(c;r) \subset A$. We show that $B(c;r) \subset A \backslash B$. To this end suppose there is a point $x \in B(c;r) \cap B$. Since $x \in B$ there is a broken line L lying in A joining a to x. Appealing to the convexity of $B(c;r)$ we infer that the line segment $\overline{xc}$ is contained in $B(c;r)$ and hence also in A. The union $L \cup \overline{xc}$ is thus a broken line lying in A joining the point a to the point c contrary to the hypothesis that $c \notin B$. This contradiction completes the proof. ■

We now prove a theorem which gives a topological classification of a wide family of convex subspaces of Euclidean spaces. We begin with a proof of an auxiliary lemma.

1.10.8. LEMMA. *If A is a compact, convex subspace of the Euclidean space $\mathbf{R}^m$ and $a \in \operatorname{int} A$ (interior relative to $\mathbf{R}^m$), then every half-line with a as endpoint intersects the boundary $\operatorname{bd} A$ in exactly one point.*

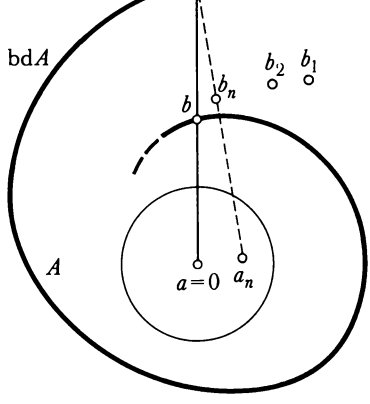


Fig.36. Illustration for the proof of Lemma 1.10.8.

PROOF. Applying an appropriate translation we may at once assume that $a = 0$. Let L be any half-line with a as endpoint. We show first that $L \cap \operatorname{bd} A$ contains at least one point. The boundary $\operatorname{bd}(L \cap A)$ relative to the half-line L is of course contained in $L \cap \operatorname{bd} A$ so it is enough to show $\operatorname{bd}(L \cap A) \neq \emptyset$. Since the set A is bounded, it is indeed enough to observe that the boundary of every non-empty bounded set on the half-line $\mathbf{R}_+$ is non-empty, but this follows immediately from the existence of the least upper bound.

Now suppose that $b, c \in L \cap \operatorname{bd} A$ with $b \neq c$. Thus $b = rc$ and we may suppose that $0 < r < 1$. Since $b \in \operatorname{bd} A$ there exists a sequence of points $b_n \in \mathbf{R}^m \backslash A$ for $n = 1, 2, \ldots$

such that $\lim_n b_n = b$. Consider the points

$$a_n = \frac{-r}{1-r}c + \frac{1}{1-r}b_n \quad \text{for} \quad n = 1, 2, \ldots$$

Then

$$\lim_n a_n = \frac{-r}{1-r}c + \frac{1}{1-r}b = \frac{-r}{1-r}c + \frac{r}{1-r}c = 0.$$

Since $0 \in A$, we have for sufficiently large indices n that $a_n \in A$. Since $b_n = (1-r)a_n + rc$ and $c \in A$ it follows from the convexity of A that $b_n \in A$ for these indices n, which contradicts the construction of the sequence $\{b_n\}$. ■

1.10.9. THEOREM. *If A is a compact, convex subspace of the Euclidean space $\mathbf{R}^m$ and the interior of the set A in the space $\mathbf{R}^m$ is non-empty, then there is a homeomorphism $h: A \to \bar{B}^m$ with $h(\operatorname{bd} A) = S^{m-1}$.*

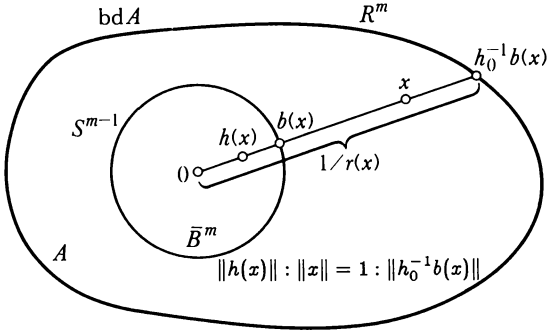


Fig.37. Illustration for the proof of Theorem 1.10.9.

PROOF. Applying if necessary a translation we may at once suppose that $0 \in \operatorname{int} A$. Consider the map $b: \mathbf{R}^m \setminus \{0\} \to S^{m-1}$ defined by the formula $b(x) = x/\|x\|$ for $x \in \mathbf{R}^m \setminus \{0\}$. From Examples 1.3.4 and 1.3.12 it follows that b is continuous. By Lemma 1.10.8 the map $h_0 = b|\operatorname{bd} A: \operatorname{bd} A \to S^{m-1}$ is bijective. Since the boundary $\operatorname{bd} A$ is a closed subset of the compact space A it is by Theorem 1.8.3 itself a compact subspace and using Theorem 1.8.15 we infer that h_0 is a homeomorphism of $\operatorname{bd} A$ onto the sphere S^{m-1}.

Now take $r(x) = 1/\|h_0^{-1}b(x)\|$ for $x \in A \setminus \{0\}$ and define

$$h(x) = \begin{cases} r(x)x, & \text{if } x \in A\setminus\{0\}, \\ 0, & \text{if } x = 0. \end{cases}$$

Since $\|x\| \leq \|h_0^{-1}b(x)\|$ we have $r(x) \leq 1/\|x\|$ for each $x \in A\setminus\{0\}$. Hence h is a map from A into $\bar{B}^m$.

The map h is continuous. Indeed, since $0 \in \operatorname{int} A$ and $h_0^{-1}b(x) \in \operatorname{bd} A$ for $x \in A\setminus\{0\}$ we deduce that there is a positive real number s such that $\|h_0^{-1}b(x)\| \geq s$ and so $r(x) \leq 1/s$ for $x \in A\setminus\{0\}$. This implies the continuity of h at the point 0 while continuity at the other points of the set A is obvious.

The map h is bijective. Indeed if $x \neq 0$, then obviously $h(x) \neq h(0) = 0$. On the other hand if $h(x') = h(x'')$ where $x' \neq 0 \neq x''$, then $r(x')x' = r(x'')x''$, hence

$b(x') = b(x'')$ and so $r(x') = r(x'')$ and finally we have $x' = x''$; thus the map is injective. Moreover if $y \in \bar{B}^m \backslash \{0\}$, then taking $x = y/r(y)$ we have $r(x) = r(y)$ and so $x = y/r(x)$, that is $y = h(x)$. Thus the map h takes the set A onto $\bar{B}^m$.

Since $h_0^{-1} b(x) = x$ for $x \in \operatorname{bd} A$, we have $h | \operatorname{bd} A = b | \operatorname{bd} A = h_0$, which completes the proof. ■

1.10.10. COROLLARY. *Any two compact, convex subspaces of the Euclidean space* $\mathbf{R}^m$ *which have non-empty interiors are homeomorphic.* ■

1.10.11. COROLLARY. *The* m*-dimensional unit cube* I^m *and the* m*-dimensional closed unit ball* $\bar{B}^m$ *are homeomorphic for each* m. ■

In order to widen the range of applicability of Theorem 1.10.9 we observe that the following lemmas hold.

1.10.12. LEMMA. *For every non-empty convex subset* A *of Euclidean space* $\mathbf{R}^m$ *there is exactly one affine subspace of* $\mathbf{R}^m$ *containing* A *relative to which* A *has non-empty interior.*

PROOF. Let the points $a_0, a_1, \ldots, a_n$ be an affine independent subset of A of largest cardinality and let H be their affine hull. For each $a \in A$ the set $a_0, a_1, \ldots, a_n, a$ is affinely dependent and by Theorem 0.4.9 is contained in an affine subspace of dimension less than $n+1$. Since H is the only affine subspace of dimension less than $n+1$ to contain $a_0, a_1, \ldots, a_n$, we have $a \in H$. We thus obtain $A \subset H$.

Now $\operatorname{conv}\{a_0, a_1, \ldots, a_n\} \subset \operatorname{conv} A = A$, so it is enough to show that the set $\operatorname{conv}\{a_0, a_1, \ldots, a_n\}$ has non-empty interior relative to H. The map sending each point of H to its barycentric coordinates relative to $a_0, a_1, \ldots, a_n$ is an affine isomorphism of H onto the n-dimensional hyperplane H_0 in $\mathbf{R}^{n+1}$ given by the equation $\sum_{j=1}^{n+1} x^j = 1$. Under this isomorphism $\operatorname{conv}\{a_0, a_1, \ldots, a_n\}$ is taken to the intersection of H_0 with the half-spaces $x^j \geq 0$ for $j = 1, 2, \ldots, n+1$. By Corollary 1.6.25 the point $(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1})$ belongs to the interior of this intersection; the interior is therefore non-empty. It follows from Corollary 1.3.27 that the interior of $\operatorname{conv}\{a_0, a_1, \ldots, an\}$ relative to H is also non-empty.

Assume now that there are two distinct affine subspaces containing the set A relative to both of which A has non-empty interior. Taking their intersection we may suppose that one of the affine subspaces is a proper subspace of the other and so according to Theorem 0.4.8 has smaller dimension. By Example 1.3.18 we may assume that one of the affine subspaces is the space $\mathbf{R}^m$ and the other is of dimension less than m. The assumption that the interior of A is non-empty then contradicts Example 1.6.15. ■

1.10.13. LEMMA. *The closure of a convex set in Euclidean space is a convex set.*

PROOF. Suppose that $A \subset \mathbf{R}^m$ is convex and that $a', a'' \in \operatorname{cl} A$. Suppose that $a' = \lim_n a'_n$ and $a'' = \lim_n a''_n$ where $a'_n, a''_n \in A$ for $n = 1, 2, \ldots$ If $a = (1-r)a' + ra''$ where $r \in I$, then by Examples 1.3.8 and 1.3.12 and Theorem 1.5.6 we obtain $a = \lim_n a_n$ where $a_n = (1-r)a'_n + ra''_n \in A$ for $n = 1, 2, \ldots$, thus $a \in \operatorname{cl} A$. ■

In view of Lemma 1.10.12 and Example 1.3.18 Theorem 1.10.9 implies the following.

1.10.14. COROLLARY. *Every non-empty, compact, convex metric subspace of the space* $\mathbf{R}^m$ *is homeomorphic to the ball* $\bar{B}^n$ *for some* $n \leq m$. ■

On the other hand an application of Lemma 1.10.13 allows us to deduce from Theorem 1.10.9 the following.

1.10.15. COROLLARY. *Every non-empty, convex, bounded region* A *in the Euclidean space* $\mathbf{R}^m$ *is homeomorphic to the open unit ball* B^m *while its boundary* $\operatorname{bd} A$ *is homeomorphic to the sphere* S^{m-1}.

PROOF. The closure $\operatorname{cl} A$ of the set A is by Assertion 1.6.8 a closed and bounded set of the space $\mathbf{R}^m$ and so by Theorem 1.10.2 is a compact subspace. By Lemma 1.10.13 it is a convex set and moreover $\emptyset \neq A = \operatorname{int} A \subset \operatorname{int}(\operatorname{cl} A)$. There is thus a homeomorphism $h: \operatorname{cl} A \to \bar{B}^m$ under which $h(\operatorname{bd} A) = S^{m-1}$. Since A is an open set, we have $h(A) = h(\operatorname{cl} A \setminus \operatorname{bd} A) = h(\operatorname{cl} A) \setminus h(\operatorname{bd} A) = \bar{B}^m \setminus S^{m-1} = B^m$, which completes the proof. ■

Making use of the properties of compact spaces we prove a final theorem on the separation of convex sets in Euclidean spaces.

1.10.16. THEOREM. *If* A *and* B *are disjoint convex sets in the Euclidean space* $\mathbf{R}^m$, *one of which is closed and the other compact, then there is a closed half-space which contains one of the sets* A *or* B *and is disjoint with the other.*

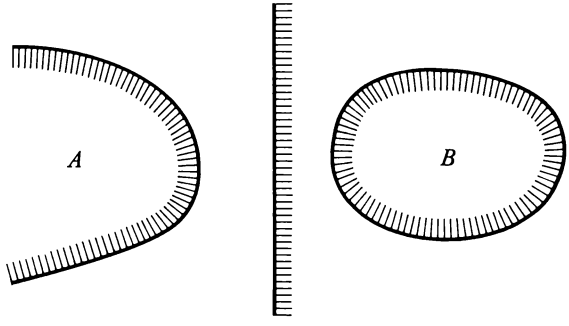


Fig.38. The set A is closed while B is compact. There exists a half-space containing B but disjoint from A (Theorem 1.10.16).

PROOF. To fix our attention we will assume that the set B is compact and that $A \neq \emptyset$. Consider a number r with the property that the set $A' = \{a \in A : \rho(a, B) \leq r\}$ is non-empty. The map $f: \mathbf{R}^m \to \mathbf{R}$ defined by $f(x) = \rho(x, B)$ for $x \in \mathbf{R}^m$ is continuous by Theorem 1.4.10, so the set $A' = A \cap f^{-1}([0, r])$ is closed. The latter is also bounded, since $\operatorname{diam} A' \leq \operatorname{diam} B + 2r$; thus A' is also compact.

Take $\epsilon = \inf\{\rho(a, b) : a \in A,\ b \in B\} = \inf\{\rho(a, b) : a \in A',\ b \in B\}$. Appealing to Theorems 1.8.5 and 1.8.16 we deduce that $\epsilon > 0$ and that there are points $a_0 \in A$ and $b_0 \in B$ for which $\rho(a_0, b_0) = \epsilon$. Now that this has been established the sets A and B play symmetric roles in the proof.

Applying if necessary an appropriate translation and rotation (cf. Example 1.3.16) we may as well assume that $a_0 = (0, \ldots, 0, c)$, $b_0 = (0, \ldots, 0, -c)$ where $c = \frac{1}{2}\epsilon$. Denote by $\mathbf{R}^m_+$ the set $\{(x^1, x^2, \ldots, x^m) \in \mathbf{R}^m : x^m \geq 0\}$; we shall show that $A \subset \mathbf{R}^m_+$ and that $B \cap \mathbf{R}^m_+ = \emptyset$, which will complete the proof. Since $a_0 \in \mathbf{R}^m_+$ and $b_0 \in \mathbf{R}^m \backslash \mathbf{R}^m_+$ it is enough by the convexity of A and B to show that both are disjoint with the hyperplane $\mathbf{R}^{m-1}_0 = \{x^1, x^2, \ldots, x^m) \in \mathbf{R}^m : x^m = 0\}$. So suppose for example that $a \in A \cap \mathbf{R}^{m-1}_0$ and let $a = (a^1, a^2, \ldots, a^{m-1}, 0)$. Obviously $a \neq a_0$ since $a_0 \in \mathbf{R}^m_+ \backslash \mathbf{R}^{m-1}_0$. For each $r \in I$ the point $(1-r)a_0 + ra$ belongs to the set A; the square of the distance

$$\begin{aligned} \rho^2(b_0, (1-r)a_0 + ra) &= \|r(a - a_0) + (a_0 - b_0)\|^2 \\ &= r^2\|a - a_0\|^2 + 2r(a - a_0) \cdot (a_0 - b_0) + \|a_0 - b_0\|^2 \end{aligned}$$

is a quadratic in r which in view of the definition of a_0 and b_0 has a minimum over the interval I occuring at $r = 0$. It follows that the minimum of this quadratic on the whole line $\mathbf{R}$ must occur at a non-positive value of r and hence the coefficient $(a - a_0) \cdot (a_0 - b_0) = (-c)(2c) = -4c^2$ must be non-negative. Thus $c = 0$, which is impossible since $c = \frac{1}{2}\epsilon > 0$. ■

1.10.17. COROLLARY. *For any point $a \in \mathbf{R}^m$ and any closed convex set $B \subset \mathbf{R}^m$ which does not contain the point a there is a closed half-space containing B which omits the point a.* ■

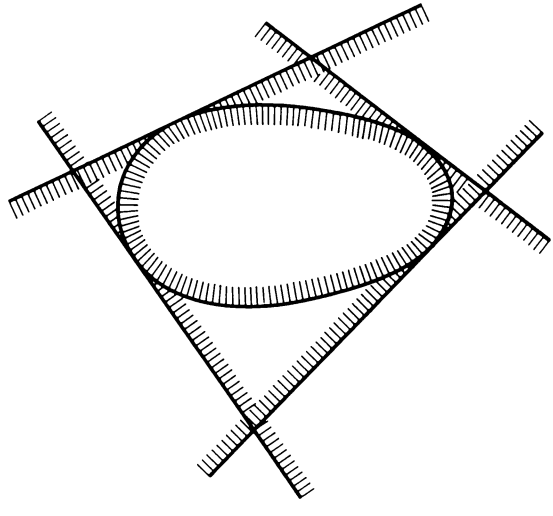


Fig.39. Every closed convex set is the intersection of all half-spaces which contain it (Corollary 1.10.18).

1.10.18. COROLLARY. *Every closed, convex set in the Euclidean space $\mathbf{R}^m$ is the intersection of the closed half-spaces which contain it.* ■

Exercises

a) Give an example of a closed and bounded set in the Hilbert space $\mathbf{R}^\omega$ which is not a compact subspace.

b) Carry out a classification of convex subsets of the real line $\mathbf{R}$ from the point of view of similarity geometry and topology.

c) Give an example of a plane region A with the property that for any positive number ϵ there are two points $a, b \in A$ such that $\rho(a,b) < \epsilon$ but the sum of the lengths of the segments of any broken line joining a to b exceeds ϵ.

d) Generalize Corollary 1.10.15 by showing that every non-empty convex region (not necessarily bounded) in the Euclidean space $\mathbf{R}^m$ is homeomorphic to an open ball.

e) Give an example showing that in Theorem 1.10.16 it is not enough to suppose that both sets A and B are closed, convex and disjoint.

1.S. Supplements

1.S.1. The concept of a metric space is due to M. Fréchet (1906), the name was suggested by F. Hausdorff (1914). Of the many generalizations of the concept of a metric particular applicability accords to the concept of a *pseudometric*, by which we mean a function $\rho: X \times X \to \mathbf{R}$ satisfying the axioms (M2) and (M3) and an axiom

(M'1) $\rho(x,x) = 0$ *for every* $x \in X$.

A set X with a pseudometric ρ is called a *pseudometric space*. Some of the metric concepts (like that of convergent sequence or of a Cauchy sequence) may be carried across to pseudometric spaces with the preservation of their essential properties (cf. [9], p. 232–234).

In Problem 1.P.2 it is shown that every pseudometric space determines in a natural way a certain metric space.

A vector space V over the field $\mathbf{R}$ is called a *normed space* if there is a function which associates with each vector $\alpha \in V$ a number $\|\alpha\|$ (called the *norm* of the vector α) such that the following conditions hold:

(N1) $\|\alpha\| = 0$ if and only if $\alpha = 0$,

(N2) $\|r\alpha\| = |r|\|\alpha\|$ for $\alpha \in V$, $r \in \mathbf{R}$,

(N3) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ for $\alpha, \beta \in V$.

It is obvious that the function ρ defined by the formula $\rho(\alpha,\beta) = \|\alpha - \beta\|$ for $\alpha, \beta \in V$ is a metric on V, we call it the metric *induced* by the norm. Convergence of sequences in the sense of this metric is called *convergence in norm.*

A metric ρ on a vector space V over the field $\mathbf{R}$ is said to be *translatable* if $\rho(\alpha+\gamma, \beta+\gamma) = \rho(\alpha,\beta)$ for $\alpha, \beta, \gamma \in V$. It is said to be *absolutely homogeneous* if $\rho(r\alpha, r\beta) = |r|\rho(\alpha,\beta)$ for $\alpha, \beta \in V$, $r \in \mathbf{R}$. It is easily seen that the metric induced by a norm is translatable and absolutely homogeneous and that conversely every translatable, absolutely homogenous metric determines a norm $\|\alpha\| = \rho(\alpha, 0)$ for $\alpha \in V$. Metrics considered in the Examples 1.1.5, 1.1.8 an in the Supplements 1.S.4, 1.S.5, 1.S.6 are all determined by some norm.

1.S.2. The Hilbert space $\mathbf{R}^\omega$ was introduced by Hilbert in connection with the theory of integral equations about 1906. It is also denoted by the symbol ℓ^2 and referred to

as the *space of square summable sequences* reserving the term Hilbert space to a wider class of spaces (the complete unitary spaces).

1.S.3. The Cauchy-Schwarz inequality was stated by Schwarz in 1885 and earlier by Cauchy (1821); in an integral form it was noticed by Buniakowski (1859), it therefore appears in the literature under various names among them the *Cauchy-Buniakowski inequality*. It is a special case of a more general result known as *Hölder's inequality*. If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for any two sequences of m real numbers $(a^1, a^2, \ldots, a^m)$ and $(b^1, b^2, \ldots, b^m)$ the inequality

$$\sum_{i=1}^{m} |a^i b^i| \leq \left(\sum_{i=1}^{m} |a^i|^p \right)^{1/p} \left(\sum_{i=1}^{m} |b^i|^q \right)^{1/q}$$

holds.

An analogous inequality for infinite sequences is also known as the *Hölder inequality*. We say that the infinite sequence of real numbers $\{a^1, a^2, \ldots\}$ is *p^{th}-power summable* where $p > 1$ if the series $\sum_{i=1}^{\infty} |a^i|^p$ is convergent. Given two infinite sequences of real numbers $\{a^1, a^2, \ldots\}$ which is p^{th}-power summable and $\{b^1, b^2, \ldots\}$ which is q^{th}-power summable where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ the series $\sum_{i=1}^{\infty} |a^i b^i|$ is summable and the inequality

$$\sum_{i=1}^{\infty} |a^i b^i| \leq \left(\sum_{i=1}^{\infty} |a^i|^p \right)^{1/p} \left(\sum_{i=1}^{\infty} |b^i|^q \right)^{1/q}$$

holds.

The Hölder inequality also has an integral analogue. If a measurable function $f: I \to \mathbf{R}$ is such that the integral $\int_0^1 |f(x)|^p dx$ is finite where $p > 1$, then the function is said to be *p^{th}-power integrable*. Given a function $f: I \to \mathbf{R}$ which is p^{th}-power integrable and a function $g: I \to \mathbf{R}$ which is q^{th}-power integrable, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the integral $\int_0^1 |f(x)g(x)| dx$ exists and the inequality

$$\int_0^1 |f(x)g(x)| dx \leq \left(\int_0^1 |f(x)|^p dx \right)^{1/p} \left(\int_0^1 |g(x)|^q dx \right)^{1/q}$$

holds.

1.S.4. A metric can be introduced in the space of p^{th}-power summable sequences (cf. Supplement 1.S.3) by the formula

$$\rho(x, y) = \left(\sum_{i=1}^{\infty} |x^i - y^i|^p \right)^{1/p}$$

for $x = \{x^1, x^2, \ldots\}$ and $y = \{y^1, y^2, \ldots\}$. The metric space so obtained is called the *space of p^{th}-power summable sequences* and is denoted ℓ^p; this is a generalization of the Hilbert space which is obtained on putting $p = 2$.

1.S.5. The set of all p^{th}-power integrable functions (in which we identify functions that differ on a set of measure zero, something that is unnecessary if we restrict attention to continuous functions) may be given a metric (more accurately speaking – a

pseudometric, but see the construction in Problem 1.P.2) by means of the formula

$$\rho(f,g) = \left(\int_0^1 |f(t) - g(t)|^p dt\right)^{1/p}.$$

The resulting metric space is known as the *space of p^{th}-power integrable functions* and is denoted L^p.

1.S.6. If W is a bounded convex set in the Euclidean space $\mathbf{R}^m$ which contains 0 in its interior and is symmetric with respect to 0, then the function ρ_W defined by the formula $\rho_W(x,y) = \inf\{r > 0 : \frac{x-y}{r} \in W\}$ for $x, y \in \mathbf{R}^m$ is a metric on $\mathbf{R}^m$ (see Problem 1.P.5). It is known as the *Minkowski metric* determined by W. It is obvious that the interior of the set W coincides with the open unit ball in the metric ρ_W.

1.S.7. The Lipschitz maps are a special case of the Hölder maps. We say that a map $f: X \to Y$ is a *Hölder map with constant $c \geq 0$ and exponent α* for $0 < \alpha \leq 1$, if

$$\rho(f(x), f(x')) \leq c\rho(x,x')^\alpha$$

for all $x, x' \in X$.

1.S.8. Lipschitz maps with constant $c < 1$ are called *contractive*; they have an important role in the theory of fixed points in complete spaces, which will be touched on in Section 6.4.

1.S.9. The definition of continuity given in the present chapter is known as the *Cauchy definition.* If the condition given in Theorem 1.5.6 is taken as a definition, it is then known as the *Heine definition.*

1.S.10. Suppose given a map $f: X \to Y$ between metric spaces. The number

$$o_f(x) = \inf\{\operatorname{diam} f(B(x;\delta)) : \delta > 0\}$$

is called the *oscillation of the map f at the point $x \in X$.* The function ω_f defined by the formula

$$\omega_f(\delta) = \sup\{\rho(f(x), f(x')) : x, x' \in X, \rho(x,x') < \delta\}$$

for $\delta > 0$ is called the *modulus of continuity of the map f.*

The connection of these two concepts with the concepts of Lipschitz map, Hölder map, uniformly continuous map and continuous map is the theme of Problems 1.P.18–1.P.21.

1.S.11. The programme proposed by F. Klein and known as the Erlangen programme, was stated in 1872 in his inaugural lecture on the occasion of taking up a professorial chair at the University of Erlangen. In the case of Euclidean spaces it also includes affine geometry – since as may be proved (see Problem 1.P.17) every similarity of a space $\mathbf{R}^m$ is an affine isomorphism and every affine isomorphism is a uniform homeomorphism (Corollary 1.3.27).

1.S.12. The diagonal map $d: X \to \bigtriangledown^m X$ described in Example 1.3.23 is a special case of the diagonal of a family of maps. If $f_i: X \to Y_i$ for $i = 1, 2, \ldots, m$ are given, then the map $f: X \to \bigtriangledown_{i=1}^m Y_i$ defined by the formula $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is called the *diagonal* of the maps f_i where $i = 1, 2, \ldots, m$ and is denoted by $\triangle_{i=1}^m f_i$. Of course the diagonal map is the diagonal of the identity maps.

1.S.13. In Example 1.1.9 we defined a metric $\tilde{\rho}$ on the set of all maps $f: X \to Y$ where Y is a bounded metric space by the formula

$$\tilde{\rho}(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\}.$$

Note that the same formula defines a metric on the set of all bounded maps $f: X \to Y$ where Y is any arbitrary metric space that is not necessarily bounded. This is because for any point $x_0 \in X$ we have

$$\begin{aligned}\rho(f(x), g(x)) &\leq \rho(f(x), f(x_0)) + \rho(f(x_0), g(x_0)) + \rho(g(x_0), g(x)) \\ &\leq \operatorname{diam} f(X) + \rho(f(x_0), g(x_0)) + \operatorname{diam} g(X);\end{aligned}$$

thus $\tilde{\rho}(f, g) < \infty$. The proof of properties (M1)–(M3) goes through identically to that in Example 1.1.9.

We study the space of bounded maps in detail in Section 6.2 (see also Supplement 7.S.21, where the compact-open topology on the space of maps is defined). We have already come across similar questions when introducing pointwise and uniform convergence and in attempting to metrize the space of all maps.

1.S.14. We say that a metric space is metrically

(1) *completely inhomogeneous* if the only isometry $f: X \to X$ is the identity map;
(2) *inhomogeneous* if there are points $a, b \in X$ for which there is no isometry $f: X \to X$ such that $f(a) = b$;
(3) *homogeneous* if for any pair of points $a, b \in X$ there is an isometry $f: X \to X$ such that $f(a) = b$;
(4) *strongly homogeneous* if for any two isometric sets $A, B \subset X$ there is an isometry $f: X \to X$ such that $f(A) = B$;
(5) *perfectly homogeneous* if for any two sets $A, B \subset X$ and any isometry $f_0: A \to B$ there is an isometry $f: X \to X$ such that $f \mid A = f_0$.

Examples of metric spaces can be given to distinguish between the classes given above (cf. Problems 1.P.16 and [1], p. 123–126).

It may be proved that the Euclidean space $\mathbf{R}^m$ is metrically perfectly homogeneous (cf. Problem 1.P.14 and [1], p. 126–129).

1.S.15. If X is a metric space, $a, b \in X$ and $r \in \mathbf{R}$, then any point $x \in X$ which satisfies: $\rho(x, a) = |r|\rho(a, b)$, $\rho(x, b) = |1 - r|\rho(a, b)$ is said to *divide the distance from a to b in the ratio $r : (1 - r)$*. This definition does not of course prejudge the existence of such a point nor its uniqueness. In particular any point which divides the distance from a to b in the ration $\frac{1}{2} : \frac{1}{2}$ is called a *midpoint* of the pair a, b. A metric space in which every pair of points has a midpoint is said to be *convex*; if every pair of points has a unique midpoint, then it is said to be *strongly convex*. It transpires (cf. Problem

1.P.25) that in the Euclidean space $\mathbf{R}^m$ the set of points dividing the distance from a to b in the ratio $r : (1 - r)$ consists of just one point, namely $(1 - r)a + rb$.

1.S.16. In a metric space (X, ρ), the metric itself is essential only in questions discussed from the point of view of metric geometry. However in any geometry based on a wider class of maps, in topology for instance, concepts like the limit of a sequence of points, or the openness of a set, may turn out to be identical, even though different metrics may have been used. This indicates that it would not be inappropriate to introduce a relation making certain metrics equivalent.

If K is a category whose objects are metric spaces, and ρ and ρ' are metrics on some set X, then we say that the metric ρ is *not stronger* than the metric ρ' (in the sense of the category K) if the identity map $\mathrm{id}_X\colon (X, \rho) \to (X, \rho')$ is a morphism of the category. If moreover it is an isomorphism, then we say that the metrics are *equivalent* (in the sense of the category K). The most interesting case appears to be that of the category of metric spaces and continuous maps; then (that is to say, when the identity map id_X is continuous, or is a homeomorphism) we drop the reference to the category. It is easy to see (cf. Theorem 1.5.6) that the metrics ρ and ρ' are equivalent if and only if they give rise to the same concept of convergence in the space X and also (cf. Theorem 1.6.24) if and only if they determine identical classes of open, or closed, sets.

1.S.17. The freedom in selecting a metric left to us by an equivalence class which we observe when we view a metric space from a topological standpoint (cf. 1.S.16) carries the suggestion that topology could be based upon other primitive notions. One approach to this question is to take the concept of a limit of a sequence of points as the primitive notion and certain obvious properties of the limit enjoyed in metric spaces (namely 1.5.1–1.5.4) as axioms. The objects thus obtained are called *L^*-spaces*. In L^*-spaces it is possible to introduce in a natural way the concept of a continuous map (cf. Theorem 1.5.6), of a closed set and then by way of complementation the concept of an open set (see [4], p. 90, 91 and [9], p. 188–204).

It turns out to be more fruitful to take a distinguished family of open sets as the primitive notion and certain obvious properties of open sets enjoyed in metric spaces (Theorems 1.6.3 and 1.6.4 and the fact that the empty set and the whole space are open) as axioms. Theorem 1.6.24 then constitutes a definition of continuous maps while closed sets may be defined as the complements of open sets. The spaces introduced in this way are called *topological spaces*; we examine this concept more closely in Chapter 7. However it is worth noting here that there exist L^*-spaces and topological spaces which are not metrizable, that is to say, for which no metric can be found that would induce the structure which was taken as primitive.

1.S.18. We may use the ideas introduced in Supplement 1.S.17 to define the *Möbius space*. For $m = 1, 2, \ldots$ denote by M^m the union of the Euclidean m-dimensional space $\mathbf{R}^m$ and a single point $p_\infty \notin \mathbf{R}^m$ known traditionally as an *improper* point or point *at infinity*. We will not define a metric on M^m, instead we take as the open sets of M^m all the open sets of the space $\mathbf{R}^m$ and sets of the form $U \cup \{p_\infty\}$ where U is the complement of a compact set in $\mathbf{R}^m$. It is easy to see that we thus obtain a topological space; we call it the *m-dimensional Möbius space*. We can also give the Möbius space M^m the structure

of an L^*-space by taking as its convergent sequences all the convergent sequences of $\mathbf{R}^m$ and those sequences of points $x_n \in \mathbf{R}^n$ for $n = 1, 2, \ldots$ for which $\lim_n \|x_n\| = \infty$ and having these converge to p_∞.

Denote by $i: \mathbf{R}^m \setminus \{0\} \to \mathbf{R}^m \setminus \{0\}$ the inversion in the $(m-1)$-dimensional unit sphere in the space $\mathbf{R}^m$ and by $s: S \setminus \{0\} \to \mathbf{R}^m$ the stereographic projection from the pole 0 of an appropriately positioned m-dimensional sphere in the space $\mathbf{R}^{m+1}$ onto the m-dimensional hyperplane H composed with an isometry of H onto the space $\mathbf{R}^m$. It turns out that (cf. Problem 1.P.22) the inversion i extends to a homeomorphism $i^*: M^m \to M^m$ such that $i^*(0) = p_\infty$ and $i^*(p_\infty) = 0$. Similarly the stereographic projection s extends to a homeomorphism $s^*: S \to M^m$ such that $s^*(0) = p_\infty$; thus the m-dimensional Möbius space M^m is homeomorphic to the m-dimensional sphere S^m.

Further properties of inversion and stereographic projection are given in Problems 1.P.23–1.P.24. The property described in Problem 1.P.23 is expressed by saying that inversion is a *spherical affinity*. The property described in Problem 1.P.24 is expressed by saying that stereographic projection is a *conformal* map. A detailed development of the theory of Möbius spaces may be found in [1].

1.S.19. The concept of a connected space is due to C. Jordan (1893); it is one of the most intuitive concepts of topology. By means of the concept of a connected space several other topological invariants may be defined. For instance the number, or more generally the cardinality of the set of components of a space is a topological concept. Another concept of this type is that of a set separating a space, to which we devote more attention in Section 4.2.

1.S.20. The concept of a compact metric space is due to M. Fréchet (1906); for more general spaces it was introduced by L. Vietoris (1921) and independently by P. Alexandrov and P. Urysohn (1923). It is indoubtedly one of the most important topological concepts. It may be proved (cf. Problem 1.P.33) that the Borel-Lebesgue Theorem characterizes the compact spaces in the class of metric spaces. This fact permits the concept of compactness to be carried over to the context of general topology, where the property mentioned in the Borel-Lebesgue Theorem is taken as defining compactness. We shall return to the concept of compactness in Chapter 7; we will also consider there a number of related concepts.

1.S.21. Complete spaces were first considered by M. Fréchet (1906). This class of spaces, though not topological in character deserves much attention in view of its numerous applications first and foremost in analysis. Wide enough to contain Euclidean spaces, Hilbert space and several important function spaces it nevertheless possesses some of the properties enjoyed by compact spaces. We shall return to complete spaces in Section 6.4 where we consider them from the angle of application to certain problems in analysis.

1.S.22. In closing we mention that in place of the symbol $\operatorname{cl} A$ also appearing in the literature (eg. [4], [9]) is the symbol $\bar{A}$ which we preferred to abandon in favour of a unified notation; similarly in place of the symbol bd the symbol Fr occurs which is an abbreviation from the french word frontière.

1.P. Problems

1.P.1. Suppose X is a metric space and the sets $A, B \subset X$ are non-empty, closed and bounded. Define

$$\operatorname{dist}(A,B) = \max(\sup\{\rho(x,B) : x \in A\},\ \sup\{\rho(y,A) : y \in B\}).$$

Show that dist is a metric on the family of non-empty, closed and bounded subsets of the space X; it is called the *Hausdorff* metric.

1.P.2. Let (X,ρ) be a pseudometric space (cf. Supplement 1.S.1). Define a relation $\sim$ on the set X by putting $x \sim x'$ when $\rho(x,x') = 0$. Show this is an equivalence relation. Let $\hat{X}$ be the set of equivalence classes of this relation. Show that the function $\hat{\rho}$ defined by $\hat{\rho}([x],[x']) = \rho(x,x')$ for $[x],[x'] \in \hat{X}$ is validly defined and is a metric on $\hat{X}$.

1.P.3. Check that the axioms for a metric hold for the space of p^{th}-power summable sequences (cf. Supplement 1.S.4).

1.P.4. Check that the axioms for a metric hold for the space of p^{th}-power integrable functions (cf. Supplement 1.S.5).

1.P.5. Check that the function ρ_W defined in Supplement 1.S.6 is a metric (the *Minkowski metric*). Prove that all Minkowski metrics in the space $\mathbf{R}^m$ are equivalent (cf. Supplement 1.S.16).

1.P.6. Show that if both identity maps $\mathrm{id}\colon (X,\rho) \to (X,\rho')$ and $\mathrm{id}'\colon (X,\rho') \to (X,\rho)$ are Lipschitz (not necessarily with the same constant), then the metrics ρ and ρ' on the set X are equivalent (cf. Supplement 1.S.16).

1.P.7. Prove that the metric of the metric product is equivalent to each of the metrics defined in Exercises (a) and (b) of Section 1.2. (Hint: Take advantage of the result in Problem 1.P.6).

1.P.8. Check that the zero-one metric on a set X is not stronger than any other metric in X (cf. Supplement 1.S.16). Investigate whether there always exists a metric for which any other metric is not stronger.

1.P.9. Prove that every metric space X is homeomorphic to a bounded space. (Hint: Consider the new metric defined by $\tilde{\rho}(x,y) = \rho(x,y)/(1+\rho(x,y))$ for $x,y \in X$ and show that it is equivalent to the metric ρ.)

1.P.10. Suppose that (a) X is an arbitrary metric space and $Y = \mathbf{R}$; (b) $X = Y = \mathbf{R}^m$; (c) $X = Y = S^m$. Prove that for every set $A \subset X$ and every non-expansive map $f\colon A \to Y$ there exists a map $f^*\colon X \to Y$ which is also non-expansive and satisfies $f^* | A = f$. (Hint: Show that, in order for a pair of metric spaces X and Y to have the desired property, it is necessary and sufficient that for any two equinumerous families of balls $\{B(x_t;r_t)\}_{t\in T}$ in X and $\{B(y_t;r_t)\}_{t\in T}$ in Y whose centres satisfy $\rho(y_t,y_{t'}) \leq \rho(x_t,x_{t'})$ for every $t,t' \in T$, the condition $\bigcap_{t\in T} B(x_t;r_t) \neq \emptyset$ should imply the condition $\bigcap_{t\in T} B(y_t;r_t) \neq \emptyset$.)

1.P.11. Show that if the system of points $a_0, a_1, \ldots, a_n \in \mathbf{R}^m$ is affinely independent and the system $b_0, b_1, \ldots, b_n$ is isometric to it, then the system $b_0, b_1, \ldots, b_n$ is also affinely independent.

1.P.12. Show that if the system of points $a_0, a_1, \ldots, a_m \in \mathbf{R}^m$ is affinely independent, then every point $x \in \mathbf{R}^m$ is uniquely determined by the distances $\rho(x, a_j)$ for $j = 0, 1, \ldots, m$.

1.P.13. Show that every pair of isometries $f, g: A \to \mathbf{R}^m$, where $A \subset \mathbf{R}^m$, which agree on some affinely independent set of points $a_0, a_1, \ldots, a_m \in A$ are identical. (Hint: Use the results of Problems 1.P.11 and 1.P.12).

1.P.14. Show that the Euclidean space $\mathbf{R}^m$ is metrically perfectly homogeneous (cf. Supplement 1.S.14 (5)). (Hint: Consider first the possibility of extending isometries defined on affine subspaces of $\mathbf{R}^m$ and hence reduce the general case to the situation when an isometry $f_0: A \to B$ is given, where $A \subset \mathbf{R}^m$ contains an affinely independent system of points $a_0, a_1, \ldots, a_m$. Make use of the result in Problem 1.P.11. Consider an affine isomorphism $f: \mathbf{R}^m \to \mathbf{R}^m$ whose existence follows from the first part of Theorem 0.4.20; check that it is an isometry. To verify that f is an extension of f_0 use the result in Problem 1.P.13).

1.P.15. Show that the n-dimensional sphere is metrically perfectly homogeneous.

1.P.16. Give examples of finite metric spaces which are metrically:
a) completely inhomogeneous,
b) inhomogeneous, but not completely inhomogeneous,
c) homogeneous, but not strongly homogeneous,
d) strongly homogeneous, but not perfectly homogeneous (cf. Supplement 1.S.14).

1.S.17. Show that isometries $f: \mathbf{R}^m \to \mathbf{R}^m$ are identical with maps described by equations of the form:

$$y^i = a_0^i + \sum_{j=1}^{m} a_j^i x^j \quad \text{for } i = 1, 2, \ldots, m, \quad \text{where} \quad \sum_{i=1}^{m} a_j^i a_k^i = \delta_{jk} \quad \text{for } j, k = 1, 2 \ldots, m.$$

(Hint: Check that the equations do in fact define an isometry. For a given isometry $f: \mathbf{R}^m \to \mathbf{R}^m$ consider its values on the orthonormal set $e_0 = 0$, $e_i = (\delta_i^1, \delta_i^2, \ldots, \delta_i^m)$ for $i = 1, 2, \ldots, m$ and put $(a_0^1, a_0^2, \ldots, a_0^m) = f(e_0)$, $(a_i^1, a_i^2, \ldots, a_i^m) = f(e_i) - f(e_0)$ for $i = 1, 2, \ldots, m$. Check that $\sum_{i=1}^m a_j^i a_k^i = \delta_{jk}$ for $j, k = 1, 2, \ldots, m$. Use the result of Problem 1.P.13.) Deduce that every isometry $f: \mathbf{R}^m \to \mathbf{R}^m$ is an affine isomorphism. Carry through an analogous analysis for similarities in $\mathbf{R}^m$ and show that they are affine isomorphisms.

1.P.18. Show that a map $f: X \to Y$ is continuous at a point $x \in X$ if and only if $o_f(x) = 0$ (cf. Supplement 1.S.10).

1.P.19. Prove that $f: X \to Y$ is a Hölder map with constant c and exponent α if and only if $\omega_f(\delta) \leq c\delta^\alpha$ for each $\delta > 0$ (cf. Supplements 1.S.7 and 1.S.10).

1.P.20. Prove that a map $f: X \to Y$ is uniformly continuous if and only if $\inf\{\omega_f(\delta) : \delta > 0\} = 0$ (cf. Supplement 1.S.10).

1.P.21. Suppose that $A \subset X$ and denote by χ the characteristic function of the set A, that is the function $\chi: X \to \mathbf{R}$ defined by:

$$\chi(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in X \setminus A. \end{cases}$$

Prove that $\operatorname{bd} A = \{x \in X : o_\chi(x) > 0\}$ (cf. Supplement 1.S.10).

1.P.22. Prove that the inversion $i: \mathbf{R}^m \setminus \{0\} \to \mathbf{R}^m \setminus \{0\}$ and the stereographic projection $s: S \setminus \{0\} \to \mathbf{R}^m$, as also the extended inversion $i^*: M^m \to M^m$ and the extended stereographic projection $s^*: S \to M^m$ (cf. Supplement 1.S.18) are homeomorphisms.

1.P.23. Suppose that the set $A \subset S^m$ is isometric with the n-dimensional sphere S^n. Investigate its image under the extended stereographic projection (cf. Supplement 1.S.18). Making use of the connection between inversion and stereographic projection, draw the corresponding conclusion regarding inversion (cf. [1], p. 300–302).

1.P.24. Show that if $s: S \setminus \{0\} \to \mathbf{R}^m$ is the stereographic projection, $x_n, y_n \in S \setminus \{0\}$ for $n = 1, 2, \ldots$ and $\lim_n x_n = \lim_n y_n = z \in S \setminus \{0\}$, then the limit

$$\lim_n \frac{\rho(x_n, y_n)}{\rho(s(x_n), s(y_n))}$$

exists and depends only on the point z (cf. [1]; p. 297–298).

1.P.25. Show that in the Euclidean space $\mathbf{R}^m$ the set of points dividing the distance between a and b in the ratio $r : (1 - r)$ (cf. Supplement 1.S.15) consists of exactly the one point $(1-r)a + rb$. (Hint: Investigate when the triangle inequality in the space $\mathbf{R}^m$ becomes an equation. For this purpose investigate when the Cauchy-Schwarz inequality becomes an equation.)

1.P.26. Show in a convex complete metric space X (cf. Supplement 1.S.15) that for every pair of points $a, b \in X$ there exists a set $A \subset X$ and an isometry f of the interval $[0, \rho(a, b)]$ onto A such that $f(0) = a$, $f(\rho(a, b)) = b$. Show that strong convexity secures the uniqueness of the set A with this property.

1.P.27. Show that every m-dimensional open ball, every m-dimensional closed ball and every $(m-1)$-dimensional sphere in the Euclidean space $\mathbf{R}^m$ uniquely determine their centre and radius. (Hint: Proof is required that if $B(c; r) = B(c'; r')$, then $c = c'$ and $r = r'$ and analogoues for closed balls and spheres. For this purpose assume on the basis of the analysis of Example 1.3.16 that $c = 0$ and $c' = (a, 0, \ldots, 0)$ where $a \geq 0$. Then show that $a + |r - r'| = 0$.)

1.P.28. Suppose X_0 is a metric subspace of a space X and $A \subset X_0$. Show that the closure of the set A in X_0 is $(\operatorname{cl} A) \cap X_0$ where $\operatorname{cl} A$ denotes the closure of the set A in the space X. State and prove the analogous result for the interior (cf. Theorems 1.6.5 and 1.6.20).

1.P.29. Show that $\operatorname{int} A$ is the union of all the open subsets of the space X which are contained in A and $\operatorname{cl} A$ is the intersection of all the closed subsets of the space which contain A.

1.P.30. Prove that the metric of Example 1.1.9 defines in the space of maps a notion of convergence for sequences of maps identical to uniform convergence.

1.P.31. Investigate whether the limit in the sense of: (1) pointwise convergence, (2) uniform convergence of a sequence of maps which are: (a) Lipschitz with constant c, (b) Lipschitz, (c) uniformly continuous, belongs to the same class of maps.

1.P.32. Prove that if a metric space X has the property that for every continuous function $f: X \to \mathbf{R}$, any two points $a, b \in X$ and any real number $r \in \mathbf{R}$ such that $f(a) \leq r \leq f(b)$ there is a point $c \in X$ satisfying $f(c) = r$, then the space X is connected (cf. the Darboux Theorem 1.7.21).

1.P.33. Prove that if in a metric space X every open covering contains a finite covering, then the space is compact (cf. the Borel-Lebesgue Theorem 1.8.12).

1.P.34. Prove that the space Q of rational numbers is not homeomorphic to any complete space. (Hint: Prove that any countable complete space possesses an isolated point).

1.P.35. Prove that every broken line joining distinct points a and b contains a broken line joining the two points that is homeomorphic to the unit interval I.

1.P.36. Prove that if a subspace $A \subset \mathbf{R}^m$ is compact, then $\operatorname{conv} A$ is also a compact subspace. Give an example to illustrate that $\operatorname{conv} A$ need not be closed for a closed set A of the space $\mathbf{R}^m$.

1.P.37. Prove the *Radon Theorem*: Every set $A \subset \mathbf{R}^m$ containing at least $m+2$ points is a union $A = B \cup C$ where $B \cap C = \emptyset$ and $\operatorname{conv} B \cap \operatorname{conv} C \neq \emptyset$.

1.P.38. Prove the *Carathéodory Theorem*: If $A \subset \mathbf{R}^m$, then each point $x \in \operatorname{conv} A$ is of the form $x = \sum_{i=0}^{m} r^i a_i$, where $a_i \in A$, $r^i \geq 0$ for $i = 0, 1, \ldots, m$ and $\sum_{i=0}^{m} r^i = 1$. (Hint: Make use of the result in Problem 1.P.37.)

1.P.39. Prove the *Helly Theorem*: If a family of at least $m+1$ convex, closed subsets of the Euclidean space $\mathbf{R}^m$ has the property that every $m+1$ many members have non-empty intersection, then the whole family has non-empty intersection. (Hint: Proof of the contrapositive by induction. If $\bigcap_j C_j = \emptyset$, then there exist indices $j_0, j_1, \ldots, j_m$ such that $B = C_{j_1} \cap C_{j_2} \cap \ldots C_{j_m} \neq \emptyset$ but $B \cap C_{j_0} = \emptyset$. Apply Theorem 1.10.16 and then the inductive hypothesis to the intersection of the sets C_j with a hyperplane.)

1.P.40. Prove that if a convex set in the space $\mathbf{R}^m$ is contained in the union of a finite number of half-spaces, then it is contained in the union of at most $m+1$ of these half-spaces. (Hint: Use the result of Problem 1.P.39.)

Chapter 2
Polyhedra

The class of polyhedra is of particular significance in general topology for two reasons. Firstly, polyhedra possess numerous properties which are regarded as paradigms. Since they may be finitely decomposed into very simple elements (simplices, cells), polyhedra may be studied by means of finite algorithms (for example by induction on the number of simplices, or on their dimension) and by means of various combinatorial methods. It was for this reason that the theory of polyhedra formed a natural foundation for the development of algebraic topology, known initially under the name of combinatorial topology. Secondly, polyhedra and the continuous maps corresponding to what are called simplicial maps form a large enough category, that by various approximation techniques (such as those associated with the notion of the nerve of a covering, or the simplicial approximation etc.) it proves possible to extend some of the results to a wider class of spaces and maps.

Section 2.1 is dedicated to simplices, which are the basic components of a polyhedron. We study various geometric and topological properties of simplices, and define the interior, boundary and faces of a simplex. In Section 2.2 we introduce the notion of a simplicial complex and subcomplex, and in the examples we discuss the nerve of a covering and the n-dimensional skeleton. Section 2.3 contains the definition of a polyhedron as the underlying space of a simplicial complex. Next we show how one can check the connectedness of a polyhedron by means of a triangulation. The Section also includes the definition of the geometric dimension of a polyhedron and the barycentric coordinates of a point in a polyhedron. We close by studying the properties of the covering of a polyhedron by the stars of the vertices of a triangulation.

In Section 2.4 we consider the subdivisions of a simplicial complex. We describe the construction of the barycentric subdivision and prove its basic properties. Section 2.5 is dedicated to simplicial maps. It includes the definition of a simplicial map, the definition of a simplicial approximation of a continuous map and a proof of the fundamental theorem on the existence of simplicial approximations. Using this theorem we prove Sperner's lemma; this will enable us to state in the next chapter some important properties of the ball $\bar{B}^m$ and the sphere S^{m-1}. The last section, 2.6, concerns cell-complexes. We prove that the union of a finite number of cells is the underlying space of a cell complex and thereafter that every cell complex has a simplicial subdivision. The resulting characterization of polyhedra as finite unions of cells is used to prove that polyhedra form a class which is closed under unions and intersections and also under metric products.

2.1. Simplices

Let the points $a_0, a_1, \ldots, a_n$ of m-dimensional Euclidean space $\mathbf{R}^m$ be an affinely independent set. The convex hull $\operatorname{conv}\{a_0, a_1, \ldots, a_n\}$ is called the *n-dimensional simplex with vertices* $a_0, a_1, \ldots, a_n$ and is denoted $\Delta(a_0, a_1, \ldots, a_n)$. Thus the 0-dimensional simplex $\Delta(a_0)$ consists of the one point a_0, the 1-dimensional simplex $\Delta(a_0, a_1)$ is a non-degenerate line segment with endpoints a_0, a_1; the 2-dimensional simplex $\Delta(a_0, a_1, a_2)$ is a triangular disk with non-collinear vertices a_0, a_1, a_2; the 3-dimensional simplex $\Delta(a_0, a_1, a_2, a_3)$ is a tetrahedron with non-co-planar vertices a_0, a_1, a_2, a_3. Moreover it is convenient to regard the empty set $\emptyset$ as the unique (-1)-dimensional simplex.

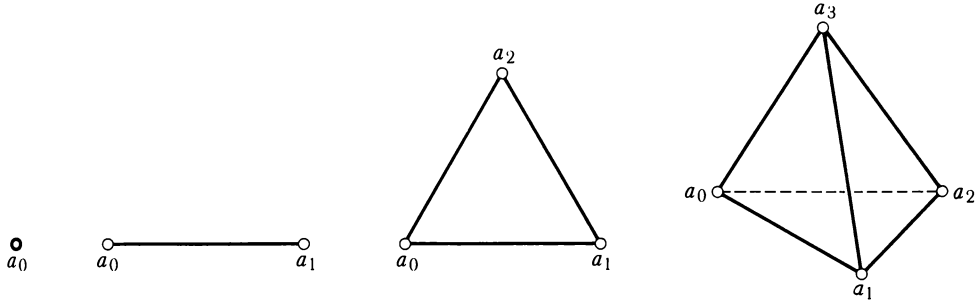


Fig.40. The n-dimensional simplex $\Delta(a_0, a_1, \ldots, a_n)$ in the cases $n = 0, 1, 2, 3$.

From Lemma 1.4.7 we have the following.

2.1.1. COROLLARY. $\operatorname{diam}\Delta(a_0, a_1, \ldots, a_n) = \operatorname{diam}\{a_0, a_1, \ldots, a_n\}$. ■

In particular we obtain the following.

2.1.2. COROLLARY. *Every simplex is bounded.* ■

Using Theorem 0.4.16 we have the following.

2.1.3. COROLLARY. *For every affinely independent set of points $a_0, a_1, \ldots, a_n \in \mathbf{R}^m$ the equation $\Delta(a_0, a_1, \ldots, a_n) = \{x \in \mathbf{R}^m : x = \sum_{j=0}^n r^j a_j$, where $r^j \geq 0$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^n r^j = 1\}$ holds.* ■

We now prove that every simplex geometrically determines its set of vertices. The following statement holds:

2.1.4. THEOREM. *The point $p \in \Delta(a_0, a_1, \ldots, a_n)$ is a vertex of the simplex $\Delta(a_0, a_1, \ldots, a_n)$ if and only if the set $\Delta(a_0, a_1, \ldots, a_n)\backslash\{p\}$ is convex.*

PROOF. Note first that the set $\Delta(a_0, a_1, \ldots, a_n)\backslash\{a_k\}$ is convex for $k = 0, 1, \ldots, n$. For, if $x = \sum_{j=0}^n r^j a_j$ and $y = \sum_{j=0}^n s^j a_j$, where $r^j, s^j \geq 0$ for $j = 0, 1, \ldots, n$, $\sum_{j=0}^n r^j = \sum_{j=0}^n s^j = 1$ and $r^k, s^k < 1$ then for every number $t \in I$ we have $(1-t)x + ty = \sum_{j=0}^n ((1-t)r^j + ts^j)a_j$, where $(1-t)r^j + ts^j \geq 0$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^n (1-t)r^j + ts^j = (1-t)\sum_{j=0}^n r^j + t\sum_{j=0}^n s^j = (1-t) + t = 1$ while $(1-t)r^k + ts^k < 1$.

Conversely, suppose $p \in \Delta(a_0, a_1, \ldots, a_n)\backslash\{a_0, a_1, \ldots, a_n\}$; then $a_0, a_1, \ldots, a_n \in \Delta(a_0, a_1, \ldots, a_n)\backslash\{p\}$. If the set $\Delta(a_0, a_1, \ldots, a_n)\backslash\{p\}$ were convex, it would follow from

the definition of a simplex that $\Delta(a_0, a_1, \ldots, a_n) \subset \Delta(a_0, a_1, \ldots, a_n) \backslash \{p\}$, contrary to hypothesis. The removal of the point p thus disturbs the convexity of the simplex $\Delta(a_0, a_1, \ldots, a_n)$. ■

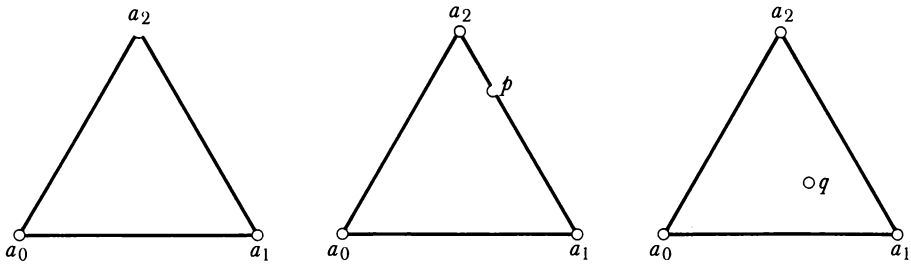


Fig.41. Removal of the vertex a_2 does not disturb the convexity of the simplex $\Delta(a_0, a_1, a_2)$; however the sets $\Delta(a_0, a_1, a_2) \backslash \{p\}$ and $\Delta(a_0, a_1, a_2) \backslash \{q\}$ are not convex (cf. Theorem 2.1.4).

In view of Theorem 2.1.4 we will often leave out the symbols for the vertices of a simplex $\Delta(a_0, a_1, \ldots, a_n)$ denoting it simply by Δ. The number of vertices of the simplex Δ less one is called its *dimension* and is denoted by $\dim \Delta$.

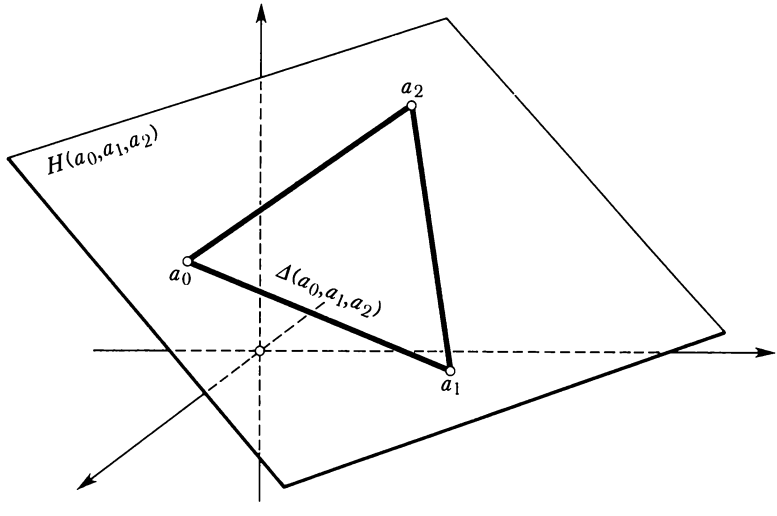


Fig.42. The carrier subspace $H(a_0, a_1, a_2)$ of the simplex $\Delta(a_0, a_1, a_2)$ in the space $\mathbf{R}^3$.

Let $H(a_0, a_1, \ldots, a_n)$ be the n-dimensional affine subspace in $\mathbf{R}^m$ uniquely determined by the set of vertices of the simplex $\Delta(a_0, a_1, \ldots, a_n)$ (see Theorem 0.4.7). Evidently, the inclusion $\Delta(a_0, a_1, \ldots, a_n) \subset H(a_0, a_1, \ldots, a_n)$ holds. The affine hull $H(a_0, a_1, \ldots, a_n)$ is said to be the *carrier subspace* of the simplex $\Delta(a_0, a_1, \ldots, a_n)$. The empty set is regarded as the carrier subspace of the (-1)-dimensional simplex. By the *barycentric coordinates* of a point p in the simplex $\Delta(a_0, a_1, \ldots, a_n)$ we shall mean its barycentric coordinates relative to the set of vertices $a_0, a_1, \ldots, a_n$ with the tacit assumption that the vertices have some fixed order.

Consider the points $e_0, e_1, \ldots, e_n$ of $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$, defined by $e_j = (\delta_j^0, \delta_j^1, \ldots, \delta_j^n)$ for $j = 0, 1, \ldots, n$. The simplex $\Delta(e_0, e_1, \ldots, e_n)$ is known

as the *unit n-dimensional simplex* and is denoted by Δ^n. Its carrier subspace has equation $\sum_{j=0}^n x^j = 1$. In this hyperplane the barycentric coordinates of a point x relative to the system of points $e_0, e_1, \ldots, e_n$ coincide with the Cartesian coordinates of the point. The map h of the subspace $H(a_0, a_1, \ldots, a_n)$ onto the subspace $H(e_0, e_1, \ldots, e_n)$ which sends the point $x = \sum_{j=0}^n r^j a_j \in H(a_0, a_1, \ldots, a_n)$, where $\sum_{j=0}^n r^j = 1$, to the point $h(x) = (r_0, r_1, \ldots, r_n) \in H(e_0, e_1, \ldots, e_n)$ is obviously an affine isomorphism. Under this isomorphism the simplex $\Delta(a_0, a_1, \ldots, a_n)$ is mapped onto the unit n-dimensional simplex Δ^n. We thus have:

2.1.5. COROLLARY. *Two simplices of the same dimension are affinely isomorphic.* ■

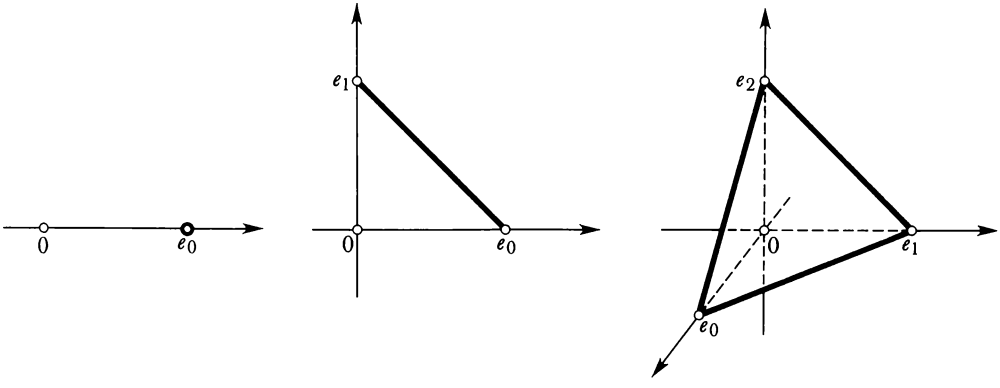


Fig.43. The n-dimensional unit simplex Δ^n in the space $\mathbf{R}^{n+1}$ for $n = 0, 1, 2$.

We now study the topology of simplices. Since the unit n-dimensional simplex consists of points $(x^0, x^1, \ldots, x^n) \in \mathbf{R}^{n+1}$ with $x^j \geq 0$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^n x^j = 1$, we have the following assertion.

2.1.6. ASSERTION. *The unit n-dimensional simplex is a closed subset of its carrier subspace. Its interior relative to this subspace is the set*

$$\{(x^0, x^1, \ldots, x^n) \in \mathbf{R}^{n+1} : x^j > 0 \text{ for } j = 0, 1, \ldots, n \text{ and } \sum_{j=0}^n x^j = 1\}. \blacksquare$$

By Corollary 1.3.27 every affine isomorphism is a homeomorphism, hence we obtain:

2.1.7. COROLLARY. *Every simplex is a closed subset of its carrier subspace. The interior of the simplex $\Delta(a_0, a_1, \ldots, a_n)$ relative to its carrier subspace is the set $\{x \in \mathbf{R}^m : x = \sum_{j=0}^n r^j a_j$, where $r^j > 0$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^n r^j = 1\}$.* ■

The first part of Corollary 2.1.7 and Corollary 2.1.2 imply by Theorem 1.10.2 the following.

2.1.8. COROLLARY. *Every simplex is compact.* ■

The interior of the simplex Δ relative to its carrier subspace is called its *geometric interior* or simply its *interior*[4] and is denoted by $\operatorname{int} \Delta$. Since, in the case of simplices,

[4] In English this is more often called the *relative interior* and denoted relint Δ.

we shall in general use the notion of interior only in the geometric sense, this use of the symbol will not lead to any misunderstanding. The boundary of the simplex Δ relative to its carrier subspace will likewise be referred to simply as the *boundary* and will be denoted by $\operatorname{bd}\Delta$. It follows from the second part of Corollary 2.1.7 that every simplex of non-negative dimension has non-empty geometric interior. For instance, the point $\frac{1}{n+1}\sum_{j=0}^{n} a_j$, known as the *barycentre* of the simplex $\Delta(a_0, a_1, \ldots, a_n)$, belongs to its geometric interior. In view of Corollary 2.1.8 we thus obtain by Theorem 1.10.9 the following.

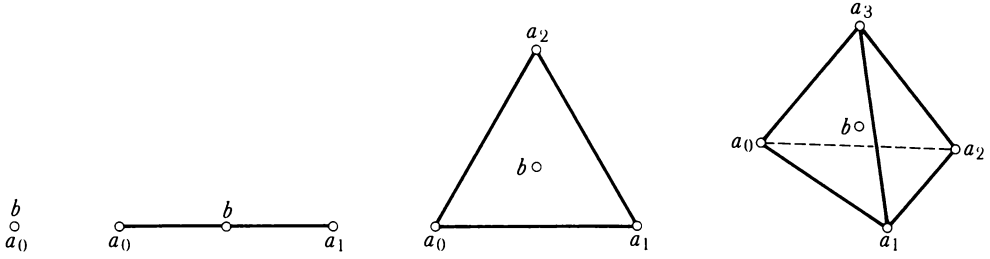


Fig.44. The barycentre $b = \frac{1}{n+1}\sum_{j=0}^{n} a_j$ of the simplex $\Delta(a_0, a_1, \ldots, a_n)$ in the cases $n = 0, 1, 2, 3$.

2.1.9. COROLLARY. *Let $n \geq 0$. Every n-dimensional simplex is homeomorphic to the n-dimensional closed unit ball. The geometric interior of the n-dimensional simplex is homeomorphic to the n-dimensional open unit ball. The boundary of the n-dimensional simplex is homeomorphic to the $(n-1)$-dimensional unit sphere.* ■

For any subset $\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}$ of an affinely independent set of points $\{a_0, a_1, \ldots, a_n\}$ in Euclidean space $\mathbf{R}^m$ the simplex $\Delta(a_{i_0}, a_{i_1}, \ldots, a_{i_k})$ is known as a *k-dimensional face* of the simplex $\Delta(a_0, a_1, \ldots, a_n)$. We also agree to regard the (-1)-dimensional simplex, viz the empty set, as a face of any simplex. Of course every simplex contains all of its faces.

The vertices of a simplex are just its 0-dimensional faces. The n-dimensional simplex itself is its only n-dimensional face. The $(n-1)$-dimensional faces of an n-dimensional simplex are known as *facets*. The 1-dimensional faces of a simplex are also known as *edges*.

If a simplex Δ' is a face of the simplex Δ'', then we write $\Delta' \leq \Delta''$; if moreover $\Delta' \neq \Delta''$ we say that Δ' is a *proper face* and write $\Delta' < \Delta''$.

We now prove the following.

2.1.10. THEOREM. *The boundary of an n-dimensional simplex, for $n \geq 0$, is the union of its facets.*

PROOF. The point x belongs to the boundary of the simplex $\Delta(a_0, a_1, \ldots, a_n)$ if and only if it belongs to the simplex but does not belong to its interior. Hence, according to Corollary 2.1.7, this occurs when $x = \sum_{j=0}^{n} r^j a_j$, where $r^j \geq 0$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^{n} r^j = 1$ and moreover there is an index $0 \leq k \leq n$ such that $r^k = 0$. This is the same as saying $x \in \Delta(a_0, a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$. ■

From this we obtain the following corollaries.

2.1.11. COROLLARY. *The boundary of a simplex is the union of its proper faces.* ■

2.1.12. COROLLARY. *Every simplex is the disjoint union of the geometric interiors of all of its faces.* ■

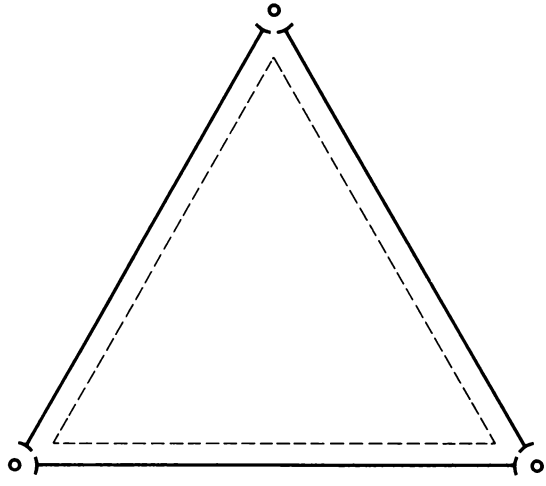


Fig.45. Every simplex is the disjoint union of the geometric interiors of all of its faces (Corollary 2.1.12).

Exercises

a) Show that if a set $\Delta' \subset \mathbf{R}^m$ is affinely isomorphic to a simplex $\Delta \subset \mathbf{R}^n$, then Δ' is also a simplex.

b) Determine the number of k-dimensional faces of an n-dimensional simplex.

c) Show that the removal of any number of faces from a simplex does not disturb its convexity.

d) Prove that the intersection of all of the k-dimensional faces of an n-dimensional simplex, for $k < n$, is empty.

e) Give conditions which must be satisfied by the integers k and m so that an n-dimensional simplex Δ necessarily contains faces Δ' and Δ'' such that $\dim \Delta' = k$, $\dim'' = m$ and $\Delta' \cap \Delta'' = \emptyset$.

2.2. Simplicial complexes

A finite family K of simplices lying in the Euclidean space $\mathbf{R}^m$ is called a *simplicial complex* or simply a *complex* if the following conditions are met:

(SC1) *the family K contains all the faces of each simplex in the family,*

(SC2) *the intersection of any pair of simplices of K is a common face.*

It follows from condition (SC1) that every non-empty simplicial complex contains the empty set, that is, the (-1)-dimensional simplex. The vertices of the simplices

making up the simplicial complex are briefly referred to as the *vertices* of the complex K. If K is a non-empty simplicial complex, the number $\dim K = \max\{\dim \Delta : \Delta \in K\}$ is called its *dimension* and the number $\operatorname{diam} K = \max\{\operatorname{diam} \Delta : \Delta \in K\}$ is called its *diameter*. Simplicial complexes of dimension not exceeding 1 are called *graphs*. (See also Supplements 2.S.1 and 2.S.4.)

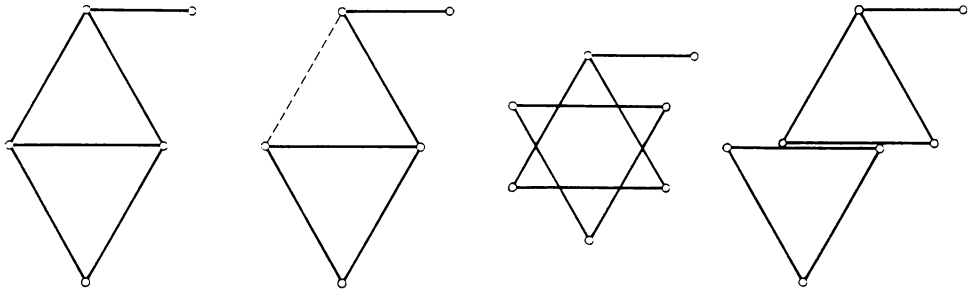


Fig.46. Of the four families of simplices represented in the figure only the first is a simplical complex; the second disobeys (SC1); the third and fourth disobey (SC2).

2.2.1. EXAMPLE. Let K_Δ be the family of all the faces of some fixed k-dimensional simplex Δ; then K_Δ is a k-dimensional simplicial complex. Indeed, property (SC1) follows from the observation that the relation $\leq$ is transitive. In order to check property (SC2) it is enough to observe on the basis of Corollary 2.1.3, that if $\Delta = \Delta(a_0, a_1, \ldots, a_n)$, $\Delta_1 = \Delta(a_{i_0}, a_{i_1}, \ldots, a_{i_q})$ and $\Delta_2 = \Delta(a_{j_0}, a_{j_1}, \ldots, a_{j_r})$, then $\Delta_1 \cap \Delta_2 = \Delta(a_{h_0}, a_{h_1}, \ldots, a_{h_p})$, where $\{h_0, h_1, \ldots, h_p\} = \{i_0, i_1, \ldots, i_q\} \cap \{j_0, j_1, \ldots, j_r\}$. ■

We now prove a theorem from which follows the equivalence of an alternative definition of simplicial complex.

2.2.2. THEOREM. *In order for a finite family K of simplices in the space $\mathbf{R}^m$ to be a simplicial complex it is necessary and sufficient that condition* (SC1) *is satisfied and also the condition:*
(SC2′) *the geometric interiors of distinct simplices in the family K are disjoint.*

PROOF. Assume that, as in condition (SC2), the intersection $\Delta_0 = \Delta_1 \cap \Delta_2$ of two simplices $\Delta_1, \Delta_2 \in K$ satisfies the relations $\Delta_0 \leq \Delta_1$ and $\Delta_0 \leq \Delta_2$. If $\Delta_1 \neq \Delta_2$, then either $\Delta_0 < \Delta_1$ or $\Delta_0 < \Delta_2$; let us suppose that $\Delta_0 < \Delta_1$. It follows from Corollary 2.1.11 that $\Delta_0 \cap \operatorname{int} \Delta_1 = \emptyset$. But $\operatorname{int} \Delta_1 \cap \operatorname{int} \Delta_2 \subset \Delta_0 \cap \operatorname{int} \Delta_1$, so condition (SC2′) is met.

Conversely, suppose that condition (SC2′) holds. If $\Delta_1 \cap \Delta_2 = \emptyset$ then (SC2) obviously holds. If $p \in \Delta_1 \cap \Delta_2$ then by Corollary 2.1.2 there exists a simplex $\Delta_1' \leq \Delta_1$ and a simplex $\Delta_2' \leq \Delta_2$ such that $p \in \operatorname{int} \Delta_1' \cap \operatorname{int} \Delta_2'$; hence $\Delta_1' = \Delta_2'$. It follows that every point in the intersection $\Delta_1 \cap \Delta_2$ belongs to a common face of the simplices Δ_1 and Δ_2. The intersection $\Delta_1 \cap \Delta_2$ is thus the union of the common faces of the simplices Δ_1 and Δ_2. Let $a_0, a_1, \ldots, a_n$ be all the common vertices of the simplices Δ_1 and Δ_2. Then $\Delta(a_0, a_1, \ldots, a_n)$ is the common face of the simplices Δ_1 and Δ_2 which contains all the other common faces. It follows that $\Delta_1 \cap \Delta_2 = \Delta(a_0, a_1, \ldots, a_n)$. ■

A subset K_0 of a simplicial complex K is said to be a *simplicial subcomplex* (or briefly a *subcomplex*) if K_0 contains all of the faces of every simplex in K_0. Thus every subcomplex is in its own right a simplicial complex.

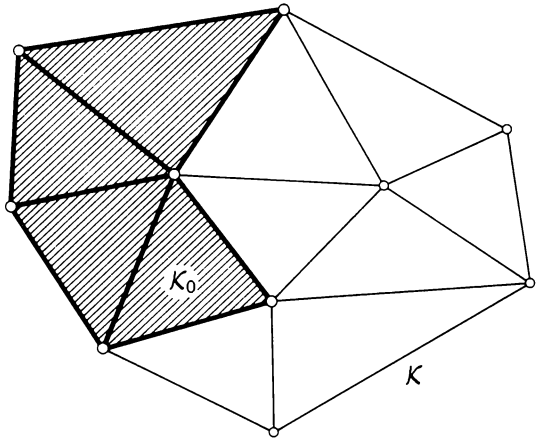


Fig.47. A subcomplex K_0 of the simplicial complex K.

2.2.3. EXAMPLE. *The n-dimensional skeleton of a simplicial complex.* Let K be a simplicial complex and n an integer. The family $K^{[n]} = \{\Delta \in K : \dim \Delta \leq n\}$ is called the *n-dimensional skeleton* of the complex K. Since the dimension of a face of a simplex does not exceed the dimension of the simplex, we see that the skeleton $K^{[n-1]}$ is a subcomplex of the skeleton $K^{[n]}$. Let us note that $K^{[-1]} = \{\emptyset\}$ and $K^{[k]} = K$, where $k = \dim K$. ■

2.2.4. EXAMPLE. *Nerve of a covering.* Let $\mathcal{A} = \{A_i\}_{i=0}^k$ be a finite covering of the set X; that is, $X = \bigcup_{i=0}^k A_i$. We say that the simplicial complex $\mathcal{N}$ is the *nerve* of the covering $\mathcal{A}$ if its vertices can be arranged in a finite sequence $a_0, a_1, \ldots, a_k$ such that $\Delta(a_{i_0}, a_{i_1}, \ldots, a_{i_n}) \in \mathcal{N}$ if and only if $A_{i_0} \cap A_{i_1} \cap \ldots A_{i_n} \neq \emptyset$. When speaking of the nerve of a covering we shall tacitly assume that its vertices are ordered in some way.

Let us observe that every finite covering has a nerve. For, let $\mathcal{A} = \{A_i\}_{i=0}^k$ be a covering of X and let $\Delta = \Delta(a_0, a_1, \ldots, a_k)$ be any k-dimensional simplex. We define a subcomplex $\mathcal{N}$ of the simplicial complex K_Δ considered in Example 2.2.1. Let $\mathcal{N} = \{\Delta(a_{i_0}, a_{i_1}, \ldots, a_{i_n}) \in K_\Delta : A_{i_0} \cap A_{i_1} \cap \ldots A_{i_n} \neq \emptyset\}$. It is easy to see that $\mathcal{N}$ is a nerve of the covering $\mathcal{A}$. We note that a nerve is not uniquely determined by a covering. (See also the Supplement 2.S.7). ■

An immediate consequence of the definition of a simplicial subcomplex is the following.

2.2.5. ASSERTION. *If K_1 and K_2 are simplicial subcomplexes of a simplicial complex K, then the union $K_1 \cup K_2$ and the intersection $K_1 \cap K_2$ are also simplicial subcomplexes of the complex K.* ■

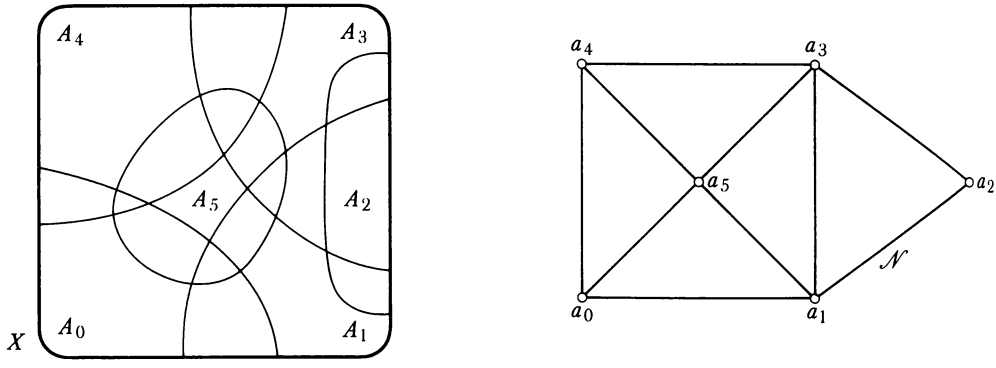


Fig.48. A covering $\mathcal{A} = \{A_i\}_{i=0}^5$ of a space X and its nerve $\mathcal{N}$.

Exercises

a) Let K be a simplicial complex and let $\Delta_0 \in K$. Show that the set $K \setminus \{\Delta \in K : \Delta_0 \leq \Delta\}$ is a subcomplex of the complex K. Is the set $\{\Delta \in K : \Delta_0 \leq \Delta\}$ a subcomplex of the complex K?

b) Let K be a simplicial complex and let $\Delta \in K$. Show that if $\dim \Delta = \dim K$, then $K \setminus \{\Delta\}$ is a simplicial subcomplex of the complex K. Is the given condition also necessary?

c) Show that if K_0 is a simplicial subcomplex of a complex K, then for every integer n the skeleton $K_0^{[n]}$ is a simplicial subcomplex of the skeleton $K^{[n]}$.

d) Let $X = \{2, 3, \ldots, 10\}$ and let A_i be the set of numbers in X which are divisible by i, where $i = 2, 3, \ldots, 7$. Determine a nerve for the covering of X by the sets $A_2, A_3, \ldots, A_7$.

2.3. Polyhedra

Let K be a simplicial complex. The set $\cup\{\Delta : \Delta \in K\}$ is called the *underlying space* of the complex K and is denoted by $|K|$. It follows from Corollary 2.1.12 that we also have $|K| = \cup\{\operatorname{int} \Delta : \Delta \in K\}$. Now by Theorem 2.2.2 the geometric interiors of distinct simplices of K are disjoint, so for every point $p \in |K|$ there is precisely one simplex $\Delta \in K$ such that $p \in \operatorname{int} \Delta$; this simplex is known as the *carrier* of the point p in the complex K.

Any subset X of Euclidean space $\mathbf{R}^m$ for which there exists a simplicial complex K with $|K| = X$ is called a *polyhedron*. We then say that the complex K is a *triangulation* of the polyhedron X. Evidently a polyhedron may have several triangulations. (See also the Supplement 2.S.2).

2.3.1. EXAMPLE. *The simplex.* Every simplex Δ is a polyhedron. For, by Example 2.2.1, the family of all of the faces of the simplex Δ constitutes a triangulation. ■

2.3.2. EXAMPLE. *The boundary of a simplex.* Let Δ be an n-dimensional simplex. By Corollary 2.1.11 the boundary $\operatorname{bd}\Delta$ is the union of all the proper faces of the simplex Δ. According to Example 2.2.3 the $(n-1)$-dimensional skeleton of the complex described in Example 2.2.1 constitutes a triangulation of the boundary $\operatorname{bd}\Delta$. ■

2.3.3. EXAMPLE. *Finite set.* Every finite set $X \subset \mathbf{R}^m$ is a polyhedron. The individual points of X treated as 0-dimensional simplices form a triangulation. ■

From Corollary 2.1.8 and in view of Lemma 1.4.9 and Theorems 1.6.19 and 1.10.2 we obtain the following.

2.3.4. COROLLARY. *Every polyhedron is a compact space.* ■

A triangulation of a polyhedron determines whether it is connected. We say that a simplicial complex K is *connected* if it cannot be expressed as the union of two subcomplexes K_1 and K_2 where $\dim K_1 \geq 0$, $\dim K_2 \geq 0$ and $K_1 \cap K_2 = \{\emptyset\}$. The following is the case.

2.3.5. LEMMA. *A simplicial complex is connected if and only if for every pair of vertices $a, b \in K$ there is a sequence of vertices $a_0, a_1, \ldots, a_k \in K$ such that $a_0 = a$, $a_k = b$ and $\Delta(a_{j-1}, a_j) \in K$ for $j = 1, 2, \ldots, k$.*

PROOF. Let a be a vertex of a connected complex K; denote by K_1^0 the set of vertices b of the complex K for which there exists a sequence of vertices $a_0, a_1, \ldots, a_k \in K$ such that $a_0 = a$, $a_k = b$ and $\Delta(a_{j-1}, a_j) \in K$ for $j = 1, 2, \ldots, k$. Let K_2^0 denote the vertices of the complex K which do not belong to K_1^0. Evidently every simplex of the complex K has all of its vertices either entirely in K_1^0 or entirely in K_2^0; the two sets are thus the vertex sets of two subcomplexes K_1 and K_2 of the complex K which satisfy $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \{\emptyset\}$. Since $a \in K_1$ and the complex K is connected it follows that K_2^0 has no vertices.

Conversely, suppose that the complex K is not connected and $K = K_1 \cup K_2$ where $\dim K_1 \geq 0$, $\dim K_2 \geq 0$ and $K_1 \cap K_2 = \{\emptyset\}$. Suppose that $a_0, a_1, \ldots, a_k \in K$ is a sequence of vertices such that $a_0 \in K_1$ and $a_k \in K_2$. Let j be the least natural number such that $a_j \in K_2$. Then $a_{j-1} \in K_1$ and if we supposed that $\Delta(a_{j-1}, a_j) \in K$, then, since K is a union of the subcomplexes K_1 and K_2, we would have either $\Delta(a_{j-1}, a_j) \in K_1$ or $\Delta(a_{j-1}, a_j) \in K_2$, so $a_{j-1} \in K_1 \cap K_2$ or $a_j \in K_1 \cap K_2$ contrary to the hypothesis that $K_1 \cap K_2 = \{\emptyset\}$. ■

We may now give a characterization of connected polyhedra in terms of triangulations.

2.3.6. THEOREM. *The polyhedron $|K|$ is connected if and only if the complex K is connected.*

PROOF. If the simplicial complex K is connected then by Lemma 2.3.5 for each pair of vertices $a, b \in K$ there is a broken line in $|K|$ joining a to b. Since every simplex is convex, it will also be the case that for any pair of points $a, b \in |K|$ there is a broken line in $|K|$ joining a to b. By Corollary 1.10.6 the polyhedron $|K|$ is connected.

Conversely, if the simplicial complex is not connected and $K = K_1 \cup K_2$, where the subcomplexes K_1, K_2 satisfy $\dim K_1 \geq 0$, $\dim K_2 \geq 0$ and $K_1 \cap K_2 = \{\emptyset\}$, then $|K| = |K_1| \cup |K_2|$ where $|K_1| \neq \emptyset \neq |K_2|$ and $|K_1| \cap |K_2| = \emptyset$. Moreover, by Corollary 2.3.4 and Theorem 1.8.4 the sets $|K_1|$ and $|K_2|$ are closed in $|K|$; hence the polyhedron $|K|$ is not connected. ■

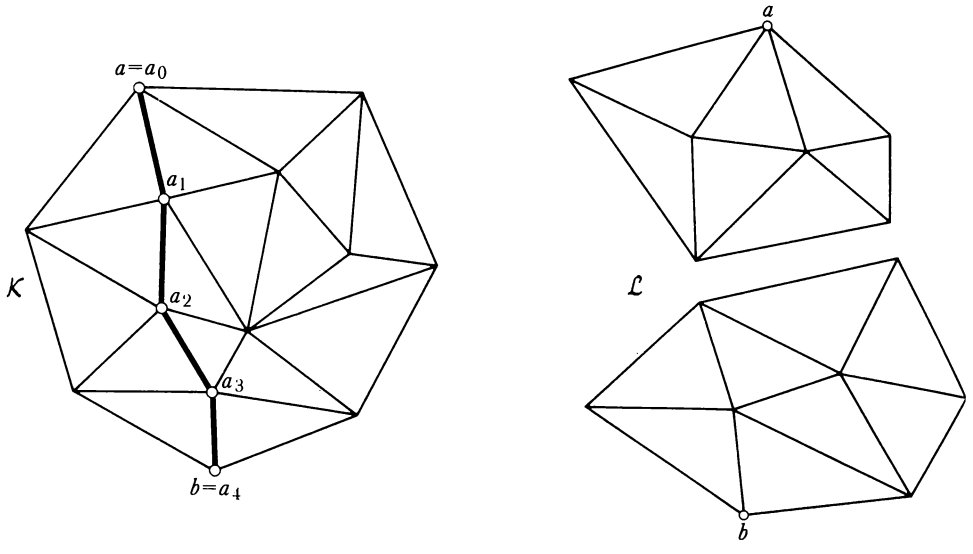


Fig.49. The complex K is connected, since for every pair of vertices $a, b \in K$ there is a sequence of vertices $a_0, a_1, \ldots, a_k \in K$ such that $a_0 = a$, $a_k = b$ and $\Delta(a_{j-1}, a_j) \in K$ for $j = 1, 2, \ldots, k$. The complex $\mathcal{L}$ does not have this property, so is not connected (Lemma 2.3.5). From Theorem 2.3.6 it follows that the polyhedron $|K|$ is connected and $|\mathcal{L}|$ is not.

We now prove the following.

2.3.7. THEOREM. *If the underlying spaces of the simplicial complexes K' and K'' satisfy $|K'| \subset |K''|$, then* $\dim K' \leq \dim K''$.

PROOF. Assume for the sake of argument that $\dim K'' = k'' < k' = \dim K'$. Let $\Delta' \in K'$ with $\dim \Delta' = k'$. By Example 1.6.15 it follows that for every simplex $\Delta'' \in K''$ the intersection $\Delta' \cap \Delta''$ has empty interior in Δ'. Hence the difference $\Delta' \setminus |K''|$ is non-empty which completes the proof. ■

From the theorem above we obtain the following.

2.3.8. COROLLARY. *If $|K'| = |K''|$, then* $\dim K' = \dim K''$. ■

Let X be a polyhedron with triangulation K. The number $\dim K$, which in view of Corollary 2.3.8 depends only on the polyhedron, is known as its *geometric dimension* or simply its *dimension* and is denoted by $\dim X$. From Theorem 2.3.7 we obtain the following.

2.3.9. COROLLARY. *If the polyhedra X' and X'' satisfy the inclusion $X' \subset X''$, then* $\dim X' \leq \dim X''$. ■

Returning to Examples 2.3.1–2.3.3 we obtain

2.3.10. EXAMPLE. If Δ is an n-dimensional simplex then $\dim \Delta = n$ and $\dim \operatorname{bd} \Delta = n - 1$. If X is a non-empty finite set, then $\dim X = 0$. ■

Let $a_0, a_1, \ldots, a_k$ be all the vertices of the simplicial complex K and let the simplex $\Delta(a_{j_0}, a_{j_1}, \ldots, a_{j_n})$ be the carrier of $p \in |K|$. The numbers $r^0, r^1, \ldots, r^k$ where $r^j = 0$ for $j \neq j_0, j_1, \ldots, j_n$ and $r^{j_0}, r^{j_1}, \ldots, r^{j_n}$ are the barycentric coordinates of the point p in the simplex $\Delta(a_{j_0}, a_{j_1}, \ldots, a_{j_n})$ are called the *barycentric coordinates* of the point p of the polyhedron $|K|$ relative to the triangulation K. We may then write $p = \sum_{j=0}^{n} r^j a_j$ where of course $r^j \geq 0$ for $j = 0, 1, \ldots, k$ and $\sum_j^k r^j = 1$. Observe however that not every system of non-negative numbers $r^0, r^1, \ldots, r^k$ satisfying $\sum_{j=0}^{k} r^j = 1$ gives a point $\sum_{j=0}^{k} r^j a_j$ lying in $|K|$.

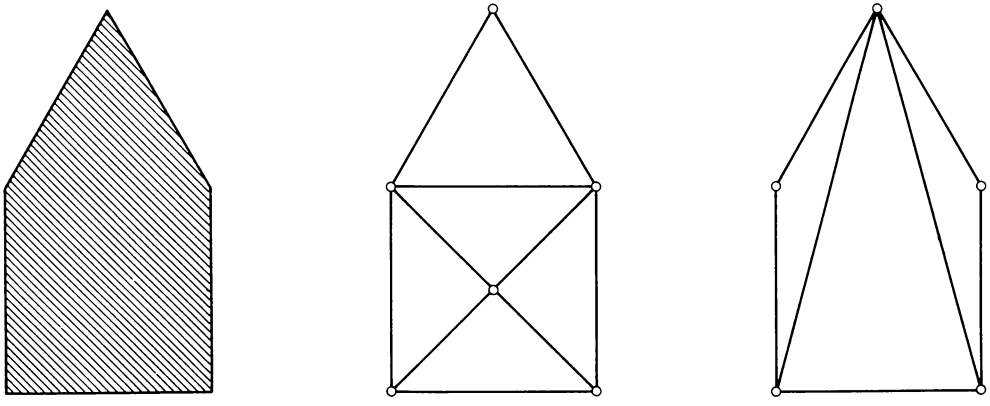


Fig.50. A polyhedron and two different triangulations of it.
As stated in Corollary 2.3.8 both have the same dimension 2.

Let $\bar{K}$ denote the simplicial complex formed from those faces $\Delta(e_{j_0}, e_{j_1}, \ldots, e_{j_n})$ of the unit simplex Δ^k for which $\Delta(a_{j_0}, a_{j_1}, \ldots, a_{j_n}) \in K$. The following is then the case.

2.3.11. THEOREM. *The map which sends each point of the polyhedron $|K|$ to the system of barycentric coordinates relative to the triangulation K is a homeomorphism of the polyhedron $|K|$ onto the polyhedron $|\bar{K}|$.*

PROOF. Let $h(p) = (r^0, r^1, \ldots, r^k) \in |\bar{K}|$ for $p = \sum_{j=0}^{k} r^j a_j \in |K|$. Then h is injective since the indices of the positive barycentric coordinates of a point p uniquely determine the carrier of p and in each simplex of K the map just defined is an affine isomorphism onto the unit simplex of appropriate dimension (compare the remarks preceding Corollary 2.1.5). We thus have $h^{-1}\colon |\bar{K}| \to |K|$, where $h^{-1}(r^0, r^1, \ldots, r^k) = \sum_{j=0}^{k} r^j a_j$ for $(r^0, r^1, \ldots, r^k) \in |\bar{K}|$. In view of the continuity of linear transformations acting on points of Euclidean space (Examples 1.3.8 and 1.3.12) and by the compactness of the polyhedron $|\bar{K}|$ we deduce from Theorem 1.8.15 that the map h^{-1} and hence also h is a homeomorphism. ■

For every vertex a of the simplicial complex K the set $\operatorname{st} a = \cup\{\operatorname{int} \Delta : a \text{ is a vertex of the simplex } \Delta \in K\}$ is known as the *star* of the vertex a in the complex K (see also

Supplement 2.S.5). Since every point of the polyhedron $|K|$ lies in the interior of its carrier in K, we have the following.

2.3.12. ASSERTION. *The stars of the vertices of a simplicial complex K form a covering of the polyhedron $|K|$.* ■

We now prove the following.

2.3.13. LEMMA. *Let the distinct points $a_0, a_1, \ldots, a_n$ be vertices of a simplicial complex K. Then $\Delta(a_0, a_1, \ldots, a_n) \in K$ if and only if $\operatorname{st} a_0 \cap \ldots \cap \operatorname{st} a_n \neq \emptyset$.*

PROOF. If $\Delta(a_0, a_1, \ldots, a_n) \in K$, then obviously $\emptyset \neq \operatorname{int} \Delta(a_0, a_1, \ldots, a_n) \subset \operatorname{st} a_0 \cap \operatorname{st} a_1 \cap \ldots \cap \operatorname{st} a_n$. Conversely, if $p \in \operatorname{st} a_0 \cap \operatorname{st} a_1 \cap \ldots \cap \operatorname{st} a_n$ and the simplex Δ is the carrier of p in the complex K, then each of the points $a_0, a_1, \ldots, a_n$ is a vertex of Δ; hence $\Delta(a_0, a_1, \ldots, a_n) \in K$. ■

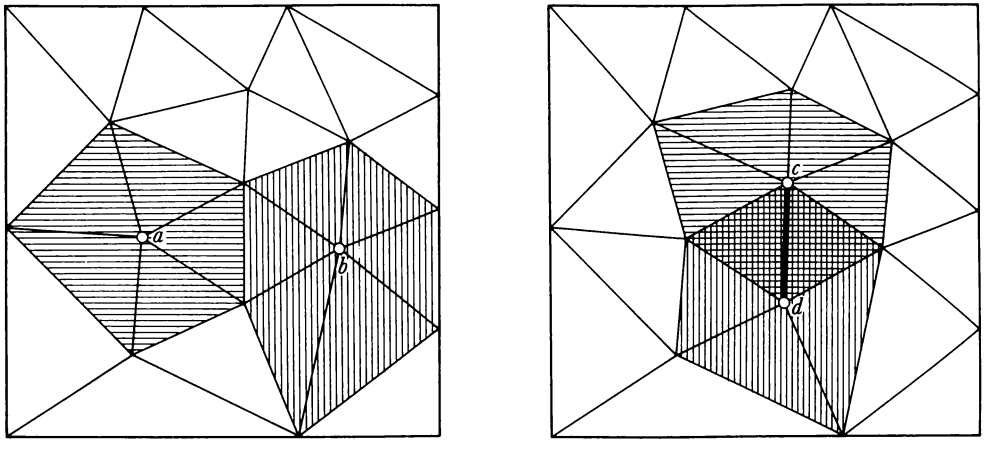


Fig.51. The stars $\operatorname{st} a$ and $\operatorname{st} b$ are disjoint since the vertices a and b do not determine a simplex; the stars $\operatorname{st} c$ and $\operatorname{st} d$ intersect since the simplex $\Delta(c, d)$ belongs to the complex.

The lemma above has the following corollary.

2.3.14. COROLLARY. *The simplicial complex K is a nerve of the covering of the polyhedron $|K|$ by of the stars of the vertices of the complex K.* ■

For each vertex a of the simplicial complex K the set $K_a = \{\Delta \in K : a \text{ is not a vertex of } \Delta\}$ is clearly a subcomplex of the simplicial complex K. In view of the obvious equality $\operatorname{st} a = |K| \setminus |K_a|$ we obtain from Corollary 2.3.4 and Theorem 1.8.4 the following.

2.3.15. ASSERTION. *The star of each vertex of a simplicial complex K is an open subset of the polyhedron $|K|$.* ■

We conclude this section with a proof of the following theorem (see Assertion 2.2.5).

2.3.16. THEOREM. *If K_0 is a subcomplex of simplicial complexes K_1 and K_2, and $|K_1| \cap |K_2| = |K_0|$, then $K_1 \cup K_2$ is a simplicial complex.*

PROOF. The union $K_1 \cup K_2$ clearly satisfies condition (SC1). To check condition (SC2) we shall use Theorem 2.2.2. Evidently it is enough to consider simplices $\Delta_1 \in K_1$ and $\Delta_2 \in K_2$. Then $\operatorname{int}\Delta_1 \cap \operatorname{int}\Delta_2 \subset |K_1| \cap |K_2| = |K_0|$. Hence there is a simplex $\Delta_0 \in K_0$ such that $\operatorname{int}\Delta_0 \cap \operatorname{int}\Delta_1 \neq \emptyset \neq \operatorname{int}\Delta_0 \cap \operatorname{int}\Delta_2$ and hence $\Delta_1 = \Delta_0 = \Delta_2$. ■

Exercises

a) Find a triangulation of the unit m-dimensional cube I^m whose vertices take the form $(x^1, x^2, \ldots, x^m)$ with $x^i = 0$ or $x^i = 1$ for $i = 1, 2, \ldots, m$.

b) Show that if K_1 and K_2 are subcomplexes of some simplicial complex, then $|K_1 \cup K_2| = |K_1| \cup |K_2|$ and $|K_1 \cap K_2| = |K_1| \cap |K_2|$ (cf. Assertion 2.2.5).

c) Let $a_0, a_1, \ldots, a_k$ be the vertices of a complex K. Show that the star $\operatorname{st} a_j$ for $j = 0, 1, \ldots, k$ is the set of points of the polyhedron $|K|$ whose j^{th}-barycentric coordinate relative to K is positive.

d) Prove that in order for a simplicial complex K to be disconnected it is necessary and sufficient that K contains a subcomplex K_0 such that $\dim K_0 \geq 0$, $K_0 \neq K$ and $\operatorname{st} a \subset |K_0|$ for each vertex $a \in K_0$.

2.4. Subdivisions

Let K' and K be simplicial complexes. We say that the complex K' is a *subdivision* of the complex K if

(1) $|K'| = |K|$,

(2) for every simplex $\Delta' \in K'$ there is a simplex $\Delta \in K$ such that $\Delta' \subset \Delta$.

We will now describe the construction of a subdivision which plays an important role in the theory of polyhedra. We begin with a proof of the following lemma.

2.4.1. LEMMA. *Let the points $a_0, a_1, \ldots, a_n \in \mathbf{R}^n$ form an affinely independent set and let b_j be the barycentre of the simplex $\Delta(a_0, a_1, \ldots, a_j)$ for $j = 0, 1, \ldots, n$. Then the points $b_0, b_1, \ldots, b_n$ form an affinely independent set.*

PROOF. Suppose $\sum_{j=0}^n s^j b_j = 0$, where $\sum_{j=0}^n s^j = 0$. Then $\sum_{j=0}^n \frac{s^j}{j+1} \sum_{h=0}^j a_h = 0$, and taking $r^h = \sum_{j=h}^n \frac{s^j}{j+1}$ for $h = 0, 1, \ldots, n$ we have $\sum_{h=0}^n r^h a_h = 0$, where $\sum_{h=0}^n r^h = \sum_{h=0}^n \sum_{j=h}^n \frac{s^j}{j+1} = \sum_{j=0}^n s^j = 0$. It follows from the affine independence of the set $\{a_0, a_1, \ldots, a_n\}$ that $r^h = 0$ for $h = 0, 1, \ldots, n$, whence $s^j = 0$ for $j = 0, 1, \ldots, n$. ■

We remark that the following is obvious.

2.4.2. ASSERTION. *If $\emptyset \neq \Delta_0 < \Delta_1 < \ldots < \Delta_q$, then there exists a rearrangement $a_0, a_1, \ldots, a_n$ of the vertices of the simplex Δ_q such that the simplices $\Delta_0, \Delta_1, \ldots, \Delta_q$ form a subsequence of the sequence $\Delta(a_0), \Delta(a_0, a_1), \ldots, \Delta(a_0, a_1, \ldots, a_n)$.* ■

In view of Lemma 2.4.1 we draw the following conclusion from the assertion.

2.4.3. COROLLARY. *Suppose $\emptyset \neq \Delta_0 < \Delta_1 < \ldots < \Delta_q$ and let b_h, for $h = 0, 1, \ldots, q$, be the barycentre of the simplex Δ_h. Then the points $b_0, b_1, \ldots, b_q$ form an affinely independent set.* ■

We will now prove a theorem which will help us to define the desired subdivision.

2.4.4. THEOREM. *The family K' of all simplices of the form $\Delta(b_0, b_1, \ldots, b_q)$, where b_h is the barycentre of the simplex $\Delta_h \in K$ for $h = 0, 1, \ldots, q$ and $\emptyset \neq \Delta_0 < \Delta_1 < \ldots < \Delta_q$, is a subdivision of the complex K.*

PROOF. It follows from Assertion 2.4.2 that $\Delta' \in K'$ if and only if there is an arrangement $a_0, a_1, \ldots, a_n$ of some of the vertices of the complex K such that $\Delta(a_0, a_1, \ldots, a_n) \in K$ and $\Delta' = \Delta(b_{j_0}, b_{j_1}, \ldots, b_{j_q})$ where $b_j = \frac{1}{j+1}\sum_{h=0}^{j} a_h$ for $j = 0, 1, \ldots, n$ and $0 \leq j_0 < j_1 < \ldots < j_q \leq n$. It follows immediately that K' satisfies condition (SC1). We now check that condition (SC2′) of Theorem 2.2.2 is met.

If $x \in \operatorname{int}\Delta(b_{j_0}, b_{j_1}, \ldots, b_{j_q})$ then $x = \sum_{j=0}^{n} s^j b_j$ with $s_j \geq 0$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^{n} s^j = 1$; moreover $s^j > 0$ if and only if $j \in \{j_0, j_1, \ldots, j_q\}$. We thus have $x = \sum_{j=0}^{n} \frac{s^j}{j+1}\sum_{h=0}^{j} a_h$ and taking $r^h = \sum_{j=h}^{n} \frac{s^j}{j+1}$ for $h = 0, 1, \ldots, n$ we obtain $x = \sum_{h=0}^{n} r^h a_h$ where $r^h \geq 0$ for $h = 0, 1, \ldots, n$ and $\sum_{h=0}^{n} r^h = \sum_{h=0}^{n}\sum_{j=h}^{n} \frac{s^j}{j+1} = \sum_{j=0}^{n} s^j = 1$. Moreover, since $r^h - r^{h+1} = \frac{s^h}{h+1}$, we have $r^{h+1} \leq r^h$ for $h = 0, 1, \ldots, n-1$, where $r^{h+1} < r^h$ if and only if $h \in \{j_0, j_1, \ldots, j_q\}$, and $0 < r^n$ if and only if $n \in \{j_0, j_1, \ldots, j_q\}$. Thus, ordering the barycentric coordinates of the point x in the simplex $\Delta(a_0, a_1, \ldots, a_n)$ by magnitude, we conclude that they determine uniquely the system of indices $\{j_0, j_1, \ldots, j_q\}$. Assuming without loss of generality that the simplex $\Delta(a_0, a_1, \ldots, a_n)$ is of minimal dimension, we deduce that $j_q = n$, so that $0 < r^n$, whence also $0 < r^h$ for $h = 0, 1, \ldots, n$; that is, $x \in \operatorname{int}\Delta(a_0, a_1, \ldots, a_n)$. Since the interiors of distinct simplices of the complex K are disjoint, we see that the family K' also satisfies condition (SC2′).

We have thus shown that the family K' is a simplicial complex. From the remark at the beginning of the proof and from the definition of a simplex it follows that this complex meets condition (2) of the definition of a subdivision. Obviously $|K'| \subset |K|$. To prove the reverse inclusion, assume that $x \in \Delta(a_0, a_1, \ldots, a_n) \in K$. Thus $x = \sum_{h=0}^{n} r^h a_h$, where $r^h \geq 0$ for $h = 0, 1, \ldots, n$ and $\sum_{h=0}^{n} r^h = 1$. Applying an appropriate permutation to the vertices we may suppose that $r^{h+1} \leq r^h$ for $h = 0, 1, \ldots, n-1$. Put $s^j = (j+1)(r^j - r^{j+1})$ for $j = 0, 1, \ldots, n-1$ and $s^n = (n+1)r^n$. Then $s^j \geq 0$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^{n} s^j = \sum_{h=0}^{n} r^h = 1$; moreover

$$x = \sum_{h=0}^{n} r^h a_h = \sum_{j=0}^{n-1}(r^j - r^{j+1})\sum_{h=0}^{j} a_h + r^n \sum_{h=0}^{n} a_h = \sum_{j=0}^{n} s^j b_j,$$

so $x \in \Delta(b_0, b_1, \ldots, b_n)$ where $b_j = \frac{1}{j+1}\sum_{h=0}^{j} a_h$ for $j = 0, 1, \ldots, n$. ■

The subdivision K' defined in Theorem 2.4.4 is called the *barycentric subdivision* of the complex K (see also Supplement 2.S.6). From the definition of a subdivision and from Corollary 2.3.8 we conclude as obvious the following.

2.4.5. COROLLARY. *If K' is the barycentric subdivision of K, then* $\dim K' = \dim K$. ■

We will now determine an upper bound for the diameter of the barycentric subdivision. We begin with the following lemma which supplements Lemma 2.4.1.

2.4.6. LEMMA. *Suppose that the points $a_0, a_1, \ldots, a_n$ of $\mathbf{R}^m$ form an affinely independent set and let b_j be the barycentre of the simplex $\Delta(a_0, a_1, \ldots, a_j)$ for $j = 0, 1, \ldots, n$. Then* $\operatorname{diam} \Delta(b_0, b_1, \ldots, b_n) \leq \frac{n}{n+1} \operatorname{diam} \Delta(a_0, a_1, \ldots, a_n)$.

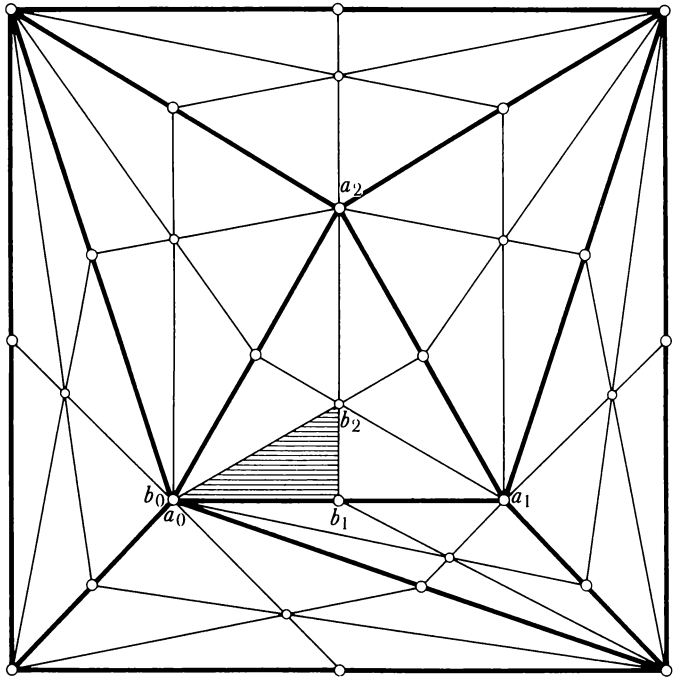


Fig.52. The barycentric subdivision K' of a simplicial complex K. The simplex $\Delta(b_0, b_1, b_2)$ belongs to K' as b_0 is the barycentre of the simplex $\Delta_0 = \Delta(a_0)$, b_1 is the barycentre of $\Delta_1 = \Delta(a_0, a_1)$ and b_2 is the barycentre of $\Delta_2 = \Delta(a_0, a_1, a_2)$. Moreover $\Delta_0 < \Delta_1 < \Delta_2$.

PROOF. For the purpose of bounding the diameter of the simplex $\Delta(b_0, b_1, \ldots, b_n)$ we suppose that $0 \leq j \leq k \leq n$ and consider the difference

$$\begin{aligned} b_j - b_k &= \sum_{h=0}^{j} \frac{1}{j+1} a_h - \sum_{h=0}^{k} \frac{1}{k+1} a_h \\ &= \left(\frac{1}{j+1} - \frac{1}{k+1}\right) \sum_{h=0}^{j} a_h - \frac{1}{k+1} \sum_{h=j+1}^{k} a_h \\ &= \frac{k-j}{k+1} \left(\frac{1}{j+1} \sum_{h=0}^{j} a_h - \frac{1}{k-j} \sum_{h=j+1}^{k} a_h \right). \end{aligned}$$

Now $\frac{1}{j+1} \sum_{h=0}^{j} a_h \in \Delta(a_0, a_1, \ldots, a_j)$ and $\frac{1}{k-j} \sum_{h=j+1}^{k} a_h \in \Delta(a_{j+1}, a_{j+2}, \ldots, a_k)$, so $|b_j - b_k| \leq \frac{k-j}{k+1} \operatorname{diam} \Delta(a_0, a_1, \ldots, a_n)$. But $\frac{k-j}{k+1} \leq \frac{k}{k+1} \leq \frac{n}{n+1}$, so using Corollary 2.1.1

we infer that

$$\operatorname{diam}\Delta(b_0, b_1, \ldots, b_n) \leq \frac{n}{n+1} \operatorname{diam}\Delta(a_0, a_1, \ldots, a_n). \blacksquare$$

In view of Assertion 2.4.2 the lemma above implies the following.

2.4.7. COROLLARY. *If K' is the barycentric subdivision of an n-dimensional simplicial complex K, then* $\operatorname{diam} K' \leq \frac{n}{n+1} \operatorname{diam} K$. ■

Let K be a simplicial complex and p a non-negative integer. We define $K^{(p)}$, the *barycentric subdivision of order p* of the complex K, inductively: $K^{(0)} = K$, $K^{(p+1)} = (K^{(p)})'$ for $p = 0, 1, \ldots$ From Corollaries 2.4.5 and 2.4.7 we deduce the following.

2.4.8. COROLLARY. *If $K^{(p)}$ is the barycentric subdivision of order p of an n-dimensional simplicial complex K, then* $\operatorname{diam} K^{(p)} \leq \left(\frac{n}{n+1}\right)^p \operatorname{diam} K$. ■

Finally, since $\lim_p \left(\frac{n}{n+1}\right)^p = 0$ for $n = 0, 1, \ldots$, we thus obtain the following.

2.4.9. COROLLARY. *If $K^{(p)}$ is the barycentric subdivision of order p of a simplicial complex K, then* $\lim_p \operatorname{diam} K^{(p)} = 0$. ■

Exercises

a) Show that if K_0 is a subcomplex of a simplicial complex K, then the barycentric subdivision K_0' is a subcomplex of the barycentric subdivision K'.

b) Show that if the subcomplexes K_1 and K_2 of the simplicial complex K satisfy $K_1 \cup K_2 = K$, then their barycentric subdivisions satisfy $K_1' \cup K_2' = K'$.

c) Investigate whether the bound given in Lemma 2.4.6 can be improved.

d) Suppose the simplicial complex K is the nerve of a covering $\mathcal{A}$ of a space X. Does there always exist a covering $\mathcal{A}'$ of the space X whose nerve is the barycentric subdivision K' of the complex K?

2.5. Simplicial maps

In defining an n-dimensional simplex we assumed that its vertices formed an affinely independent set, so that they were distinct. When listing the elements of the vertex set we tacitly used the generally accepted convention that every element of the set appears exactly once in the list; that is, the points $a_0, a_1, \ldots, a_n$ appearing in the symbol $\Delta(a_0, a_1, \ldots, a_n)$ are pairwise distinct.

In certain situations, however, listings of the vertex set of a simplex occur in which the same element may appear more than once. A simplex with an affinely independent vertex set $\{a_0, a_1, \ldots, a_n\}$, where the listing $\{a_0, a_1, \ldots, a_n\}$ allows repetitions, will be denoted by $\Delta\{a_0, a_1, \ldots, a_n\}$. The form of the symbol $\Delta\{a_0, a_1, \ldots, a_n\}$ will no longer in general indicate the dimension of the simplex; it may first be necessary to delete from the listing superfluous occurrences of points.

Let K and $\mathcal{L}$ be simplicial complexes with respective vertex sets K^0 and $\mathcal{L}^0$ and let $K^0 = \{a_0, a_1, \ldots, a_k\}$. A map $\varphi^0: K^0 \to \mathcal{L}^0$ is called a *simplicial map of the vertices* if for each set of vertices $a_{j_0}, a_{j_1}, \ldots, a_{j_n}$ which determines a simplex $\Delta(a_{j_0}, a_{j_1}, \ldots, a_{j_n}) \in K$ the vertices $\varphi(a_{j_0}), \varphi(a_{j_1}), \ldots, \varphi(a_{j_n})$ determine a simplex $\Delta\{\varphi(a_{j_0}), \varphi(a_{j_1}), \ldots, \varphi(a_{j_n})\} \in \mathcal{L}$. It thus follows that every simplicial map of the vertices $\varphi^0: K^0 \to \mathcal{L}^0$ extends uniquely to a map $\varphi: K \to \mathcal{L}$ defined by the formula $\varphi(\Delta(a_{j_0}, a_{j_1}, \ldots, a_{j_n})) = \Delta\{\varphi(a_{j_0}), \varphi(a_{j_1}), \ldots, \varphi(a_{j_n})\}$. Maps of this kind are called *simplicial* (see also Supplements 2.S.9 and 2.S.10). If a map $\varphi: K \to \mathcal{L}$ is simplicial, then obviously $\dim \varphi(\Delta) \leq \dim \Delta$ for every $\Delta \in K$. The following evidently holds.

2.5.1. ASSERTION. *The composition of two simplicial maps is a simplicial map.* ■

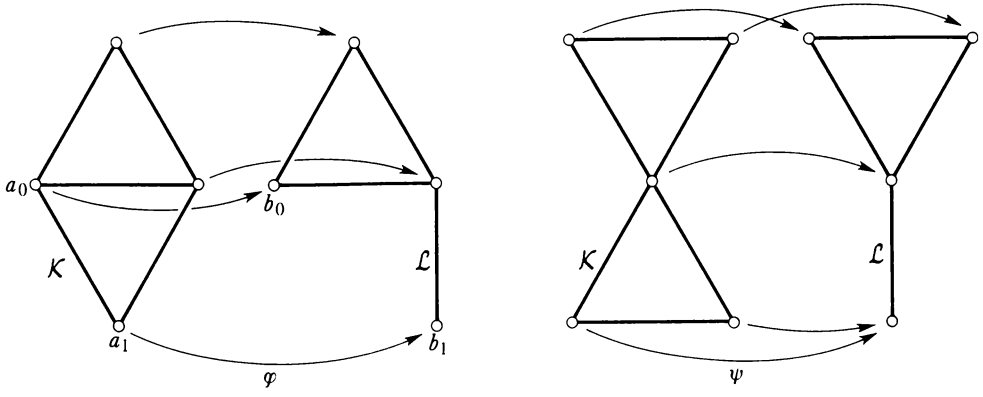


Fig.53. The map φ is not simplicial since $\Delta(a_0, a_1) \in K$ whereas the images $b_0 = \varphi(a_0)$ and $b_1 = \varphi(a_1)$ do not determine a simplex of $\mathcal{L}$. The map ψ is simplicial.

One-to-one simplicial maps are called *simplicial isomorphisms.* The following is the case.

2.5.2. ASSERTION. *The composition of two simplicial isomorphisms is a simplicial isomorphism. The identity map on any complex is a simplicial isomorphism. The inverse map of a simplicial isomorphism is a simplicial isomorphism.* ■

If a simplicial isomorphism from a complex K onto a complex $\mathcal{L}$ exists, then we say that the complexes are *simplicially isomorphic.*

2.5.3. EXAMPLE. Every simplicial complex which has exactly $k+1$ vertices is simplicially isomorphic to a subcomplex of the complex K_{Δ^k} (see Example 2.2.1).For, if $a_0, a_1, \ldots, a_k$ are the vertices of the complex K, put $\varphi^0(a_j) = e_j$ for $j = 0, 1, \ldots, k$. It is easy to see that $\varphi^0: \{a_0, a_1, \ldots, a_k\} \to \{e_0, e_1, \ldots, e_k\}$ is a simplicial map of the vertices. Since it is one-to-one the simplicial map $\varphi: K \to K_{\Delta^k}$ which it determines is a simplicial isomorphism from the complex K onto the subcomplex $\varphi(K)$ of the complex K_{Δ^k}. ■

2.5.4. EXAMPLE. *Every n-dimensional simplicial complex is simplicially isomorphic to a complex in the space $\mathbf{R}^{2n+1}$.* Indeed, let the n-dimensional complex K have vertices $a_0, a_1, \ldots, a_k$. In the space $\mathbf{R}^{2n+1}$ consider an arbitrary set of points $b_0, b_1, \ldots, b_k$ in general position. We define a family $\mathcal{L}$ of simplices in $\mathbf{R}^{2n+1}$ by letting $\Delta(b_{h_0}, b_{h_1}, \ldots,$

$b_{h_p}) \in \mathcal{L}$ if and only if $\Delta(a_{h_0}, a_{h_1}, \ldots, a_{h_p}) \in K$. Since $n+1 < 2n+2$ and every set of $2n+2$ points from among $b_0, b_1, \ldots, b_k$ is affinely independent, the definition is valid; furthermore condition (SC1) is obviously satisfied. To check that condition (SC2′) of Theorem 2.2.2 is met we observe that a point x which lies in the interior of two simplices $\Delta_1, \Delta_2 \in \mathcal{L}$ may be expressed in the form $x = \sum_{j=0}^{k} r^j b_j$ and also $x = \sum_{j=0}^{k} s^j b_j$, where $\sum_{j=0}^{k} r^j = 1 = \sum_{j=0}^{k} s^j$, and no more than $n+1$ coefficients r^j and no more than $n+1$ coefficients s^j are non-zero. Writing $t^j = r^j - s^j$ for $j = 0, 1, \ldots, k$ we have $\sum_{j=0}^{k} t^j b_j = 0$ with $\sum_{j=0}^{k} t^j = 0$ and, moreover, not more than $(n+1)+(n+1) = 2n+2$ of the coefficients t^j are non-zero. Since the points $b_0, b_1, \ldots, b_k$ are in general position it follows that $t^j = 0$; that is, $r^j = s^j$ for $j = 0, 1, \ldots, k$. Since b_j is a vertex of Δ_1 if and only if $r^j > 0$, and b_j is a vertex of Δ_2 if and only if $s^j > 0$, we have $\Delta_1 = \Delta_2$. The family $\mathcal{L}$ is thus a simplicial complex which is simplicially isomorphic to K. ■

Let $\varphi\colon K \to \mathcal{L}$ be a simplicial map and let $K^0 = \{a_0, a_1, \ldots, a_k\}$ and $\mathcal{L}^0 = \{b_0, b_1, \ldots, b_l\}$ be vertex sets of K and $\mathcal{L}$, respectively. The map φ induces a map $|\varphi|\colon |K| \to |\mathcal{L}|$ via the barycentric coordinates defined by $|\varphi|(\sum_{h=0}^{k} r^h a_h) = \sum_{j=0}^{l} s^j b_j$ where s^j equals the sum of all the r^h for which $\varphi(a_h) = b_j$, if any such exist, and is otherwise zero. Thus we have $s^j \geq 0$ for $j = 0, 1, \ldots, l$ and $\sum_{j=0}^{l} s^j = \sum_{h=0}^{k} r^h = 1$. Furthermore, since φ is simplicial, we have $|\varphi|(x) \in |\mathcal{L}|$ whenever $x \in |K|$.

We prove the following.

2.5.5. THEOREM. *For every simplicial map $\varphi\colon K \to \mathcal{L}$ the induced map $|\varphi|\colon |K| \to |\mathcal{L}|$ is continuous.*

PROOF. We follow the notation introduced in the definition of $|\varphi|$. Let $\bar{K}$ denote the simplicial complex consisting of those faces $\Delta(e_{h_0}, e_{h_1}, \ldots, e_{h_n})$ of the unit simplex Δ^k for which $\Delta(a_{h_0}, a_{h_1}, \ldots, a_{h_n}) \in K$. Similarly let $\bar{\mathcal{L}}$ denote the simplicial complex consisting of those faces $\Delta(e_{j_0}, e_{j_1}, \ldots, e_{j_n})$ of the unit simplex Δ^l for which $\Delta(b_{j_0}, b_{j_1}, \ldots, b_{j_n}) \in \mathcal{L}$. Let the maps $g\colon |K| \to |\bar{K}|$ and $h\colon |\mathcal{L}| \to |\bar{\mathcal{L}}|$ send each point to the vector of its barycentric coordinates; by Theorem 2.3.11 both maps g and h are homeomorphisms. The map $|\bar{\varphi}|\colon |\bar{K}| \to |\bar{\mathcal{L}}|$ defined by the formula $|\bar{\varphi}|(r^0, r^1, \ldots, r^k) = (s^0, s^1, \ldots, s^l)$ is obviously continuous. Since $|\varphi| = h^{-1}|\bar{\varphi}|g$, we see that $|\varphi|$ is also continuous. ■

The operation that takes the simplicial map φ to the induced map $|\varphi|$ has the following obvious properties.

2.5.6. ASSERTION. *If $\varphi\colon K \to \mathcal{L}$ and $\psi\colon \mathcal{L} \to \mathcal{M}$, then $|\psi\varphi| = |\psi||\varphi|$. Furthermore, $|\mathrm{id}_K| = \mathrm{id}_{|K|}$.* ■

2.5.7. COROLLARY. *If φ is a simplicial isomorphism, then $|\varphi|$ is a homeomorphism.* ■

A simplicial map $\varphi\colon K \to \mathcal{L}$ is said to be a *simplicial approximation* of the map $f\colon |K| \to |\mathcal{L}|$ if $f(\mathrm{st}\, a) \subset \mathrm{st}\, \varphi(a)$ for each vertex a of the complex K.

The following holds.

2.5.8. THEOREM. *If the simplicial map $\varphi\colon K \to \mathcal{L}$ is a simplicial approximation of the map $f\colon |K| \to |\mathcal{L}|$, then for each point $x \in |K|$ there is a simplex $\Delta \in \mathcal{L}$ such that $f(x) \in \mathrm{int}\, \Delta$ and $|\varphi|(x) \in \Delta$.*

PROOF. Let $a_0, a_1, \ldots, a_n$ be the vertices of the carrier of the point x in the polyhedron $|K|$. Then $x \in \operatorname{st} a_0 \cap \operatorname{st} a_1 \cap \ldots \cap \operatorname{st} a_n$ and we have $f(x) \in \operatorname{st} \varphi(a_0) \cap \operatorname{st} \varphi(a_1) \cap \ldots \cap \operatorname{st} \varphi(a_n)$. Let Δ be the carrier of the point $f(x)$ in the polyhedron $|\mathcal{L}|$. We thus have $|\varphi|(x) \in \Delta\{\varphi(a_0), \varphi(a_1), \ldots, \varphi(a_n)\} \subset \Delta$. Of course $f(x) \in \operatorname{int} \Delta$ and this clinches the proof. ■

The condition stated in Theorem 2.5.8 is also sufficient to ensure that the map φ is a simplicial approximation to the map f (see Problem 2.P.2).

Theorem 2.5.8 implies the following.

2.5.9. COROLLARY. *If the simplicial map $\varphi: K \to \mathcal{L}$ is a simplicial approximation of the map $f: |K| \to |\mathcal{L}|$, then $\rho(f(x), |\varphi|(x)) \leq \operatorname{diam} \mathcal{L}$ for each $x \in |K|$.* ■

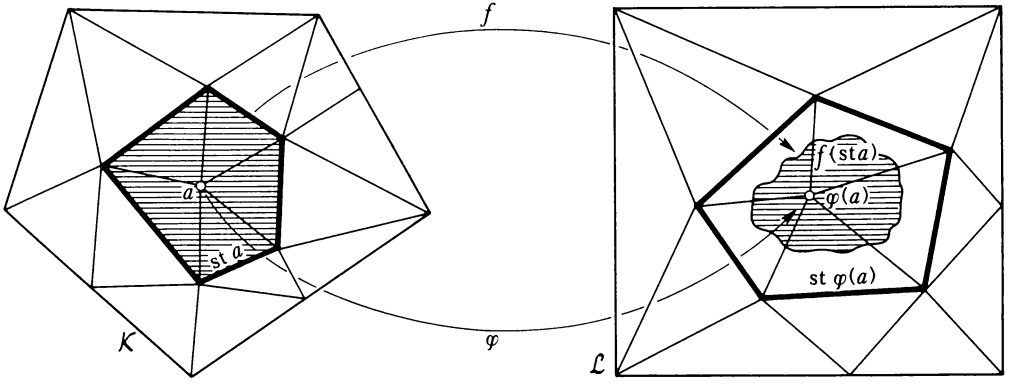


Fig.54. The simplicial map $\varphi: K \to \mathcal{L}$ is a simplicial approximation of the map $f: |K| \to |\mathcal{L}|$ when $f(\operatorname{st} a) \subset \operatorname{st} \varphi(a)$ for each vertex $a \in K$.

We now prove two fundamental theorems on the existence of simplicial approximations.

2.5.10. THEOREM. *For each continuous map $f: |K| \to |\mathcal{L}|$ there is a non-negative integer p such that f has a simplicial approximation φ defined on the barycentric subdivision of order p of the complex K.*

PROOF. In view of Assertion 2.3.12 and 2.3.15 the stars $\operatorname{st} b$, where b runs through the vertices of the complex $\mathcal{L}$, form an open covering of the polyhedron $|\mathcal{L}|$. Hence the inverse images $f^{-1}(\operatorname{st} b)$ form an open covering of the polyhedron $|K|$. Let $\lambda > 0$ be a Lebesgue number of this covering. Applying Corollary 2.4.9 we choose a non-negative integer p such that $\operatorname{diam} K^{(p)} < \frac{1}{2}\lambda$. Each star $\operatorname{st} a$, for a vertex a of the complex $K^{(p)}$, thus has diameter less than λ and is contained in one of the sets $f^{-1}(\operatorname{st} b)$, with b a vertex of $\mathcal{L}$.

Letting $\varphi^0(a) = b$ we obtain a map of the vertex set of $K^{(p)}$ into the vertex set of $\mathcal{L}$, which is in fact a simplicial map of the vertices. For, if $\Delta(a_0, a_1, \ldots, a_n) \in K^{(p)}$, then by Lemma 2.3.13 we have $\operatorname{st} a_0 \cap \operatorname{st} a_1 \cap \ldots \operatorname{st} a_n \neq \emptyset$. But then $\emptyset \neq \operatorname{st} a_0 \cap \operatorname{st} a_1 \cap \ldots \cap \operatorname{st} a_n \subset f^{-1}(\operatorname{st} \varphi^0(a_0)) \cap f^{-1}(\operatorname{st} \varphi^0(a_1)) \cap \ldots \cap f^{-1}(\operatorname{st} \varphi^0(a_n)) = f^{-1}(\operatorname{st} \varphi^0(a_0) \cap \operatorname{st} \varphi^0(a_1) \cap \ldots \cap \operatorname{st} \varphi^0(a_n))$, so $\operatorname{st} \varphi^0(a_0) \cap \operatorname{st} \varphi^0(a_1) \cap \ldots \cap \operatorname{st} \varphi^0(a_n) \neq \emptyset$. Appealing again to Lemma 2.3.13 we conclude that $\Delta\{\varphi^0(a_0), \varphi^0(a_1), \ldots, \varphi^0(a_n)\} \in \mathcal{L}$.

Extending the simplicial map of the vertices φ^0 to a simplicial map $\varphi: K^{(p)} \to \mathcal{L}$ yields the desired simplicial approximation to the map f. ■

2.5.11. THEOREM. *For each continuous map $f: X \to Y$ between polyhedra and every real number $\epsilon > 0$, there exist triangulations K and $\mathcal{L}$ of the polyhedra X and Y respectively and a simplicial approximation $\varphi: K \to \mathcal{L}$ of the map f such that $\rho(f(x), |\varphi|(x)) < \epsilon$ for each point $x \in X$.*

PROOF. Applying Corollary 2.4.9, choose a triangulation $\mathcal{L}$ of the polyhedron Y with diameter less than ϵ. Next, appealing to Theorem 2.5.10, take a triangulation K of the polyhedron X and a simplicial approximation $\varphi: K \to \mathcal{L}$ of f. To complete the proof it is enough to quote Corollary 2.5.9. ■

We now prove a fundamental lemma.

2.5.12. LEMMA (Sperner). *Let K be the simplicial complex consisting of a simplex Δ and all its faces. Let K' be the barycentric subdivision of the complex K. If a simplicial map $\varphi: K' \to K$ has the property that $a \in \operatorname{st} \varphi(a)$ for every vertex a of the complex K', then there exists a simplex $\Delta' \in K'$ such that $\varphi(\Delta') = \Delta$.*

PROOF. Proceeding by induction on the dimension n of the simplex Δ we prove a stronger proposition, namely that the number r_n of simplices $\Delta' \in K'$ satisfying the equation $\varphi(\Delta') = \Delta$ is odd. The proposition is true for $n = 0$ since $r_0 = 1$. Assume that the proposition is true when the dimension is $n - 1$ with $n > 0$ and suppose $\Delta = \Delta(a_0, a_1, \ldots, a_n)$. Let $\Delta_0 = \Delta(a_0, a_1, \ldots, a_{n-1})$.

Let $\Delta'_1, \Delta'_2, \ldots, \Delta'_q$ be all the n-dimensional simplices belonging to K'. For $k = 1, 2, \ldots, q$, let s_k denote the number of $(n-1)$-dimensional faces Δ' of the simplex Δ'_k which satisfy $\varphi(\Delta') = \Delta_0$. If $\dim \varphi(\Delta'_k) = n$, then $\varphi(\Delta'_k) = \Delta$ and we have $s_k = 1$. If $\dim \varphi(\Delta'_k) = n - 1$, then either $\varphi(\Delta'_k) = \Delta_0$ and then $s_k = 2$ or $\varphi(\Delta'_k) \neq \Delta_0$ and then $s_k = 0$. Finally if $\dim \varphi(\Delta'_k) < n - 1$, then evidently $s_k = 0$. These remarks imply that the sum $\sum_{k=1}^{q} s_k$ has the same parity as r_n.

Let K_0 denote the simplicial complex consisting of the simplex Δ_0 and all of its faces. The complex $K'_0 = \{\Delta' \in K' : \Delta' \subset \Delta_0\}$ is the barycentric subdivision of the complex K_0. Now $a \in \operatorname{st} \varphi(a)$ for each vertex a of the complex K', so $\varphi_0 = \varphi | K'_0$ is a simplicial map of K'_0 into K_0 which satifies the condition $a \in \operatorname{st} \varphi_0(a)$ for each vertex a of the complex K'_0. Moreover if $\Delta' \in K'$, $\Delta' \subset \operatorname{bd} \Delta$ and $\varphi(\Delta') = \Delta_0$, then $\Delta' \in K'_0$. But every $(n-1)$-dimensional simplex $\Delta' \in K'$ is the face of one or two simplices in the sequence $\Delta'_1, \Delta'_2, \ldots, \Delta'_q$ depending on whether $\Delta' \subset \operatorname{bd} \Delta$ or $\Delta' \cap \operatorname{int} \Delta \neq \emptyset$ respectively, so the sum $\sum_{k=1}^{q} s_k$ has the same parity as the number r_{n-1} of simplices satisfying the equation $\varphi_0(\Delta'_0) = \Delta_0$.

We have shown that the numbers r_n and r_{n-1} are of the same parity, hence in view of our inductive hypothesis the proof is complete. ■

It is worth noting that Sperner's Lemma continues to hold also in the case when K' denotes an arbitrary, not necessarily barycentric, subdivision of the complex K. The proof goes over formally without any changes, although the final remark (that every $(n-1)$-dimensional simplex $\Delta' \in K'$ is a facet of one or of two simplices in the

sequence $\Delta_1', \Delta_2', \ldots, \Delta_q'$), is no longer an easily verifiable proposition – its proof requires properties of Euclidean space which we will learn later (see Theorem 5.1.20).

Exercises

a) Show that if $\varphi: K \to \mathcal{L}$ is a simplicial map and K_0 is a subcomplex of the simplicial complex K then $\varphi | K_0$ is also a simplicial map.

b) Suppose that the simplicial complex K is the union of two subcomplexes K_1 and K_2 and $\varphi: K \to \mathcal{L}$, with $\mathcal{L}$ a simplicial complex. Show that if the maps $\varphi | K_1$ and $\varphi | K_2$ are simplicial then φ is also simplicial.

c) Show that the composition of simplicial approximations is itself a simplicial approximation of the appropriate composition.

2.6. Cell complexes

The class of polyhedra introduced in Section 2.3 is closed under the operations of union and intersection and of metric product. Proofs of these properties, however, are complicated if direct use is made of the definition of a polyhedron as the underlying space of a simplicial complex. It is easy to see that even for two simplices their union, intersection and metric product are not in general simplices and, though they are polyhedra, they have no natural triangulation. The object of this section is to generalize the notions of simplex and simplicial complex so as to remove these inadequacies.

Any set in m-dimensional Euclidean space $\mathbf{R}^m$ which is the intersection of a finite number of closed half-spaces and is bounded is called a *cell.*

2.6.1. EXAMPLE. *Every simplex is a cell.* To prove this it suffices to consider the simplex $\Delta(a_0, a_1, \ldots, a_m)$ in the space $\mathbf{R}^m$. Let $H_i = H(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)$ for $i = 0, 1, \ldots, m$ and let P_i denote the one, of the two closed half-spaces defined by the hyperplane H_i, which contains the point a_i. Then $\Delta(a_0, a_1, \ldots, a_m) = \bigcap_{i=0}^m P_i$. The boundedness of the simplex was noted in Corollary 2.1.2. ■

2.6.2. EXAMPLE. *Every cube is a cell.* For the proof it suffices to consider the cube $[a, b]^m$ in the space $\mathbf{R}^m$. Let $P_i' = \{(x^1, x^2, \ldots, x^m) \in \mathbf{R}^m : x^i \geq a\}$ and $P_i'' = \{(x^1, x^2, \ldots, x^m) \in \mathbf{R}^m : x^i \leq b\}$ for $i = 1, 2, \ldots, m$. Then $[a, b]^m = \bigcap_{i=1}^m (P_i' \cap P_i'')$. Boundedness follows from Corollary 1.4.4. ■

The following assertions are consequences of the definition of a cell.

2.6.3. ASSERTION. *The intersection of two cells is a cell.* ■

Moreover, since the metric product of a half-space of $\mathbf{R}^{m_1}$ and a half-space of $\mathbf{R}^{m_2}$ is the intersection of half-spaces of $\mathbf{R}^{m_1+m_2}$, we have the following.

2.6.4. ASSERTION. *If Q_1 is a cell in the space $\mathbf{R}^{m_1}$ and Q_2 is a cell in the space $\mathbf{R}^{m_2}$, then the metric product $Q_1 \times Q_2$ is a cell in the space $\mathbf{R}^{m_1+m_2}$.* ■

Every closed half-space is a convex set (Example 0.4.14), so from Assertion 0.4.15 we conclude:

2.6.5. ASSERTION. *Every cell is convex.* ■

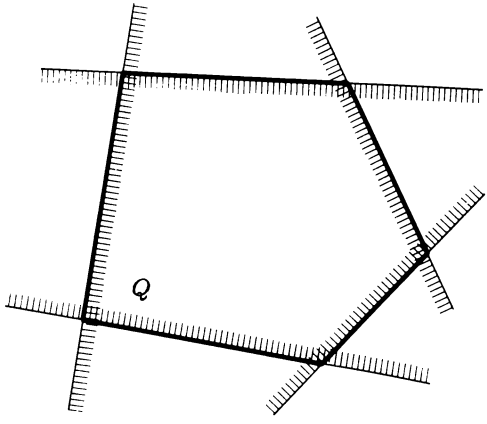


Fig.55. The cell Q is the intersection of five closed half-spaces.

Since every closed half-space is a closed subset of Euclidean space (Corollary 1.6.25), we have in view of Theorems 1.6.18 and 1.10.2 the following.

2.6.6. ASSERTION. *Every cell is compact.* ■

If Q is a cell in the space $\mathbf{R}^m$ and the points $a_0, a_1, \ldots, a_n \in Q$ form an affinely independent set of maximal cardinality in Q, then by Theorem 0.4.9 it follows that $Q \subset H(a_0, a_1, \ldots, a_n)$. On the other hand from the convexity of Q we have $\Delta(a_0, a_1, \ldots, a_n) \subset Q$. This implies that for every non-empty cell $Q \subset \mathbf{R}^m$ there exists an affine subspace H containing Q relative to which Q has non-empty interior. These properties of course determine H uniquely. The affine subspace H is known as the *carrier subspace* of the cell Q and its dimension is the *geometric dimension* of the cell Q and is denoted by $\dim Q$. We regard the empty set as the carrier subspace of the empty cell and we put $\dim \emptyset = -1$. The *geometric interior* $\operatorname{int} Q$ and the *boundary* $\operatorname{bd} Q$ of the cell Q will means its interior and boundary relative to the carrier subspace.

From Theorem 1.10.9 and in view of Assertions 2.6.5 and 2.6.6 we obtain the following generalization of Corollary 2.1.9.

2.6.7. COROLLARY. *Let $n \geq 0$. Every n-dimensional cell is homeomorphic to the n-dimensional closed unit ball. The geometric interior of an n-dimensional cell is homeomorphic to the n-dimensional open unit ball. The boundary of an n-dimensional cell is homeomorphic to the $(n-1)$-dimensional unit sphere.* ■

A finite family K of cells in the Euclidean space $\mathbf{R}^m$ is a *cell complex* if the following conditions are fulfilled:

(CC1) *the boundary of each cell $Q \in K$ is a union of cells belonging to K,*

(CC2) *the intersection of two cells $Q_1, Q_2 \in K$ is in K,*

(CC3) *the geometric interiors of distinct cells of K are disjoint.*

2.6.8. EXAMPLE. *Every simplicial complex is a cell complex.* For, it follows from Example 2.6.1 that every simplex is a cell. Condition (CC1) follows from Corollary 2.1.11 and condition (SC1). Condition (CC2) is a consequence of conditions (SC2) and (SC1). Condition (CC3) follows from condition (SC2′) (see Theorem 2.2.2). ■

Let K be a cell complex. The number $\dim K = \max\{\dim Q : Q \in K\}$ is called the *dimension* of the complex K. A subset K_0 of a cell complex K is called a *cell subcomplex* (or just a *subcomplex*) if for each cell $Q \in K_0$ the set K_0 contains all the cells of K lying in the boundary of Q. Thus every cell subcomplex is in its own rights a cell complex. The set $\bigcup\{Q : Q \in K\}$ is known as the *underlying space* of the complex K and is denoted by $|K|$ (see also Supplement 2.S.11).

We now prove the following.

2.6.9. THEOREM. *If a set X is the union of a finite family of cells $\mathcal{R}$, then there is a cell complex K such that $X = |K|$ and whenever the cells $Q \in \mathcal{R}$ and $Q' \in K$ satisfy the condition $Q \cap \operatorname{int} Q' \neq \emptyset$, then $Q' \subset Q$.*

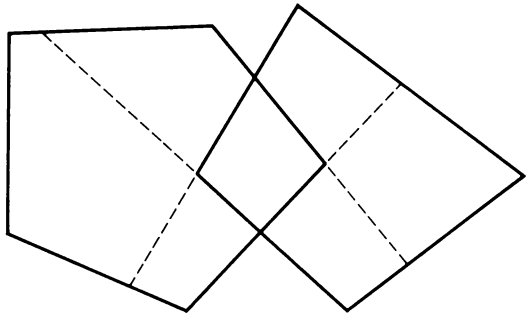


Fig.56. The union of a finite number of cells is the underlying space of a cell complex (see Theorem 2.6.9).

PROOF. Let $X = \bigcup_{k=1}^{p} Q_k$ where $Q_k = \bigcap_{l=1}^{q_k} P_{k,l}$ for $k = 1, 2, \ldots, p$ and $P_{k,l}$ is a closed half-space of the space $\mathbf{R}^m$ for $k = 1, 2, \ldots, p$, and $l = 1, 2, \ldots, q_k$. Let $\mathcal{P}$ denote the family of half-spaces $P_{k,l}$ for $k = 1, 2, \ldots, p$ and $l = 1, 2, \ldots, q_k$ and also the complementary half-spaces. For each point $x \in X$ put $Q_x = \bigcap\{P : x \in P \in \mathcal{P}\}$. Of course this defines only a finite number of distinct sets; let us denote the family of these sets by K. We shall show that this is the desired cell complex.

We note that if $x \in Q_k$ for $1 \leq k \leq p$, then $x \in Q_x \subset Q_k$. It follows that each of the sets Q_x is bounded, hence is a cell. Moreover we have $X = \bigcup_{x \in X} Q_x = \bigcup\{Q : Q \in K\}$.

Let $x \in \operatorname{bd} Q$ with $Q \in K$. There then exists a half-space $P \in \mathcal{P}$ determined by a hyperplane H such that $x \in H$ and $\emptyset \neq \operatorname{int} Q \subset P \setminus H$. Thus $Q_x \subset H$ and since the condition $x \in Q$ implies $Q_x \subset Q$ we have $Q_x \subset Q \cap H$; that is, $Q_x \subset \operatorname{bd} Q$. We conclude that condition (CC1) is satisfied.

To check condition (CC2) observe that if a cell Q is the intersection of certain half-spaces of the family $\mathcal{P}$, then $Q \in K$ if and only if for each half-space $P \in \mathcal{P}$ and its complementary half-space P^* at least one of the inclusions $Q \subset P$ or $Q \subset P^*$ holds. This property evidently is inherited from arbitrary cells $Q, Q' \in K$ by their intersection $Q' \cap Q''$.

Before checking condition (CC3) we note that if $x \in \operatorname{int} Q$, where $Q \in K$, then $Q = Q_x$. For, if $x \in \operatorname{int} Q$ we immediately obtain the inclusion $Q_x \subset Q$. To prove the reverse inclusion assume that $y \in Q$. Then for every half-space $P \in \mathcal{P}$ if $x \in P$, then $y \in P$ and hence $y \in Q_x$.

Appealing to the above property we note that if $Q', Q'' \in K$ and $x \in \operatorname{int} Q' \cap \operatorname{int} Q''$ then $Q' = Q_x = Q''$ and this proves that condition (CC3) holds.

To complete the proof of the theorem, suppose that $Q \in \mathcal{R}$, $Q' \in K$ and $x \in Q \cap \operatorname{int} Q'$. Then $Q' = Q_x$ and $Q_x \subset Q$ and so $Q' \subset Q$. ■

A cell complex K' is said to be a *subdivision* of the cell complex K, when

(1) $|K'| = |K|$, and

(2) for each cell $Q' \in K'$ there is a cell $Q \in K$ with $Q' \subset Q$.

The following holds.

2.6.10. THEOREM. *Every cell complex has a subdivision which is a simplicial complex.*

PROOF. We proceed by a double induction on the dimension n of the complex K and the number k of cells of dimension n in K. If $n = 0$, and k is arbitrary, then the theorem is obviously true, since every 0-dimensional cell complex is a simplicial complex. So suppose that $n, k > 0$ and that the theorem is true for all cell complexes of dimension less than n and also for all cell complexes of dimension n which have less than k n-dimensional cells.

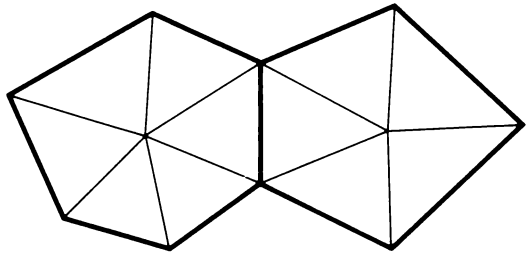


Fig.57. Every cell complex has a simplicial subdivision (Theorem 2.6.10).

Let K be an n-dimensional cell complex K which has k n-dimensional cells and let $Q \in K$ with $\dim Q = n$. Since the cell Q does not lie on the boundary of any cell of K, it follows that $K_1 = K \setminus \{Q\}$ is a subcomplex of the cell complex K; moreover, either $\dim K_1 \leq n-1$, or $\dim K_1 = n$ and K_1 contains $k-1$ n-dimensional cells. Appealing to the inductive hypotheses we obtain a simplicial subdivision K_1' of the complex K_1.

Now the boundary $\operatorname{bd} Q$ is a union of cells of K_1 and each such cell is a union of simplices of the complex K_1' so there exists a simplicial subcomplex K_0' of the complex K_1' such that $|K_0'| = \operatorname{bd} Q$.

Let $b \in \operatorname{int} Q$. Observe that if $\Delta(a_0, a_1, \ldots, a_p) \in K_0'$, then $b \notin H(a_0, a_1, \ldots, a_p)$ so the points b, $a_0, a_1, \ldots, a_p$ form an affinely independent set. Put

$$K_2' = \{b\} \cup K_0' \cup \{\Delta(b, a_0, a_1, \ldots, a_p) : \Delta(a_0, a_1, \ldots, a_p) \in K_0'\}.$$

We will show that K_2' is a triangulation of the cell Q. The condition (SC1) is satisfied in an obvious way. To verify condition (SC2) it is enough to observe that if

$$\{i_0, i_1, \ldots, i_q\} \cap \{j_0, j_1, \ldots, j_r\} = \{h_0, h_1, \ldots, h_p\},$$

then

$$\Delta(b, a_{i_0}, a_{i_1}, \ldots, a_{i_q}) \cap \Delta(b, a_{j_0}, a_{j_1}, \ldots, a_{j_r}) = \Delta(b, a_{h_0}, a_{h_1}, \ldots, a_{h_p}),$$

because the remaining cases are obvious. The family K_2' is thus a simplicial complex. From the convexity of Q we have $|K_2'| \subset Q$. Conversely, if $x \in Q$ and $x \neq b$, then by Lemma 1.10.8 it follows that the half-line with endpoint b passing through x meets $\operatorname{bd} Q$ in exactly one point x'. If $x' \in \Delta(a_0, a_1, \ldots, a_p) \in K_0'$ then $x \in \Delta(b, a_0, a_1, \ldots, a_p) \in K_2'$. Hence we have $Q \subset |K_2'|$.

Since K_0' is a simplicial subcomplex of the complexes K_1' and K_2' and $|K_1'| \cap |K_2'| = (|K| \setminus \operatorname{int} Q) \cap Q = \operatorname{bd} Q = |K_0'|$, we have by Theorem 2.3.16 that $K' = K_1' \cup K_2'$ is a simplicial complex.

Moreover $|K'| = |K_1'| \cup |K_2'| = (|K| \setminus \operatorname{int} Q) \cup Q = |K|$. If a simplex $\Delta \in K'$ belongs to K_1', then it is contained in a cell of the complex K_1; if however $\Delta \in K_2'$, then $\Delta \subset Q$. The simplicial complex K' is thus a subdivision of the cell complex K and this completes the proof. ■

From Theorems 2.6.9 and 2.6.10 we obtain the following.

2.6.11. THEOREM. *A finite union of cells is a polyhedron.* ■

From the theorem we conclude the following.

2.6.12. COROLLARY. *The union of two polyhedra is a polyhedron.* ■

On the other hand, using Assertions 2.6.3 and 2.6.4 we obtain the following corollaries (see also the Supplement 2.S.8):

2.6.13. COROLLARY. *The intersection of two polyhedra is a polyhedron.* ■

2.6.14. COROLLARY. *The metric product of two polyhedra is a polyhedron.* ■

Applying Theorems 2.6.9 and 2.6.10 we also prove the following.

2.6.15. THEOREM. *If $A \subset X$, where A and X are polyhedra, then there exists a triangulation K of the polyhedron X and a triangulation $\mathcal{L}$ of the polyhedron A such that $\mathcal{L}$ is a subcomplex of K.*

PROOF. Let K_A be any triangulation of A and let K_X be any triangulation of X. Put $\mathcal{R} = K_A \cup K_X$. We infer from Theorem 2.6.9 that there exists a cell complex $\bar{K}$ such that $X = |\bar{K}|$ and if $Q \in \bar{K}$, $\Delta \in \mathcal{R}$ and $\Delta \cap \operatorname{int} Q \neq \emptyset$ then $Q \subset \Delta$. It follows from this that if $Q \in \bar{K}$ and $A \cap \operatorname{int} Q \neq \emptyset$ then $Q \subset A$. Applying Theorem 2.6.10 we obtain a simplicial subdivision K of the complex $\bar{K}$. From the definition of a subdivision of a cell complex it follows that for every simplex $\Delta \in K$ there exists a cell $Q \in \bar{K}$ such that $\Delta \subset Q$ and $\operatorname{int} \Delta \subset \operatorname{int} Q$. So if $\Delta \in K$ and $A \cap \operatorname{int} \Delta \neq \emptyset$ then $\Delta \subset A$. It now follows that $\mathcal{L} = \{\Delta \in K : A \cap \operatorname{int} \Delta \neq \emptyset\}$ is a subcomplex of the simplicial complex K and $A = |\mathcal{L}|$. ■

Exercises

a) Show that the intersection of a cell and an affine subspace is a cell. Is the intersection of a simplex with an affine subspace always a simplex?

b) Prove that if a set $Q' \subset \mathbf{R}^m$ is affinely isomorphic to a cell $Q \subset \mathbf{R}^n$, then Q' is also a cell.

c) Apply the construction given in the proof of Theorem 2.6.10 to obtain a triangulation of the 3-dimensional cube.

2.S. Supplements

2.S.1. The systematic study of simplicial complexes was begun by J. W. Alexander and H. Poincaré, who thus laid the foundations of what was then called combinatorial topology (1899). Further contributions to the development of the theory came from P. S. Alexandrov (who introduced the concept of the nerve of a covering in 1928), L. Vietoris, H. Freudenthal, H. Hopf, S. Lefschetz and others.

Since several important topological invariants of polyhedra are defined by means of triangulations, the problem was posed quite early whether the topology of polyhedra can be reduced to properties of simplicial complexes. For many years the so-called *fundamental hypothesis of combinatorial topology (Hauptvermutung)*, according to which any two triangulations of homeomorphic polyhedra have simplically isomorphic subdivisions, remained unresolved. The hypothesis turned out to be false (the appropriate 7-dimensional complexes were constructed by J. Milnor, *Two complexes which are homeomorphic but combinatorially distinct*, Ann. of Math. 74 (1961), 575–590), which served to emphasize the significance of approximation methods as represented by, say, the theorem on simplicial approximation.

2.S.2. The notion of a polyhedron is not of course topological. However, a more general class of spaces – those which are homeomorphic to polyhedra – may be considered. If there exists a homeomorphism h of a polyhedron $|K|$ onto a metric space X then we say that X is a *curvilinear polyhedron*; the images $h(\Delta)$ for $\Delta \in K$ are then called *curvilinear simplices* in X and we say that they form a *curvilinear triangulation* of the space. For example the n-dimensional unit ball and the $(n-1)$-dimensional unit sphere are curvilinear polyhedra and their curvilinear triangulations may easily be obtained by applying Corollary 2.1.9 and the Examples 2.3.1 and 2.3.2. It may be shown (cf. Problem 2.P.11 and also Example 5.3.9) that m-dimensional projective space is a curvilinear polyhedron.

The definition given above of a curvilinear polyhedron, as a space which is homeomorphic to a polyhedron, is extrinsic in character. The problem of finding an intrinsic characterization of the curvilinear polyhedra within the class of compact metric spaces is still unsolved except for the cases of dimension 1 and 2. (For dimension 1 the characterization may easily be obtained by recourse to the order of a space at a point; see for example [10], Section 51. For dimension 2 see A. Kosiński, *A topological characterization of 2-polytopes*, Bull. Acad. Polon. Sci. 2 (1954), 321–323.)

2.S.3. Let K be an n-dimensional simplicial complex and for $j = 0, 1, \ldots, n$ let α_j denote the number of j-dimensional simplices in K. The number $\chi(K) = \sum_{j=0}^{n}(-1)^j \alpha_j$ is called the *Euler-Poincaré characteristic* of the complex K. It turns out that if the polyhedra $|K_1|$ and $|K_2|$ are homeomorphic then $\chi(K_1) = \chi(K_2)$; the Euler-Poincaré characteristic is thus a topological invariant associated with the polyhedron. It is also closely connected with the Betti numbers of a polyhedron which are studied in algebraic topology (see e.g. [5], p. 167). Elementary properties of the Euler-Poincaré characteristic are given in Problem 2.P.13.

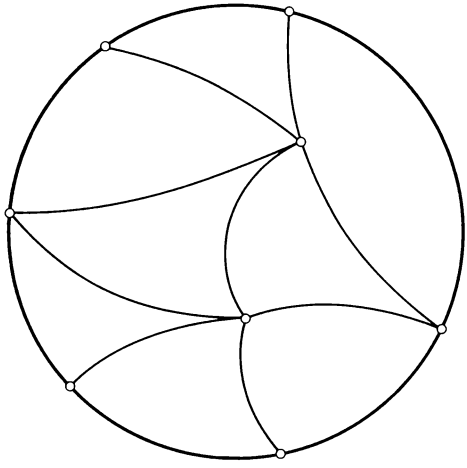


Fig.58. Curvilinear triangulation of a disc.

2.S.4. In place of the symbol "diam K", for K a simplicial complex, the symbol "mesh K" is also used in the literature.

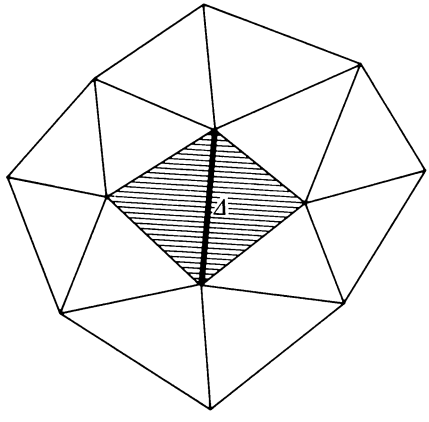


Fig. 59. Star of the simplex Δ in the complex K.

2.S.5. The notion of the *closed star* of a vertex of a complex K also appears in the literature; this is the set $\bigcup\{\Delta \in K : a \text{ is a vertex of the simplex } \Delta\}$. The closed star of a vertex a is the underlying space of a subcomplex of the complex K (namely the

subcomplex consisting of all simplices which have a as a vertex together with all their faces). Hence the closed star is a closed subset of $|K|$ (see also Problem 2.P.14).

Let Δ be a simplex in the simplicial complex K. The set $\operatorname{st}\Delta = \bigcup\{\operatorname{int}\Gamma : \Gamma \in K, \Delta \leq \Gamma\}$ is known as the *star of the simplex* Δ *in the complex* K. It is of course a generalization of the concept of the star of a vertex in K. In Problems 2.P.15 and 2.P.16 we give properties of the stars of simplices corresponding to Lemma 2.3.13 and Assertion 2.3.15.

2.S.6. The operation of forming the barycentric subdivision defined in Section 2.4 can also be described by induction on the dimension of the complex and the number of simplices of maximal dimension. If $\dim K = 0$ we put $K' = K$. Suppose the operation of barycentric subdivision has been defined on all complexes K such that $\dim K \leq n-1$, or $\dim K = n$ and K contains less than k simplices of dimension n. Let K be a simplicial complex of dimension n which contains k simplices of dimension n. If $\Delta \in K$ and $\dim\Delta = n$, then $K_1 = K\setminus\{\Delta\}$ is a subcomplex of the complex K for which, according to the inductive hypothesis, we have already defined the barycentric subdivision K_1'. The subdivision defines a triangulation K_0' of the boundary $\operatorname{bd}\Delta$. Let b denote the barycentre of the simplex Δ; then, for each simplex $\Delta(a_0, a_1, \ldots, a_p) \in K_0'$ the points $b, a_0, a_1, \ldots, a_p$ form an affinely independent set. Taking

$$K_2' = \{b\} \cup K_0' \cup \{\Delta(b, a_0, a_1, \ldots, a_p) : \Delta(a_0, a_1, \ldots, a_p) \in K_0'\}$$

we obtain a triangulation of the simplex Δ. It follows from Theorem 2.3.16 that $K' = K_1' \cup K_2'$ is a simplicial complex. It turns out (see Problem 2.P.17) that the operation so defined coincides with the operation of barycentric subdivision as defined in Section 2.4.

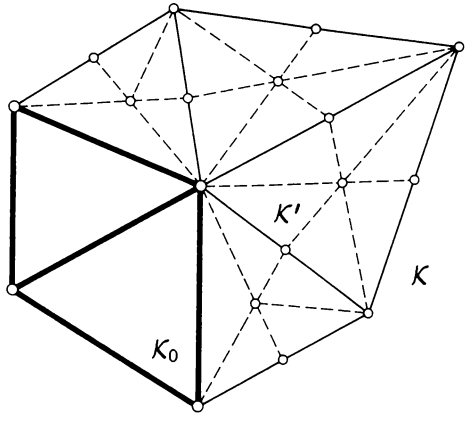


Fig.60. Relative barycentric subdivision K' mod K_0.

The operation of barycentric subdivision may be generalized by introducing the relative barycentric subdivision. Let K_0 be a simplicial subcomplex of a complex K and let $\Delta < \Delta_0 < \Delta_1 < \ldots < \Delta_q$ with $\Delta \in K_0$ and $\Delta_h \in K\setminus K_0$ for $h = 0, 1, \ldots, q$. Let $\Delta = \Delta(a_0, a_1, \ldots, a_n)$ and let b_h be the barycentre of the simplex Δ_h for $h = 0, 1, \ldots, q$. It turns out that the points $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_q$ form an affinely independent set and all the simplices of the form $\Delta(a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_q)$ constructed by the process

above form a simplicial complex. This complex is called the *barycentric subdivision of the complex K relative to the subcomplex K_0* and is denoted $K' \bmod K_0$. Obviously $K' \bmod \{\emptyset\} = K'$ and $K' \bmod K = K$.

2.S.7. Some of the notions and results concerning simplicial complexes become more transparent and permit natural generalizations if abstract complexes are introduced. An *abstract complex* or *vertex complex* is a pair $(V; K)$ where V is an arbitrary set, possibly infinite, and K is a family of finite subsets of V satisfying the following axioms:

(AC1) *if* $a \in V$, *then* $\{a\} \in K$,

(AC2) *if* $A \in K$ *and* $B \subset A$, *then* $B \in K$.

The elements of V are known as the *abstract vertices* and the elements of the family K are known as *abstract simplices*. In view of axiom (AC1) the abstract complex $(V; K)$ is often referred to by the single letter K. If the set V is finite we say that the abstract complex $(V; K)$ is *finite*. The number of vertices of an abstract simplex less 1 is known as the *dimension of the simplex*; by the *dimension of an abstract complex* we mean the supremum of the dimensions of its constituent abstract simplices (it is thus an integer not less than -1 or the symbol ∞).

Of course every simplicial complex defines a finite, abstract complex in a natural way. The inverse problem, that is the problem of obtaining the realization of an abstract complex in the form of a simplicial complex, is solved by analogy with Examples 2.5.3 and 2.5.4: every abstract complex with $k+1$ vertices has a realization in the family of faces of the simplex Δ^k; every finite, abstract complex of dimension n has a realization in the Euclidean space $\mathbf{R}^{2n+1}$.

By the *nerve* of a covering $\mathcal{A}$ (finite or infinite) of a space X we now mean an abstract complex whose set of vertices is $\mathcal{A}$ with the sets $A_0, A_1, \ldots, A_n \in \mathcal{A}$ forming an abstract simplex if and only if $A_0 \cap A_1 \cap \ldots \cap A_n \neq \emptyset$. In the case of a finite covering the notion is close to that introduced in Example 2.2.4; the difference lies in the absence of a simplicial realization which needs to be constructed whenever necessary.

The concept of an abstract complex may also be used in the construction of the barycentric subdivision. By the *barycentric subdivision* of an abstract complex (V, K) we mean a pair $(V'; K')$ where $V' = K$ and

$$K' = \{(A_0, A_1, \ldots, A_n) : A_j \in K \text{ for } j = 0, 1, \ldots, n, \quad A_0 \subsetneqq A_1 \subsetneqq \ldots \subsetneqq A_n\}.$$

It is not difficult to see that in the case of finite complexes this is an abstract analogue of the barycentric subdivision of a simplicial complex and so for every realization of an abstract complex $(V; K)$ there also exists a realization of the subdivision $(V'; K')$ with the same underlying space as the given realization.

To each abstract complex $(V; K)$ one can associate a metric space $|K|$ which we will call the *underlying space of the complex*. Its elements are the functions $r: V \to \mathbf{R}_+$ with the property that $\{a \in V : r(a) \neq 0\} \in K$ and $\sum_{a \in V} r(a) = 1$. The metric ρ is defined by the formula $\rho(r, s) = \sqrt{\sum_{a \in V} (r(a) - s(a))^2}$ for $r, s \in |K|$. It is easy to see that if the simplicial complex K_s is a realization of the finite abstract complex K_a then the underlying spaces $|K_s|$ and $|K_a|$ are homeomorphic (Problem 2.P.21). Among the abstract complexes a significant role is played by those which are *locally finite*, i.e.

those for which every vertex belongs only to a finite number of abstract simplices (see Problems 2.P.22 and Supplement 7.S.16).

2.S.8. An abstract complex $(V; K)$ (see Supplement 2.S.7) is said to be *ordered* if the set V is equipped with a linear ordering $\leq$. The ordering of course induces a linear ordering of the vertices of each abstract simplex of the family K. Let $(V_1; K_1)$ and $(V_2; K_2)$ be ordered abstract complexes. We can equip $V = V_1 \times V_2$ with an ordering $\leq$ (which in general is not linear) by taking $(a_1, a_2) \leq (b_1, b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$. Let $K = \{A \subset V : A \subset A_1 \times A_2,\ A_1 \in K_1,\ A_2 \in K_2,\ A \text{ is linearly ordered by the relation } \leq\}$. It turns out (see Problem 2.P.27) that the pair $(V; K)$ is an abstract complex; we call it the *Cartesian product* of the ordered complexes $(V_1; K_1)$ and $(V_1; K_2)$ and we denote it by $(V_1; K_1) \times (V_2; K_2)$. Furthermore, if W_1 is the underlying space of a realization of the complex $(V_1; K_1)$ and W_2 is the underlying space of a realization of the complex $(V_2; K_2)$ then there exists a triangulation of the metric product $W_1 \times W_2$ which is a realization of the Cartesian product of complexes $(V_1; K_1) \times (V_2; K_2)$. This gives an alternative proof of Corollary 2.6.14.

2.S.9. The set of simplicial maps between a fixed pair of complexes may be equipped with a relation of neighbourliness. Two simplicial maps $\varphi, \psi: K \to \mathcal{L}$ are said to be *neighbourly* if for every simplex $\Delta \in K$ there is a simplex $\Gamma \in \mathcal{L}$ such that $\varphi(\Delta)$ and $\psi(\Delta)$ are faces of Γ. For instance any two simplicial approximations of one map $f: |K| \to |\mathcal{L}|$ are neighbourly. The relation of neighbourliness is reflexive and symmetric but not in general transitive (see Problem 2.P.23). We say that the simplicial maps $\varphi, \psi: K \to \mathcal{L}$ are *neighbourly in the wider sense* if there is a sequence of simplicial maps $\varphi_j: K \to \mathcal{L}$ where $j = 1, 2, \ldots, n$ such that $\varphi = \varphi_1$, $\psi = \varphi_n$ and the maps φ_j, φ_{j+1} are neighbourly for $j = 1, 2, \ldots, n-1$. Evidently the relation of neighbourliness in the wider sense is an equivalence.

2.S.10. A simplicial map $\varphi: K \to \mathcal{L}$ induces a simplical map of the barycentric subdivisions. Define $\varphi': K' \to \mathcal{L}'$ as follows. Let $\Delta(b_0, b_1, \ldots, b_q) \in K'$ where b_h is the barycentre of a simplex $\Delta_h \in K$ for $h = 0, 1, \ldots, q$ and $\emptyset \neq \Delta_0 < \Delta_1 < \ldots < \Delta_q$. Let $\varphi'(b_h)$ be the barycentre of the simplex $\varphi(\Delta_h)$ for $h = 0, 1, \ldots, q$. Since $\varphi(\Delta_0) \leq \varphi(\Delta_1) \leq \ldots \leq \varphi(\Delta_q)$, we may define $\varphi'(\Delta(b_0, b_1, \ldots, b_h)) = \Delta\{\varphi'(b_0), \varphi'(b_1), \ldots, \varphi'(b_q)\} \in \mathcal{L}'$.

2.S.11. The theory of cells and cell complexes may be described in an equivalent way which is a direct generalization of our presentation of the theory of simplices and simplicial complexes. Let $a_0, a_1, \ldots, a_k$ be points of Euclidean space $\mathbf{R}^m$ (which may form an affinely dependent set). It is easily seen (see Problem 2.P.28) that the set $\operatorname{conv}\{a_0, a_1, \ldots, a_k\}$ is a cell. Any hyperplane which intersects a cell $Q \subset \mathbf{R}^m$ but is disjoint from its interior is called a *supporting hyperplane*. The intersection of a cell Q with any supporting hyperplane is called a *face* of Q. From this definition it follows that the faces of a cell Q are themselves cells. We regard the empty set as a (-1)-dimensional face of every cell and the cell itself as an improper face. The 0-dimensional faces of a cell are called its *vertices*. It turns out (see Problem 2.P.28) that every cell has a finite number of vertices $v_0, v_1, \ldots, v_n$ and if $Q = \operatorname{conv}\{a_0, a_1, \ldots, a_k\}$ then $\{v_0, v_1, \ldots, v_n\} \subset \{a_0, a_1, \ldots, a_k\}$ and $Q = \operatorname{conv}\{v_0, v_1, \ldots, v_n\}$. Moreover every face of the cell Q has the form $\operatorname{conv}\{b_0, b_1, \ldots, b_p\}$, where $\{b_0, b_1, \ldots, b_p\} \subset \{v_0, v_1, \ldots, v_n\}$;

thus a cell has finitely many faces. However, in general not every subset of the vertices of a cell Q defines a face. It may be shown (see Problem 2.P.29 and Corollary 2.1.12) that every cell is the union of the geometric interiors of all of its faces, these interiors being pairwise disjoint.

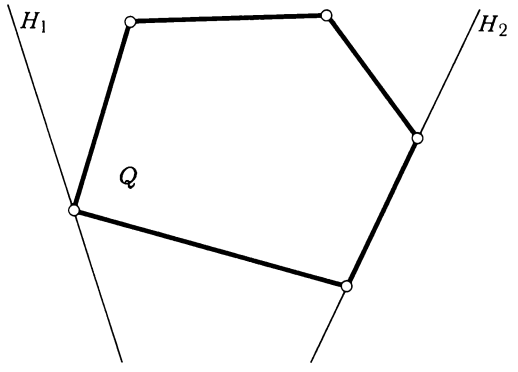


Fig.61. Supporting hyperplanes H_1 and H_2 of the cell Q.
The intersection $H_1 \cap Q$ is a vertex and $H_2 \cap Q$ is a 1-dimensional face of the cell Q.

It turns out (see Problem 2.P.30 and conditions (SC1) and (SC2)) that if a family K of cells in Euclidean space $\mathbf{R}^m$ satisfies the following conditions:

(1) the family K contains all the faces of each cell in K, and

(2) the intersection of two cells of the family K is their common face,

then K is a cell complex.

2.P. Problems

2.P.1. Show that for $n = 0, 1, \ldots$ there is an n-dimensional simplicial complex which is not simplicially isomorphic to any complex in the Euclidean space $\mathbf{R}^{2n}$. (Hint: Consider the n-dimensional skeleton of the complex consisting of the faces of the $(2n+2)$-dimensional simplex.)

2.P.2. Prove that the condition given in Theorem 2.5.8 characterizes simplicial approximations.

2.P.3. Suppose the map $f\colon |K| \to |\mathcal{L}|$ satisfies the condition $f(|K_0|) \subset |\mathcal{L}_0|$ where K_0 is a subcomplex of K and $\mathcal{L}_0$ is a subcomplex of $\mathcal{L}$. Show that if φ is a simplicial approximation of the map f, then $\varphi(K_0) \subset \mathcal{L}_0$ and $\varphi | K_0$ is a simplicial approximation of $f | |K_0|$.

2.P.4. Let $\Delta = \Delta(a_0, a_1, \ldots, a_n)$ and $\Delta_j = \Delta(a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$ for $j = 0, 1, \ldots, n$. Show that if the continuous map $f\colon \Delta \to \Delta$ satisfies the condition $f(\Delta_j) \subset \Delta_j$ for $j = 0, 1, \ldots, n$, then $f(\Delta) = \Delta$. (Hint: Apply Sperner's Lemma.)

2.P.5. Prove that if a system $F_0, F_1, \ldots, F_n$ of closed subsets of $\Delta(a_0, a_1, \ldots, a_n)$ satisfies the condition $\Delta(a_{i_0}, a_{i_1}, \ldots, a_{i_k}) \subset F_{i_0} \cup F_{i_1} \cup \ldots \cup F_{i_k}$ for any system of indices with $0 \leq i_0 < i_1 < \ldots < i_k \leq n$, then $\bigcap_{j=0}^{n} F_j \neq \emptyset$. (Hint: Apply Sperner's Lemma.)

2.P.6. Prove that if a system $F_0, F_1, \ldots, F_n$ of closed subsets of $\Delta(a_0, a_1, \ldots, a_n)$ satisfies the condition $F_0 \cup F_1 \cup \ldots \cup F_n = \Delta(a_0, a_1, \ldots, a_n)$ and $\Delta(a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) \subset F_j$ for $j = 0, 1, \ldots, n$, then $\bigcap_{j=0}^{n} F_j \neq \emptyset$. (Hint: Use the result in Problem 2.P.5.)

2.P.7. Show that for every bounded set $A \subset \mathbf{R}^m$ and every real number $\epsilon > 0$ there is a polyhedron X such that $A \subset X \subset B(A; \epsilon)$.

2.P.8. Show that the geometric dimension of a metric product of non-empty polyhedra equals the sum of the geometric dimensions of the factors.

2.P.9. Show that if X and Y are polyhedra in Euclidean space $\mathbf{R}^m$, then the set $\mathrm{cl}(X \backslash Y)$ is also a polyhedron.

2.P.10. Show that if X is a polyhedron in Euclidean space $\mathbf{R}^m$, then its boundary in $\mathbf{R}^m$, $\mathrm{bd}\, X$, is also a polyhedron.

2.P.11. Show that the m-dimensional projective space P^m is a curvilinear polyhedron (see Supplement 2.S.2). (Hint: Let $a_i = (\delta_i^1, \delta_i^2, \ldots, \delta_i^{m+1})$ and let $b_i = -a_i$ for $i = 1, 2, \ldots, m+1$. Let $Q = \mathrm{conv}\{a_1, a_2, \ldots, a_{m+1}, b_1, b_2, \ldots, b_{m+1}\}$. Consider the natural triangulation of the boundary of Q, take its barycentric subdivision and then make the appropriate identifications. Compare also Example 5.3.9 and [5] p. 133, 134.)

2.P.12. Compute the Euler-Poincaré characteristic of the n-dimensional sphere S^n and of the m-dimensional projective space P^m (see Supplement 2.S.3).

2.P.13. Prove that:

a) If K_1 and K_2 are subcomplexes of a simplicial complex K, then $\chi(K_1 \cup K_2) + \chi(K_1 \cap K_2) = \chi(K_1) + \chi(K_2)$.

b) If K' is the barycentric subdivision of the simplicial complex K, then $\chi(K') = \chi(K)$.

c) If K_1 and K_2 are simplicial complexes, then for some triangulation of the polyhedron $|K_1| \times |K_2|$ (see e.g. Supplement 2.S.8) we have $\chi(|K_1| \times |K_2|) = \chi(K_1)\chi(K_2)$ (see Supplement 2.S.3).

2.P.14. Show that every simplicial complex is the nerve of the covering of its underlying space by the closed stars of those vertices of the barycentric subdivision which belong to the complex (see Supplement 2.S.5).

2.P.15. Show that if $\Delta_0, \Delta_1, \ldots, \Delta_k \in K$, then $\mathrm{st}\, \Delta_0 \cap \mathrm{st}\, \Delta_1 \cap \ldots \mathrm{st}\, \Delta_k \neq \emptyset$ if and only if there is a simplex $\Delta \in K$ such that $\Delta_j \leq \Delta$ for $j = 0, 1, \ldots, k$ (see Supplement 2.S.5; compare Lemma 2.3.13).

2.P.16. Show that, for every simplex Δ of a simplicial complex K, the star $\operatorname{st}\Delta$ is an open set in $|K|$ (see Supplement 2.S.5; cf. Assertion 2.3.15).

2.P.17. Show that the inductive method of defining the barycentric subdivision described in Supplement 2.S.6 is equivalent to the definition used in Section 2.4.

2.P.18. Check that the construction of the relative barycentric subdivision $K' \bmod K_0$ presented in Supplement 2.S.6 is valid. Define inductively the relative barycentric subdivision of order p, $K^{(p)} \bmod K_0$. Show that if K_0 is a simplicial subcomplex of the complex K, then for every real number $\epsilon > 0$ there is a number p such that every simplex Δ of the subdivision $K^{(p)} \bmod K_0$ either has diameter less than ϵ or is contained in $B(|K_0|, \epsilon)$.

2.P.19. Suppose that $\varphi: K \to \mathcal{L}$ is a simplicial map, K_0 is a simplicial subcomplex of the complex K and $\mathcal{L}_0$ is a simplicial subcomplex of the complex $\mathcal{L}$. Find assumptions under which φ induces a simplicial map $\varphi': K' \bmod K_0 \to \mathcal{L}' \bmod \mathcal{L}_0$ (see Supplement 2.S.6) which generalizes the construction presented in Supplement 2.S.10.

2.P.20. Show that an abstract complex K is finite (cf. Supplement 2.S.7) if and only if the underlying space $|K|$ is compact.

2.P.21. Show that if the simplicial complex K_s is a realization of an abstract complex K_a (cf. Supplement 2.S.7), then the underlying spaces $|K_s|$ and $|K_a|$ are homeomorphic.

2.P.22. Let $|K|$ be the underlying space of an abstract complex K; for each abstract simplex $A \in K$ regard the space $|A|$ as a subset of the space $|K|$ (see Supplement 2.S.7). Let $\mathcal{U}$ be the family of subsets U of the space $|K|$ which have the property that $U \cap |A|$ is open in $|A|$ for each abstract simplex $A \in K$. Show that for $\mathcal{U}$ to be the family of all open sets of $|K|$ it is necessary and sufficient that K be locally finite.

2.P.23. Give an example of simplicial maps $\varphi, \psi, \pi: K \to \mathcal{L}$ such that φ and ψ are neighbourly and ψ and π are neighbourly but φ and π are not neighbourly (see Supplement 2.S.9).

2.P.24. Show that for every polyhedron $|\mathcal{L}|$ there is a real number $\epsilon > 0$ so that for any polyhedron $|K|$ and arbitrary maps $f, g: |K| \to |\mathcal{L}|$ satisfying the inequality $\rho(f(x), g(x)) < \epsilon$ for $x \in |K|$ there is an integer $p \geq 0$ and a common simplicial approximation $\varphi: K^{(p)} \to \mathcal{L}$.

2.P.25. Show that the boundary of an n-dimensional cell is a polyhedron of dimension $n-1$.

2.P.26. Let the polyhedron X_i be the underlying space of some realization of an ordered abstract complex $(V_i; K_i)$ for $i = 1, 2$ (cf. Supplement 2.S.7). Show that the Cartesian product $(V_1; K_1) \times (V_2; K_2)$ defined in Supplement 2.S.8 is an abstract complex which has a realization in the form of a triangulation of the metric product $X_1 \times X_2$.

2.P.27. Suppose the cells $Q_1, Q_2 \ldots, Q_k$ form a cell complex in Euclidean space $\mathbf{R}^m$ and suppose the cells $Q'_1, Q'_2, \ldots, Q'_{k'}$ also form a cell complex in Euclidean space $\mathbf{R}^{m'}$. Investigate whether the cells $Q_i \times Q'_{i'}$ for $i = 1, 2, \ldots, k$ and $i' = 1, 2, \ldots, k'$ form a cell complex in the space $\mathbf{R}^{m+m'}$.

2.P.28. Let $a_0, a_1, \ldots, a_k \in \mathbf{R}^m$. Show that the set $Q = \operatorname{conv}\{a_0, a_1, \ldots, a_k\}$ is a cell. Next show that if $v_0, v_1, \ldots, v_n$ are all the vertices of Q, then $\{v_0, v_1, \ldots, v_n\} \subset \{a_0, a_1, \ldots, a_k\}$, $Q = \operatorname{conv}\{v_0, v_1, \ldots, v_n\}$ and every face of the cell Q is of the form $\operatorname{conv}\{b_0, b_1, \ldots, b_p\}$ where $\{b_0, b_1, \ldots, b_p\} \subset \{v_0, v_1, \ldots, v_n\}$ (cf. Supplement 2.S.11).

2.P.29. Show that the boundary of a cell is the union of all the proper faces. Show that a cell is the union of the geometric interiors of all its faces, and that the interiors are pairwise disjoint (see Supplement 2.S.11).

2.P.30. Show that for a family K of cells to be a cell complex it is sufficient that it satisfies conditions (1) and (2) of Supplement 2.S.11. Are these conditions necessary?

2.P.31. Prove that for any two triangulations K_1 and K_2 of a polyhedron X there is a triangulation K' of the polyhedron, which is a common subdivision of K_1 and K_2.

Chapter 3

Homotopy

The concept of homotopy finds application in almost all branches of contemporary topology. The topic of this chapter is thus broad; apart from homotopy theory it also embraces those problems of map theory for which the natural tool of investigation is homotopy.

Section 3.1 is dedicated to continuous extensions of maps. We prove three theorems which are among the fundamental theorems of topology, namely: Tietze's Theorem, the theorem asserting the non-existence of a retraction of the ball onto its boundary, and Brouwer's Fixed Point Theorem. Moreover, in this section we will also be looking at the notion of pathwise connectedness.

Section 3.2 takes up the notion of homotopic map. After studying the basic properties of this notion we move on to consider the relation of homotopy to the extendability of maps and prove two theorems of Borsuk on homotopy extensions. Next we define the class of contractible spaces and prove among other things that spheres are not contractible. We conclude the section with a discussion of deformation retracts and homotopy type.

In Section 3.3 we introduce the notion of a fibration with special attention given to the case of a covering. Among the various examples, we describe the Hopf fibration. Our study of fibration theory in a chapter on homotopy is justified by the theorem on homotopy lifting, which we prove both in its absolute and in its relativised formulation. We close the section by considering the uniqueness problem of continuous liftings in the case of covering maps.

Section 3.4 is concerned with the fundamental group. We begin by defining operations on loops, then we introduce the concept of the fundamental group and identify the fundamental group of the circle. We continue by considering the nature of the dependence of the fundamental group on the choice of the base point, we study the homomorphisms of the fundamental group induced by continuous maps of the space and prove that the fundamental group is an invariant of the homotopy type; we also determine the fundamental group of the metric product of spaces. Next we develop the theory of the edge-group of a polyhedron, prove the fundamental theorem on its isomorphism with the fundamental group, and develop an algorithm which enables us to determine the fundamental group of an arbitrary polyhedron. Using these methods we prove Van Kampen's Theorem on the fundamental group of a union of polyhedra. The section closes with an application in which we use the fundamental group to solve the existence problem for continuous liftings in the case of covering maps.

3.1. Extensions of continuous maps

If $A \subset X$ and $f: A \to Y$, then any continuous map $f^*: X \to Y$ such that $f^* | A = f$ is called a *continuous extension* of the map f from the set A onto the space X. In other words, if we denote by $i: A \to X$ the inclusion map of the set A into the space X, then the continuous map $f^*: X \to Y$ is a continuous extension of the map $f: A \to Y$ if $f^* i = f$.

3.1.1. EXAMPLE. The constant map $c: A \to Y$ where $A \subset X$ and $c(a) = y_0$ for all $a \in A$ always has a continuous extension $c^*: X \to Y$ defined by the formula $c^*(x) = y_0$ for $x \in X$. ■

3.1.2. EXAMPLE. The identity map $f: \{0,1\} \to \{0,1\}$ does not have a continuous extension $f^*: I \to \{0,1\}$ since it would map the connected space I onto the disconnected space $\{0,1\}$. ■

We now prove a fundamental theorem on the existence of continuous extensions. We begin with the following lemma.

3.1.3. LEMMA. *Let A be a non-empty closed subset of a metric space X. For each point $x \in X \backslash A$ define the function $g_x: A \to \mathbf{R}$ by the formula $g_x(a) = \frac{\rho(x,a)}{\rho(x,A)} - 1$ for $a \in A$ and let $A_x = \{a \in A : \rho(x,a) < 2\rho(x,A)\}$. Then*

(1) $\inf g_x = 0$ *for each* $x \in X \backslash A$,

(2) $\inf g_x | (A \backslash A_x) \geq 1$ *for each* $x \in X \backslash A$,

(3) $\inf g_x | A_x = 0$ *for each* $x \in X \backslash A$,

(4) $|g_x(a) - g_{x'}(a)| \leq 3\rho(x,x')/r$ *for each* $x \in X \backslash A$, $x' \in X \backslash B(A;r)$, *where* $r > 0$ *and* $a \in A_x$,

(5) *for every* $a_0 \in A$ *and* $\epsilon > 0$ *there is a real number* $\delta > 0$ *such that if* $A_0 = A \cap B(a_0, \epsilon)$ *then we have* $\inf g_x | (A \backslash A_0) \geq 1$ *and* $\inf g_x | A_0 = 0$ *for each* $x \in (X \backslash A) \cap B(a_0; \delta)$.

PROOF. Conditions (1) and (2) follow immediately from the definitions of g_x and A_x. Condition (3) is a consequence of (1) and (2).

To prove condition (4) we observe that

$$\begin{aligned} |g_x(a) - g_{x'}(a)| &= \left| \frac{\rho(x,a)}{\rho(x,A)} - \frac{\rho(x',a)}{\rho(x',A)} \right| \\ &= \left| \frac{\rho(x,a)}{\rho(x,A)} \cdot \frac{\rho(x',A) - \rho(x,A)}{\rho(x',A)} + \frac{\rho(x,a) - \rho(x',a)}{\rho(x',A)} \right| \\ &\leq \frac{\rho(x,a)}{\rho(x,A)} \cdot \frac{|\rho(x',A) - \rho(x,A)|}{\rho(x',A)} + \frac{|\rho(x,a) - \rho(x',a)|}{\rho(x',A)}, \end{aligned}$$

so applying Theorem 1.4.10 and the triangle inequality we obtain

$$|g_x(a) - g_{x'}(a)| \leq \frac{\rho(x,a)}{\rho(x,A)} \cdot \frac{\rho(x,x')}{\rho(x',A)} + \frac{\rho(x,x')}{\rho(x',A)}.$$

By hypothesis $\frac{\rho(x,a)}{\rho(x,A)} < 2$ and $r \leq \rho(x',A)$, hence the required inequality.

To prove condition (5) we take $\delta = \frac{1}{3}\epsilon$. Then whenever $x \in (X\backslash A) \cap B(a_0;\delta)$, we have for each $a \in A_x$ that

$$\rho(a,a_0) \leq \rho(x,a) + \rho(x,a_0) < 2\rho(x,A) + \rho(x,a_0) \leq 3\rho(x,a_0) < 3\delta = \epsilon,$$

and so $a \in A_0$; that is, $A_x \subset A_0$. The proposition follows from conditions (2) and (3). ■

We now prove the promised theorem on the extension of a continuous map.

3.1.4. THEOREM (Tietze). *Let A be a closed subset of a metric space X. Every continuous map $f: A \to I$ has a continuous extension $f^*: X \to I$.*

PROOF. We may of course assume that $A \neq \emptyset$. We will follow the notation of Lemma 3.1.3 and in the course of the proof we will also make use of the well-known inequalities:

(6) $\inf f + \inf g \leq \inf(f+g) \leq \sup f + \inf g$,

(7) $|\inf f - \inf g| \leq \sup|f-g|$.

Put $f^*(x) = f(x)$ for $x \in A$ and $f^*(x) = \inf(f+g_x)$ for $x \in X\backslash A$. Using (6) and (1) we obtain

$$0 \leq \inf(f+g_x) \leq \sup f + \inf g_x \leq 1 + 0 = 1$$

for $x \in X\backslash A$ so $f^*: X \to I$.

To prove continuity of the function f^* on the set $X\backslash A$ observe that condition (2) implies the equation $f^*(x) = \inf(f+g_x)|A_x$ for $x \in X\backslash A$. Assume that $f^*(x) \leq f^*(x')$ where $x, x' \in X\backslash A$. Then using (7) we obtain

$$\begin{aligned} f^*(x') - f^*(x) &= \inf(f+g_{x'}) - \inf(f+g_x)|A_x \\ &\leq \inf(f+g_{x'})|A_x - \inf(f+g_x)|A_x \\ &\leq \sup|g_{x'}|A_x - g_x|A_x|. \end{aligned}$$

If $x' \in X\backslash B(A;r)$, then from condition (4) we obtain $f^*(x') - f^*(x) \leq 3\rho(x,x')/r$, and so the function f^* is continuous on $X\backslash A$.

Now consider a point $a_0 \in A$ and a real number $\eta > 0$. Let the real number $\epsilon > 0$ have the property that, if $a \in A$ and $\rho(a,a_0) < \epsilon$, then $|f(a) - f(a_0)| < \frac{1}{2}\eta$. Choose a real number δ as in (5) and let $x \in (X\backslash A) \cap B(a_0,\delta)$. In view of (6) and (7) we have $\inf f|A_0 \leq \inf(f+g_x)|A_0 - \inf g_x|A_0 \leq \sup f|A_0$, but (5) implies that $\inf(f+g_x)|A_0 - \inf g_x|A_0 = \inf(f+g_x)|A_0 = \inf(f+g_x) = f^*(x)$; thus $\inf f|A_0 \leq f^*(x) \leq \sup f|A_0$ whence $|f^*(x) - f(a_0)| \leq \eta$. The map f^* is therefore continuous. ■

We now prove a variant of Tietze's Theorem.

3.1.5. THEOREM. *Let A be a closed subset of a metric space X. Every continuous function $f: A \to \mathbf{R}$ has a continuous extension $f^*: X \to \mathbf{R}$.*

PROOF. Since the real line $\mathbf{R}$ is homeomorphic with the open interval $(0,1)$ it suffices to prove that every continuous function $f: A \to (0,1)$ has a continuous extension $f^*: X \to (0,1)$. Let i denote the inclusion map of the interval $(0,1)$ into the interval

I. By Tietze's Theorem the composition map $if: A \to I$ has a continuous extension $f_1: X \to I$. Now the set $B = f_1^{-1}(\{0,1\})$ is closed and disjoint from A so using Lemma 1.6.26 we obtain a function $g: X \to I$ such that $g(x) = 0$ for $x \in A$ and $g(x) = 1$ for $x \in B$. Taking $f^*(x) = f_1(x)(1 - g(x)) + \frac{1}{2}g(x)$ for $x \in X$ we obtain a function $f^*: X \to (0,1)$ which is a continuous extension of the function f. ■

Tietze's Theorem also implies the following.

3.1.6. COROLLARY. *Let A be a closed subset of a metric space X. Every continuous map $f: A \to I^m$ has a continuous extension $f^*: X \to I^m$.*

PROOF. Suppose $f(x) = (f^1(x), f^2(x), \ldots, f^m(x))$ for $x \in A$ where $f^i: A \to I$ for $i = 1,2,\ldots,m$. By Tietze's Theorem each function f^i for $i = 1,2,\ldots,m$ has a continuous extension $f^{i*}: X \to I$. Taking $f^*(x) = (f^{1*}(x), f^{2*}(x), \ldots, f^{m*}(x))$ for $x \in X$ we obtain a continuous extension $f^*: X \to I$ of the map f. ■

Similarly, applying Theorem 3.1.5 we obtain the following.

3.1.7. COROLLARY. *Let A be a closed subset of a metric space X. Every continuous map $f: A \to \mathbf{R}^m$ has a continuous extension $f^*: X \to \mathbf{R}^m$.* ■

The possibility of replacing the unit interval in Tietze's Theorem by other spaces $(\mathbf{R}, I^m, \mathbf{R}^m)$ established in Theorem 3.1.5 and Corollaries 3.1.6 and 3.1.7 suggests that a class of spaces could be defined in this way. We shall carry this out in Chapter 6 when we introduce the concept of an absolute extensor (see Supplement 6.S.15) which coincides with the concept of an absolute retract (compare Theorems 6.6.1, 6.6.2, 6.6.5 and Supplement 6.S.15).

The next theorem is concerned with the problem of the uniqueness of a continuous extension.

3.1.8. THEOREM. *Let A be a dense subset of a metric space X. Every continuous map $f: A \to Y$ has at most one continuous extension $f^*: X \to Y$.*

PROOF. Let $f_1^*, f_2^*: X \to Y$ be continuous extensions of the map f and let $x \in X$. Since A is dense in X, there exists a sequence of points $a_n \in A$ for $n = 1,2,\ldots$ such that $\lim_n a_n = x$. Then $f_1^*(a_n) = f(a_n) = f_2^*(a_n)$ for $n = 1,2,\ldots$, and so $f_1^*(x) = \lim_n f_1^*(a_n) = \lim_n f_2^*(a_n) = f_2^*(x)$. It follows that $f_1^* = f_2^*$. ■

We now concern ourselves with the question of continuously extending an identity map defined on a subset of a space. Let X be any metric space and let $A \subset X$. Any continuous map $r: X \to A$ which satisfies the equation $r(a) = a$ for $a \in A$ is called a *retraction* of the space X onto the set A. A retraction $r: X \to A$ is thus a continuous extension of the identity map $\mathrm{id}_A: A \to A$ onto the space X. If a retraction of the space X onto a set A exists, then we say that the set A is a *retract* of the space X.

3.1.9. EXAMPLE. *The constant map $r: X \to \{x_0\}$, where $x_0 \in X$, is a retraction. Every point of a space is its retract.* ■

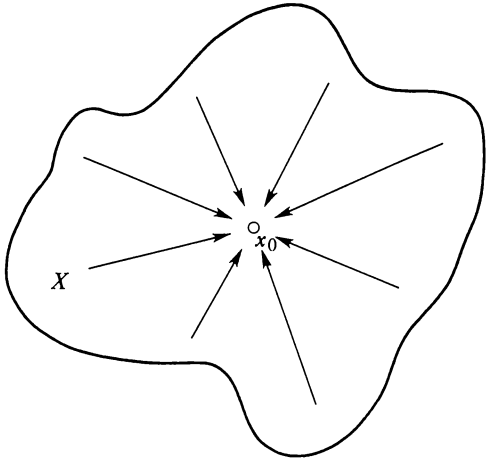


Fig.62. A constant map of a space X onto an arbitrary point x_0 of the space is a retraction (Example 3.1.9).

3.1.10. EXAMPLE. Let $X = X_1 \times X_2$ and suppose $A = \{(x_1, x_2) \in X : x_2 = \bar{x}_2\}$, where $\bar{x}_2$ is a fixed point of the space X_2. *The map* $r: X \to A$ *defined by the formula* $r(x_1, x_2) = (x_1, \bar{x}_2)$ *for* $(x_1, x_2) \in X$ *is a retraction.* In particular the projection of a square onto its side is a retraction; the side of a square is thus a retract of the square. ■

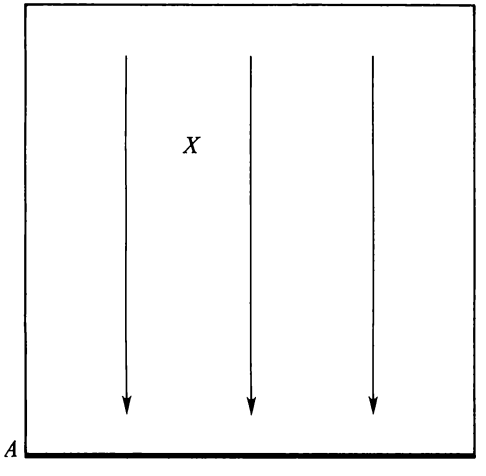


Fig.63. The side A of the square X is its retract (Example 3.1.10).

3.1.11. EXAMPLE. *The* $(m-1)$*-dimensional sphere* S^{m-1} *is a retract of the complement* $\bar{B}^m \setminus \{0\}$ *for each* $m > 0$. The map $r: \bar{B}^m \setminus \{0\} \to S^{m-1}$ defined by $r(x) = x/\|x\|$ for $x \in \bar{B}^m \{0\}$ is the desired retraction. It follows immediately from this example that the *boundary* $\operatorname{bd} \Delta$ *of any simplex* Δ *is a retract of the set* $\Delta \setminus \{b\}$ *whenever* b *is a point of the geometric interior* $\operatorname{int} \Delta$. ■

3.1.12. EXAMPLE. *The hemisphere* $S^{m-1}_+ = \{(x^1, x^2, \ldots, x^m) \in S^{m-1} : x^m \geq 0\}$ *is a retract of the half-ball* $\bar{B}^m_+ = \{(x^1, x^2, \ldots, x^m) \in \bar{B}^m : x^m \geq 0\}$ *for each* $m > 0$. For it is easy to check, that taking

$$r(x) = \frac{2(x^m + 1)}{\|x\| + 2x^m + 1}\left(x^1, x^2, \ldots, x^{m-1}, x^m + \frac{1 - \|x\|^2}{2(x^m + 1)}\right)$$

for each $x = (x^1, x^2, \ldots, x^m) \in \bar{B}^m_+$ we have $r(x) \in S^{m-1}_+$ and that $r(x) = x$ when $x \in S^{m-1}_+$.

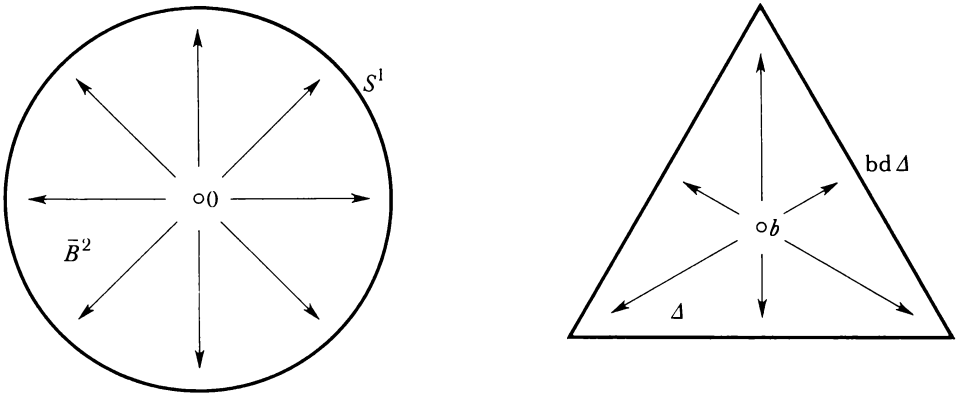


Fig.64. The circumference S^1 is a retract of the disc $\bar{B}^2$ with the point 0 removed; hence the boundary of a triangle is a retract of the complement $\Delta \setminus \{b\}$ where $b \in \operatorname{int} \Delta$ (Example 3.1.11 for the case $m = 2$).

Let Δ be any $(m-1)$-dimensional simplex with $m \geq 0$. Since it is easy to find a homeomorphism h of the metric product $\Delta \times I$ onto the half-ball $\bar{B}^m_+$ such that $h(\Delta \times \{0\} \cup (\operatorname{bd} \Delta) \times I) = S^{m-1}_+$, it follows from the last example that *the set* $\Delta \times \{0\} \cup (\operatorname{bd} \Delta) \times I$ *is a retract of the product* $\Delta \times I$. ■

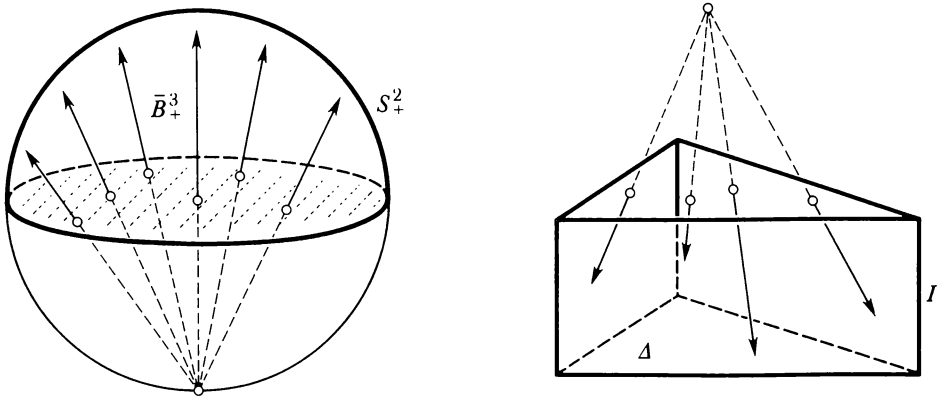


Fig.65. The hemisphere S^2_+ is a retract of the half-ball B^3_+; hence the set $\Delta \times \{0\} \cup (\operatorname{bd} \Delta) \times I$ is a retract of the product $\Delta \times I$, where Δ is the 2-dimensional simplex (Example 3.1.12 for the case $m = 3$).

As a generalization of this example, we prove the following.

3.1.13. THEOREM. *If $\mathcal{L}$ is a simplicial subcomplex of a simplicial complex K, then the set $|K| \times \{0\} \cup |\mathcal{L}| \times I$ is a retract of the metric product $|K| \times I$.*

PROOF. The proof will proceed by induction on the number of simplices in $K \setminus \mathcal{L}$. If $k = 0$, then $K = \mathcal{L}$ and the proposition obviously holds. Assume that the theorem is true for all pairs K, $\mathcal{L}$ for which $K \setminus \mathcal{L}$ consists of k simplices with $k \geq 0$. Consider a simplicial complex K and a subcomplex $\mathcal{L}$ such that $K \setminus \mathcal{L}$ consists of $k+1$ simplices. Let Δ be a simplex of maximal dimension in $K \setminus \mathcal{L}$; then Δ is not a proper face of any simplex in the complex K; taking $K_0 = K \setminus \{\Delta\}$ we obtain a simplicial complex of which $\mathcal{L}$ is a subcomplex. By the inductive hypothesis there exists a retraction $r_0 \colon |K_0| \times I \to |K_0| \times \{0\} \cup |\mathcal{L}| \times I$. Now taking $r_0(x,0) = (x,0)$ for $x \in \Delta$, we may straightaway assume that r_0 is a retraction of the set $\Delta \times \{0\} \cup |K_0| \times I$ onto $|K| \times \{0\} \cup |\mathcal{L}| \times I$. According to Example 3.1.12 there is a retraction $r_1 \colon \Delta \times I \to \Delta \times \{0\} \cup (\operatorname{bd} \Delta) \times I$. Moreover, taking $r_1(x,s) = (x,s)$ for $(x,s) \in |K_0| \times I$ we may at once assume that r_1 is a retraction of the set $|K| \times I$ onto $\Delta \times \{0\} \cup |K_0| \times I$. Then the composition $r = r_0 r_1$ is a retraction of the set $|K| \times I$ onto $|K| \times \{0\} \cup |\mathcal{L}| \times I$. This completes the proof. ■

We now prove the following.

3.1.14. THEOREM. *Every retract of a space X is a closed subset.*

PROOF. Let A be a retract of X and $r \colon X \to A$ any retraction. By Example 1.3.7 the map $f \colon X \to \mathbf{R}$ defined by the formula $f(x) = \rho(x, r(x))$ for $x \in X$ is continuous. Since $A = f^{-1}(0)$, A is a closed set in X by Theorem 1.6.24. ■

We now prove an important theorem which on the one hand generalizes the property verified in Example 3.1.2 and on the other shows that Example 3.1.11 cannot be improved by substituting the whole ball $\bar{B}^m$ for the set $\bar{B}^m \setminus \{0\}$.

3.1.15. THEOREM. *The sphere S^{m-1} is not a retract of the ball $\bar{B}^m$ for any m.*

PROOF. By Corollary 2.1.9 it suffices to prove that the boundary of the n-dimensional simplex Δ is not a retract of Δ for any $m \geq 0$. Suppose for the sake of argument that there is a retraction $r \colon \Delta \to \operatorname{bd} \Delta$. Let K be the triangulation of the simplex Δ consisting of all of its faces (see Example 2.3.1). Let K_0 be the triangulation of the boundary $\operatorname{bd} \Delta$ consisting of the proper faces of Δ (see Example 2.3.2). By Theorem 2.5.10 there is a non-negative integer p such that r has a simplicial approximation $\varphi \colon K^{(p)} \to K_0$. We regard this approximation as a simplicial map $\varphi \colon K^{(p)} \to K$. Let a be any vertex of the complex $K^{(p)}$. If $a \in \operatorname{int} \Delta$, then of course $a \in \operatorname{st} \varphi(a)$ since $\varphi(a)$ is one of the vertices of the simplex Δ. If however $a \in \operatorname{bd} \Delta$, then $a = r(a) \in r(\operatorname{st} a) \subset \operatorname{st} \varphi(a)$ in view of the definition of a retraction and of a simplicial approximation. Thus in either case $a \in \operatorname{st} \varphi(a)$ and by Sperner's Lemma (2.5.12) there is a simplex $\Delta' \in K^{(p)}$ such that $\varphi(\Delta') = \Delta$. But this is impossible since $\Delta \notin K_0$. ■

From the above theorem we easily obtain the following.

3.1.16. THEOREM (Brouwer). *For every continuous map $f \colon \bar{B}^m \to \bar{B}^m$ there is a point $x \in \bar{B}^m$ such that $f(x) = x$.*

PROOF. Suppose that $f(x) \neq x$ for every point $x \in \bar{B}^m$. For each $x \in \bar{B}^m$ consider the half-line with endpoint $f(x)$ passing through x. It is easy to check that the half-line cuts the sphere S^{m-1} in exactly one point $r(x) = x + s(x)g(x)$ where $g(x) = (x - f(x))/\|x - f(x)\|$ and $s(x) = -x \cdot g(x) + \sqrt{1 - x \cdot x + (x \cdot g(x))^2}$. We thus obtain a continuous map $r: \bar{B}^m \to S^{m-1}$. If $x \in S^{m-1}$, then $x \cdot x = 1$ and $s(x) = 0$ so $r(x) = x$. The map r is therefore a retraction of the ball $\bar{B}^m$ onto the sphere S^{m-1} contrary to Theorem 3.1.15. ■

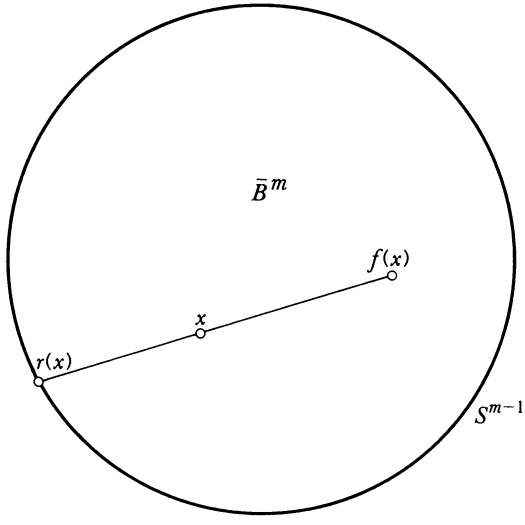


Fig.66. If a map $f: B^m \to B^m$ with no fixed point existed, then there would exist a retraction $r: B^m \to S^{m-1}$ (see the proof of Brouwer's Theorem – 3.1.16).

If $f: X \to X$, then every point $x \in X$ satisfying the equation $f(x) = x$ is called a *fixed point* of the map f. Brouwer's Theorem thus states that every continuous map of the m-dimensional closed sphere $\bar{B}^m$ into itself has a fixed point.

If every continuous map of a metric space X into itself has a fixed point, then we say that the space has the *fixed point property*. Brouwer's Theorem may thus be stated more briefly as saying that the ball $\bar{B}^m$ for each m has the fixed point property. The fixed point property is obviously topological in character and so holds in any space which is homeomorphic to the ball $\bar{B}^m$; in particular it holds in the m-dimensional cube I^m for each m.

Observe that if $r: X \to A$ is a retraction of X onto A and a map $f: A \to A$ does not have a fixed point, then the composition $fr: X \to A \subset X$ also does not have a fixed point. This has the following consequence.

3.1.17. ASSERTION. *If a metric space has the fixed point property, then each retract of it has the property.* ■

3.1.18. COROLLARY. *Every retract of the ball $\bar{B}^m$ has the fixed point property.* ■

If a closed subset A of a space X is the retract of an open set U of X which contains A, then we say that A is a *neighbourhood retract* of the space X. Thus every retract of the space X is by Theorem 3.1.14 a neighbourhood retract of the space. However,

Example 3.1.11 shows that the sphere S^{m-1} is a neighbourhood retract of the ball $\bar{B}^m$, though by Theorem 3.1.15 it is not its retract.

Theorem 3.1.15 also implies that not every continuous map $f: A \to S^{m-1}$, with A a closed subspace of a metric space X, has a continuous extension to the whole space X; thus in Theorem 3.1.6 the cube I^m cannot be replaced by the sphere S^{m-1}. The following however is true.

3.1.19. THEOREM. *Let A be a closed subset of a metric space X. For every continuous map $f: A \to S^{m-1}$ with $m > 0$ there is an open set U of X containing A and a continuous extension $f^*: U \to S^{m-1}$.*

PROOF. Let i be the inclusion map of S^{m-1} into the closed sphere $\bar{B}^m$. By Theorem 3.1.6 there exists a continuous extension $f_1: X \to \bar{B}^m$ of the composition $if: A \to \bar{B}^m$. Let $r: \bar{B}^m \setminus \{0\} \to S^{m-1}$ be the retraction defined by $r(x) = x/\|x\|$ for $x \in \bar{B}^m \setminus \{0\}$ and let $U = f_1^{-1}(\bar{B}^m \setminus \{0\})$. Thus U is an open set containing A and the map $f^* = rf_1|U$ is a continuous extension of f onto U. ■

The property of the sphere S^{m-1} formulated in Theorem 3.1.19 will be used in Chapter 6 to define the class of absolute neighbourhood extensors (see Supplement 6.S.15), which coincide with the absolute neighbourhood retracts (see Theorem 6.6.3, 6.6.4, 6.6.7 and Supplement 6.S.15).

We now consider the problem of continuous extensions of maps from the two endpoints of the unit interval onto the whole interval. Let $x_0, x_1 \in X$. Every continuous map $d: I \to X$ such that $d(0) = x_0$ and $d(1) = x_1$ is called a *path* in the space X from the point x_0 to the point x_1; the point x_0 is called the *beginning* of the path and the point x_1 the *end* of the path. The path d from the point x_0 to the point x_1 is thus a continuous extension from the set $A = \{0, 1\}$ to the whole of the interval I of the map $d_A: A \to X$ defined by the formulas $d_A(0) = x_0$ and $d_A(1) = x_1$.

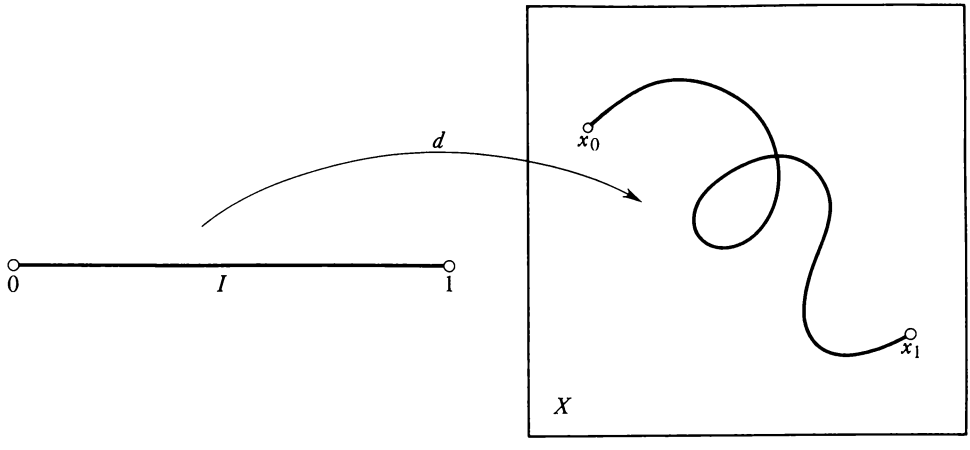


Fig.67. A path d from the point x_0 to the point x_1 in the space X.

3.1.20. EXAMPLE. Let X be a convex subset of the m-dimensional Euclidean space $\mathbf{R}^m$ and let $x_0, x_1 \in X$. The map $f: I \to X$ defined by the formula $f(x) = (1 - r)x_0 + rx_1$

for $r \in I$ is a path from the point x_0 to the point x_1. ■

If for any two points $x_0, x_1 \in X$ there is a path in the space X from x_0 to x_1, then we say that the space X is *pathwise connected* (see Supplement 3.S.3). Since the image of the unit interval I under any path from x_0 to x_1 is a connected set containing the points, we have from Theorem 1.7.6 the following.

3.1.21. ASSERTION. *Every pathwise connected space is connected.* ■

The next example shows that this implication cannot be reversed.

3.1.22. EXAMPLE. Let

$$A = \{(x^1, x^2) \in \mathbf{R} : x^2 = \sin \frac{1}{x^1},\ 0 < x_1 \leq 1\},$$

$$B = \{(x^1, x^2) \in \mathbf{R} : x^1 = 0,\ -1 \leq x^2 \leq 1\},$$

and let $X = A \cup B \subset \mathbf{R}^2$. The space X is connected, since the set A is connected and dense in X (see Theorem 1.7.12). We show that the space X is not pathwise connected.

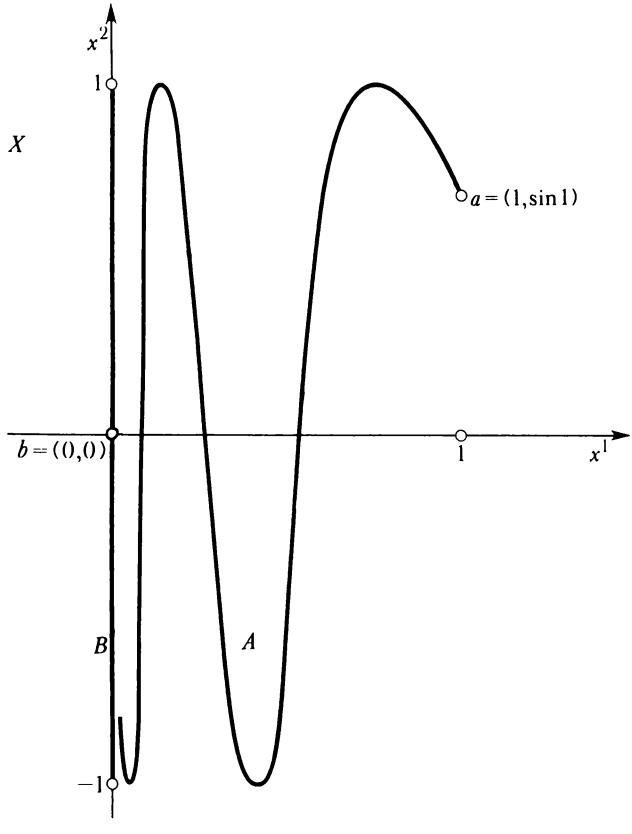


Fig.68. In the space X there is no path from the point $a = (1, \sin 1)$ to the point $b = (0, 0)$; thus the space is not pathwise connected (Example 3.1.22).

With this aim in mind observe that in X there is no path from the point $a = (1, \sin 1)$ to the point $b = (0,0)$. For, suppose that d is such a path and let $r_0 = \inf\{r \in I : d(r) \in B\}$; evidently $d(r_0) \in B$. Since the image $d([0, r_0])$ is connected, and whereas for each point $x \in A \setminus \{a\}$ the complement $X \setminus \{x\}$ is the union of two components of which one contains the point a and the other the point $d(r_0)$, we have $A \subset d([0, r_0])$. In view of the denseness of A in the space X we should have $X = d([0, r_0])$ which is impossible, since by definition of r_0 it follows that $B \cap d([0, r_0]) = \{d(r_0)\}$. ■

The following however is true.

3.1.23. THEOREM. *For an open subset of Euclidean space to be pathwise connected it is necessary and sufficient that it be connected.*

PROOF. Necessity follows from Assertion 3.1.21. To prove sufficiency suppose that $a, b \in X \subset \mathbf{R}^m$ with X open and connected. By Theorem 1.10.7 there is a broken line $L = \bigcup_{j=0}^{n-1} \overline{x_j x_{j+1}} \subset X$, with $x_0 = a$, $x_n = b$. The map $d: I \to X$ defined by the formula $d(r) = (j - nr)x_{j-1} + (1 - j + nr)x_j$ for $(j-1)/n \leq r \leq j/n$, where $j = 1, 2, \ldots, n$, is a path from a to b in X. ■

3.1.24. THEOREM. *For a polyhedron to be pathwise connected it is necessary and sufficient that it be connected.*

PROOF. Necessity follows from Assertion 3.1.21; its sufficiency comes from Theorem 2.3.6 and Lemma 2.3.5. ■

Exercises

a) Give an example to show that the assumption that A is closed in Tietze's Theorem (3.1.4) is essential.

b) Prove that the union of three line segments which are disjoint, except for one common endpoint, has the fixed point property.

c) From Brouwer's Fixed Point Theorem (3.1.16) deduce the theorem that there does not exist a retraction of the ball onto its boundary (3.1.15).

d) Show that if a subspace A is a retract (a neighbourhood retract) of a space X and a set $B \subset A$ is a retract (a neighbourhood retract) of the space A, then B is a retract (a neighbourhood retract) of the space X.

e) Give an example of a metric space X, a closed subset $A \subset X$ and a continuous map $f: A \to X'$, where X' is the space constructed in Example 3.1.22, such that the map f does not have a continuous extension onto any open set U containing A in the space X (see Theorem 3.1.19).

f) Give an example of a plane region whose closure in the plane is not pathwise connected.

3.2. Homotopic maps

Let X and Y be metric spaces and let the maps $f_0, f_1: X \to Y$ be continuous. We say that the map f_0 is homotopic to the map f_1 if there exists a continuous map

$H: X \times I \to Y$, known as a *homotopy* from f_0 to f_1, such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for each $x \in X$. We then write $f_0 \underset{H}{\simeq} f_1$ or $f_0 \simeq f_1$. A homotopy from f_0 to some other map is called a *homotopy of the map* f_0.

3.2.1. EXAMPLE. If X is a compact space and we equip the set P of continuous maps from the space X into a space Y with a metric $\tilde{\rho}$ defined by the formula

$$\tilde{\rho}(f,g) = \sup\{\rho(f(x),g(x)) : x \in X\} \quad \text{for} \quad f,g \in P$$

(see Example 1.1.9 and Section 6.2), then a homotopy H from a map f_0 to a map f_1, where $f_0, f_1: X \to Y$, may be identified with a path in P from f_0 to f_1. For suppose we define the map $F: I \to P$ by the formula $F(r)(x) = H(x,r)$ for $x \in I$, $r \in I$. Since H is uniformly continuous (see Theorems 1.8.5 and 1.8.14) and

$$\tilde{\rho}(F(r_0),F(r)) = \sup\{\rho(H(x,r_0),H(x,r)) : x \in X\},$$

the map F is continuous. Conversely, if $F: I \to P$ is a path in P, then the map $H: X \times I \to P$ defined by the formula $H(x,r) = F(r)(x)$ for $x \in X$, $r \in I$ is continuous. Indeed, if $(x_0,r_0) \in X \times I$ and $\epsilon > 0$, then there exists a real number $\delta > 0$ such that if $|r_0 - r| < \delta$, then

$$\sup\{\rho(H(x,r_0),H(x,r)) : x \in X\} < \tfrac{1}{2}\epsilon$$

and if $\rho(x_0,x) < \delta$, then $\rho(H(x_0,r_0),H(x,r_0)) < \frac{1}{2}\epsilon$. So if $\sqrt{\rho(x_0,x)^2 + (r_0 - r)^2} < \delta$, then $\rho(H(x_0,r_0),H(x,r)) < \epsilon$. ■

3.2.2. EXAMPLE. A homotopy may also be regarded as the extension of a certain map. Call the metric product $X \times I$ the *metric cylinder* over the space X with base $X \times \{0\}$ and top $X \times \{1\}$. Let the maps $i_0: X \times \{0\} \to X$ and $i_1: X \times \{1\} \to X$ be defined by the formulas $i_0(x,0) = x$ and $i_1(x,1) = x$ for $x \in X$. Then the homotopy H from the map f_0 to the map f_1, where $f_0, f_1: X \to Y$, is a continuous extension from the union of the base and top to the cylinder $X \times I$ of the map $f: X \times \{0\} \cup X \times \{1\} \to Y$ defined by the formulas $f|X \times \{0\} = f_0 i_0$ and $f|X \times \{1\} = f_1 i_1$. ■

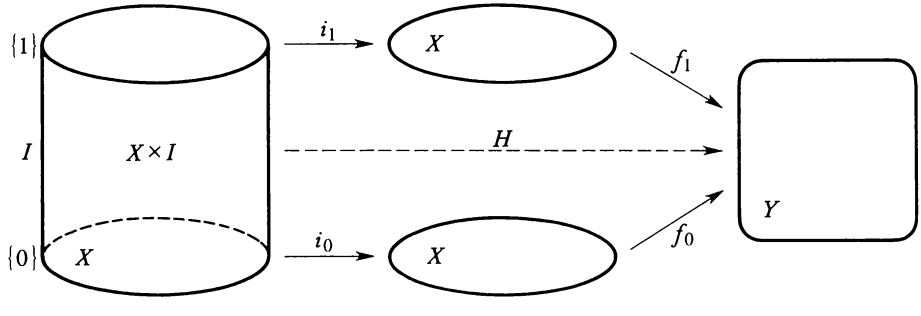


Fig.69. A homotopy from f_0 to f_1 is an extension of the maps $f_0 i_0$ and $f_1 i_1$ from the base $X \times \{0\}$ and top $X \times \{1\}$ to the whole cylinder.

We now prove the following.

3.2.3 THEOREM. *The relation of homotopy is an equivalence in the set of continuous maps of one metric space into another.*

PROOF. We shall study the relation of homotopy in the set of continuous maps of a space X into a space Y. To prove that the relation is reflexive, it is enough to observe that $f \underset{H}{\simeq} f$ where $H(x,r) = f(x)$ for all $x \in X$, $r \in I$.

To prove that the relation is symmetric, suppose that $f_0, f_1 : X \to Y$ and let $f_0 \underset{H}{\simeq} f_1$. Then $f_1 \underset{\bar{H}}{\simeq} f_0$, where $\bar{H}(x,r) = H(x,1-r)$ for $x \in X$, $r \in I$.

It remains to prove that the relation of homotopy is transitive. Let $f_0, f_1, f_2 : X \to Y$ and let $f_0 \underset{H_1}{\simeq} f_1$ and $f_1 \underset{H_2}{\simeq} f_2$. Put

$$H(x,r) = \begin{cases} H_1(x,2r), & \text{for } x \in X,\ 0 \le r \le \frac{1}{2}, \\ H_2(x,2r-1), & \text{for } x \in X,\ \frac{1}{2} \le r \le 1. \end{cases}$$

Then $H(x,0) = H_1(x,0) = f_0(x)$ and $H(x,1) = H_2(x,1) = f_2(x)$ and since H is continuous by Corollary 1.6.29, we have $f_0 \underset{H}{\simeq} f_2$. ■

Appealing to the symmetry of the relation of homotopy we shall often refer to the homotopy from a map f_0 to a map f_1 as a homotopy *between* f_0 *and* f_1, and when such a homotopy exists we shall call the maps f_0 and f_1 *homotopic*. The homotopy equivalence class of the map $f : X \to Y$ will be called the *homotopy class* of the map, denoted by $[f]$.

We now give examples of homotopic maps.

3.2.4. EXAMPLE. *Any two continuous maps into a convex subset of Euclidean space are homotopic.* For suppose $f_0, f_1 : X \to Y \subset \mathbf{R}^m$ and Y is convex. Taking $H(x,r) = (1-r)f_0(x) + rf_1(x)$ for $x \in X$, $r \in I$ we obtain a continuous map $H : X \times I \to Y$ and since $H(x_0) = f_0(x)$ and $H(x,1) = f_1(x)$ for $x \in X$, we have $f_0 \underset{H}{\simeq} f_1$. ■

3.2.5. EXAMPLE. *Let $\varphi : K \to \mathcal{L}$ be a simplicial approximation of a continuous map $f : |K| \to |\mathcal{L}|$. Then $|\varphi| \simeq f$.* For by Theorem 2.5.8 there is for each point $x \in |K|$ a simplex $\Delta \in \mathcal{L}$ such that $f(x) \in \operatorname{int} \Delta$ and $|\varphi|(x) \in \Delta$. Taking $H(x,r) = (1-r)|\varphi|(x) + rf(x)$ for $x \in |K|$, $r \in I$ we obtain, because of the convexity of simplices, a continuous map $H : |K| \times I \to |\mathcal{L}|$. And, since $H(x,0) = |\varphi|(x)$ and $H(x,1) = f(x)$ for $x \in |K|$ we have $|\varphi| \underset{H}{\simeq} f$. ■

3.2.6. EXAMPLE. Consider two arbitrary points $y_0, y_1 \in Y$ and the constant maps $f_0, f_1 : X \to Y$ with $f_0(X) = \{y_0\}$ and $f_1(X) = \{y_1\}$. If in the space Y there is a path d from the point y_0 to the point y_1, then $f_0 \simeq f_1$. Indeed, take $H(x,r) = d(r)$ for $x \in X$, $r \in I$. Then $H(x,0) = d(0) = y_0 = f_0(x)$ and $H(x,1) = d(1) = y_1 = f_1(x)$ for each $x \in X$, so $f_0 \underset{H}{\simeq} f_1$. ■

We will now show that in certain situations the existence of a continuous extension of a map depends on the homotopy class of the map. For this purpose we take up the problem of extending homotopic maps with the simultaneous extension of the homotopy which connects them. Let A be a closed subset of a metric space X. We say that the

pair (X, A) has the *homotopy extension property relative to the metric space* Y if for every map $f: A \to Y$ which has a continuous extension $f^*: X \to Y$, any homotopy F of the map f has an extension F^* which is a homotopy of f^*.

3.2.7. THEOREM. *If X and A are polyhedra, then the pair (X, A) has the homotopy extension property relative to any metric space Y.*

PROOF. Let $f^*: X \to Y$ be a continuous extension, and $F: A \times I \to Y$ a homotopy of the map $f: A \to Y$. Put $G = X \times \{0\} \cup A \times I$ and define the map $F': G \to Y$ by the formulas $F'(x,0) = f^*(x)$ for $(x,0) \in X \times \{0\}$ and $F'(x,s) = F(x,s)$ when $(x,s) \in A \times I$. Now $f^*(x) = f(x) = F(x,0)$ for $x \in A$, so by Corollary 1.6.29 the map F' is continuous. Applying Theorems 2.6.15 and 3.1.13 we infer that there exists a retraction $r: X \times I \to G$. Taking $F^* = F'r: X \times I \to Y$ we obtain $F^*|G = F'$, hence $F^*(x,s) = F(x,s)$ for $(x,s) \in A \times I$ and $F^*(x,0) = f^*(x)$ for $x \in X$, which completes the proof. ■

3.2.8. THEOREM (Borsuk). *If a metric space Y has the property that for every metric space X, for every closed subset A of X and for every continuous map $f: A \to Y$ there is an open set U of X containing A and a continuous extension $f^*: U \to Y$, then every pair (X, A) with A a closed subset of the metric space X has the homotopy extension property relative to Y.*

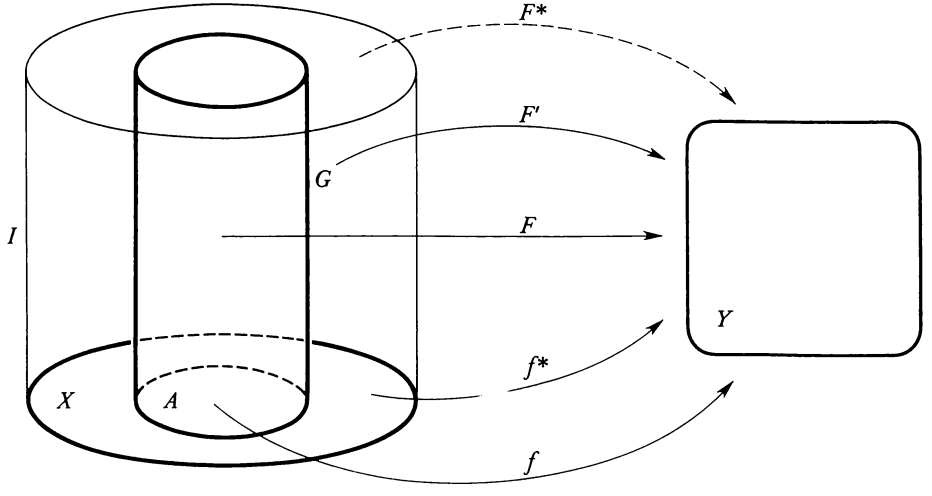


Fig.70. The pair (X, A) has the homotopy extension property relative to Y when for every map $f: A \to Y$, every extension $f^*: X \to Y$ and every homotopy F of f which jointly define a map F' of the top-hat G into Y, there exists an extension F^* of the map F' to the whole cylinder $X \times I$. F^* is then an extension of the map F and a homotopy of the map f^* (see proofs of Theorems 3.2.7 and 3.2.8).

Proof. Let A be a closed subset of a metric space X, let $f^*: X \to Y$ be a continuous extension, and let $F: A \times I \to Y$ be a homotopy of the map $f: A \to Y$. Put $G = X \times \{0\} \cup A \times I$ and define the map $F': G \to Y$ by the formulas: $F'(x,0) = f^*(x)$ when $(x,0) \in X \times \{0\}$ and $F'(x,r) = F(x,r)$ when $(x,r) \in A \times I$. Now $f^*(x) = f(x) = F(x,0)$ for $x \in A$, so by Corollary 1.6.29 the map F' is continuous. Since the set G is closed in

the product $X \times I$, it follows from our hypotheses that there is an open set U in $X \times I$ containing G and a continuous extension $F'': U \to Y$ of the map F'.

Each point $(a, r) \in \{a\} \times I$ has a neighbourhood in the space $X \times I$ contained in U which is of the form $V_{a,r} \times I_{a,r}$, where $V_{a,r}$ is a neighbourhood of the point a in the space X, and $I_{a,r}$ is a neighbourhood of the real number r in the interval I. By its compactness the set $\{a\} \times I$ is contained in a finite union of sets $V_{a,r_j} \times I_{a,r_j}$ where $j = 1, 2, \ldots, k$. Now the intersection $V_a = \bigcap_{j=1}^{k} V_{a,r_j}$ is a neighbourhood of the point a in the space X such that $V_a \times I \subset U$. Taking $V = \bigcup_{a \in A} V_a$ we obtain an open set in X such that $A \subset V$ and $V \times I \subset U$.

Now the sets $X \setminus V$ and A are closed and disjoint, so by Lemma 1.6.26 there is a function $t: X \to I$ such that $t(x) = 0$ for $x \in X \setminus V$ and $t(x) = 1$ for $x \in A$. It follows that $(x, rt(x)) \in U$ for each $(x, r) \in X \times I$. Taking $F^*(x, r) = F''(x, rt(x))$ for $(x, r) \in X \times I$ we have $F^* | G = F'' | G = F'$. Thus F^* is a homotopy of the map f^* and an extension of the homotopy F. ■

The set G appearing in the proofs of Theorems 3.2.7 and 3.2.8 is very graphically named a top-hat. Hence the theorems are sometimes called the *top-hat extension theorems* or the *homotopy extension theorems.*

The rather involved assumptions of Theorem 3.2.8 may be more briefly expressed by the use of concepts to be introduced in Chapter 6 (see the Supplement 6.S.15); the assumptions in fact signify that Y is an absolute neighbourhood extensor for metric spaces, or, alternatively, is an absolute neighbourhood retract (compare Theorem 6.6.8).

In view of Theorem 3.1.19 we obtain from Theorem 3.2.8 the following.

3.2.9. COROLLARY. *Every pair (X, A), where A is a closed subset of a metric space X, has the homotopy extension property relative to the sphere S^{m-1} where $m > 0$.* ■

Let $x_0 \in X$. We say that the space X is *contractible to the point x_0* if the identity map $\mathrm{id}: X \to X$ is homotopic to the constant map $c: X \to X$, where $c(X) = \{x_0\}$. If the space X is contractible to the point x_0 then obviously for each point $x_1 \in X$ there is in X a path from x_1 to x_0; in view of Example 3.2.6 it follows that the space X is also contractible to the point x_1. So we will simply say that the space X is *contractible.* We also infer the following.

3.2.10. ASSERTION. *Every contractible space is pathwise connected.* ■

We now give examples of contractible and non-contractible spaces.

3.2.11. EXAMPLE. From Example 3.2.4 it follows that *every convex subset of a Euclidean space is a contractible space.* In particular, *the m-dimensional closed unit ball $\bar{B}^m$ is contractible.* ■

3.2.12. THEOREM. *The sphere S^{m-1} is not a contractible space for any m.*

PROOF. Suppose there exists a homotopy H from the identity map $i: S^{m-1} \to S^{m-1}$ to the constant map $c: S^{m-1} \to S^{m-1}$ where $c(S^{m-1}) = \{x_0\}$. We thus have $H: S^{m-1} \times I \to S^{m-1}$, where $H(x, 0) = x$ and $H(x, 1) = x_0$ for $x \in S^{m-1}$. We define a map $r: \bar{B}^m \to S^{m-1}$ by the formulas $r(x) = H(x/\|x\|, 1 - \|x\|)$ for $x \neq 0$ and $r(0) = x_0$.

This map is continuous. It is in fact enough to prove continuity at the point 0. If $0 \neq x_n \in \bar{B}^m$ for $n = 1, 2, \ldots$ and $\lim_n x_n = 0$, then appealing to the compactness of S^{m-1} (compare also Theorem 1.5.4) we may straight off assume that the sequence $\{x_n/\|x_n\|\}$ converges to some point x_0', while $\lim_n r(x_n) = \lim_n H(x_n/\|x_n\|, 1 - \|x_n\|) = H(x_0', 1) = x_0$. Moreover if $x \in S^{m-1}$, then $\|x\| = 1$ so $r(x) = H(x, 0) = x$. The map r is therefore a retraction of the ball $\bar{B}^m$ onto the sphere S^{m-1} despite Theorem 3.1.15. ■

3.2.13. EXAMPLE. *The metric cone over a space.* Let (X, ρ) be any metric space for which $\operatorname{diam} X \leq 1$. We define a function $\hat{\rho}$ on the Cartesian product $X \times I$ by the formula

$$\hat{\rho}((x,r),(y,s)) = \min(1-r, 1-s)\rho(x,y) + |r-s| \quad \text{for} \quad (x,r),(y,s) \in X \times I.$$

This function is not in general a metric on $X \times I$ since $\hat{\rho}((x,r),(y,s)) = 0$ if and only if $x = y$, $r = s$ or if $r = 1 = s$.

The function $\hat{\rho}$ satisfies the other two axioms for a metric. The symmetry axiom is satisfied very obviously. To prove the triangle inequality consider any three points $(x,r),(y,s),(z,t) \in X \times I$ and write $a = \rho(x,y)$, $b = \rho(y,z)$. Since ρ satisfies the triangle inequality, it is enough to prove that

$$\min(1-r, 1-s)a + |r-s| + \min(1-s, 1-t)b + |s-t| - \\ - \min(1-r, 1-t)(a+b) - |r-t| \geq 0.$$

Without loss of generality we may suppose that $r \leq t$. Consider the three following cases: $0 \leq s \leq r$, $r \leq s \leq t$, $t \leq s \leq 1$. The inqualities correspond as follows:

$$(t-r)a + 2(r-s) \geq 0, \quad a(t-s) \geq 0, \quad (2-a-b)(s-t) \geq 0,$$

of which the first two are obvious and the third results from the estimates $a, b \leq \operatorname{diam} X \leq 1$.

Observe moreover that if $s = 1$, so that $(y,s) \in X \times \{1\}$, then $\hat{\rho}((x,r),(y,s)) = 1 - r$ for any point $(x,r) \in X \times I$. From the above properties it follows that if we denote by $C_m(X)$ the collection of the singleton points of the set $X \times [0,1)$ and the one element $v = X \times \{1\}$, then the function $\hat{\rho}$ induces a metric on $C_m(X)$ which, without fear of confusion, we shall also denote by $\hat{\rho}$. The set $C_m(X)$ with metric $\hat{\rho}$ is called the *metric cone* over the space X, the point v is called its *vertex* and the set $X \times \{0\}$ is called the *base* of the cone. We use the symbol $C_m(X)$ to distinguish the metric cone from the topological cone $C_t(X)$ which we shall introduce in Example 7.4.50.

Observe now that for every metric space X with $\operatorname{diam} X \leq 1$ the metric cone $C_m(X)$ is contractible. For, taking $H((x,r),s) = (x, r(1-s)+s)$ for $(x,r) \in X \times [0,1)$, $s \in I$ and $H(v,s) = v$ for $s \in I$, we obtain a homotopy from the identity map on $C_m(X)$ to the map sending the whole of $C_m(X)$ onto the vertex v. ■

The notion of a metric cone over a space X which was introduced in the Example above may also be used to study the contractibility of the space X. The following is the case.

3.2.14. THEOREM. *For a metric space X with $\operatorname{diam} X \leq 1$ to be contractible it is necessary and sufficient that the base of the metric cone $C_m(X)$ be a retract of the cone.*

PROOF. Let $H: X \times I \to X$ be a homotopy from the identity map $\mathrm{id}: X \to X$ to a constant map $c: X \to X$ where $c(X) = \{x_0\} \subset X$. Define a map $f: C_m(X) \to X \times \{0\}$ by the formulas: $f(x,r) = (H(x,r),0)$ for $(x,r) \in X \times [0,1)$ and $f(v) = (x_0,0)$. Since $H(x,1) = x_0$ for $x \in X$ the map f is continuous. Furthermore $f(x,0) = (H(x,0),0) = (x,0)$ for $x \in X$ and so f is a retraction of the cone $C_m(X)$ onto the set $X \times \{0\}$.

Conversely, let f be a retraction of the cone $C_m(X)$ onto the set $X \times \{0\}$ and let $f(v) = (x_0,0)$. The map $p: X \times I \to C_m(X)$ defined by the formulas: $p(x,r) = (x,r)$ for $(x,r) \in X \times \langle 0,1)$ and $p(x,1) = v$ for $x \in X$ is obviously continuous. Taking $H(x,r) = fp: X \times I \to X \times \{0\}$, yields $H(x,0) = f(x,0) = (x,0)$ and $H(x,1) = f(v) = (x_0,0)$ for $x \in X$. Thus H determines in an obvious way a homotopy from the identity map $\mathrm{id}: X \to X$ to the constant map into the point x_0. ■

Let A be a subset of a metric space X. A continuous map $r: X \to A$ is called a *deformation retraction* of X onto A if r is a retraction of X onto A and $ir \simeq \mathrm{id}_X$, where $i: A \to X$ denotes the inclusion map of the set A into the space X. If a deformation retraction of the space X onto the set A exists, then we say that A is a *deformation retract* of the space X (see also Supplement 3.S.2).

3.2.15. EXAMPLE. *The sphere S^{m-1} is a deformation retract of the complement $\bar{B}^m \setminus \{0\}$* (compare Example 3.1.11). In fact, taking $H(x,s) = x/(\|x\| + s(1 - \|x\|))$ for $x \in \bar{B}^m \setminus \{0\}$, $s \in I$ gives a map $H: (\bar{B}^m \setminus \{0\}) \times I \to \bar{B}^m \setminus \{0\}$ such that $H(x,0) = x/\|x\|$ and $H(x,1) = x$ for $x \in \bar{B}^m \setminus \{0\}$.

Considering the map H defined by the same formula but with $x \in \mathbf{R}^m \setminus \{0\}$, $s \in I$ we deduce that *the sphere S^{m-1} is a deformation retract of the complement $\mathbf{R}^m \setminus \{0\}$.*

Taking $H'(x,s) = H(x,s)$ for $x \in \mathbf{R}^m \setminus B^m$ and $H'(x,s) = x$ for $x \in \bar{B}^m$, $s \in I$ we deduce that *the ball $\bar{B}^m$ is a deformation retract of the space $\mathbf{R}^m$.* ■

From the definition of a deformation retract we immediately obtain the following.

3.2.16. ASSERTION. *For a metric space X to be contractible to the point $x_0 \in X$ it is necessary and sufficient that the set $\{x_0\}$ be a deformation retract of X.* ■

Hence from Theorem 3.2.12 it follows that a singleton set is not a deformation retract of the sphere S^{m-1} although it is its retract.

Using the notion of homotopy of maps we shall now introduce certain relations into the family of all metric spaces. Let X and Y be metric spaces. We say that the space X *homotopically dominates* the space Y or that the space Y is *homotopically dominated* by the space X if there exist continuous maps $f: X \to Y$ and $g: Y \to X$ such that $fg \simeq \mathrm{id}_Y$; we then write $Y \underset{h}{\leq} X$ and we say that the map g is a *right homotopic inverse* of the map f. If moreover $gf \simeq \mathrm{id}_X$, then we say that the spaces X and Y *have the same homotopy type* and we write $X \simeq Y$. The map f is then called a *homotopic equivalence* and g the *homotopic inverse of the map f* (see also Supplement 3.S.2).

3.2.17. EXAMPLE. *If a set A is a retract of a space X, then $A \underset{h}{\leq} X$.* For, writing $i: A \to X$ for the inclusion map and $r: X \to A$ for the retraction, we have $ri = \mathrm{id}_A$. ■

3.2.18. EXAMPLE. *If a set A is a deformation retract of a space X, then A and X have*

the same homotopy type. Again writing $i: A \to X$ for the inclusion map and $r: X \to A$ for the deformation retraction we have $ir \simeq \mathrm{id}_X$ and $ri = \mathrm{id}_A$. ■

3.2.19. EXAMPLE. *If the spaces X and Y are homeomorphic, then they have the same homotopy type.* For, any homeomorphism $h: X \to Y$ is a homotopic equivalence with a homotopic inverse $h^{-1}: Y \to X$, since $hh^{-1} = \mathrm{id}_Y$ and $h^{-1}h = \mathrm{id}_X$. ■

We now show that the relation of having the same homotopy type is an equivalence in the family of metric spaces. For this purpose we first prove the following.

3.2.20. LEMMA. *Let $g_0, g_1: X \to Y$ and suppose $g_0 \simeq g_1$. Then for any continuous maps $f: W \to X$ and $h: Y \to Z$ we have $g_0 f \simeq g_1 f$ and $hg_0 \simeq hg_1$.*

PROOF. Let $g_0 \underset{G}{\simeq} g_1$. Put $G_f(w,r) = G(f(w),r)$ for $w \in W$, $r \in I$; then we have $g_0 f \underset{G_f}{\simeq} g_1 f$. On the other hand taking $G_h(x,r) = hG(x,r)$ for $x \in I$, $r \in I$ we obtain $hg_0 \underset{G_h}{\simeq} hg_1$. ■

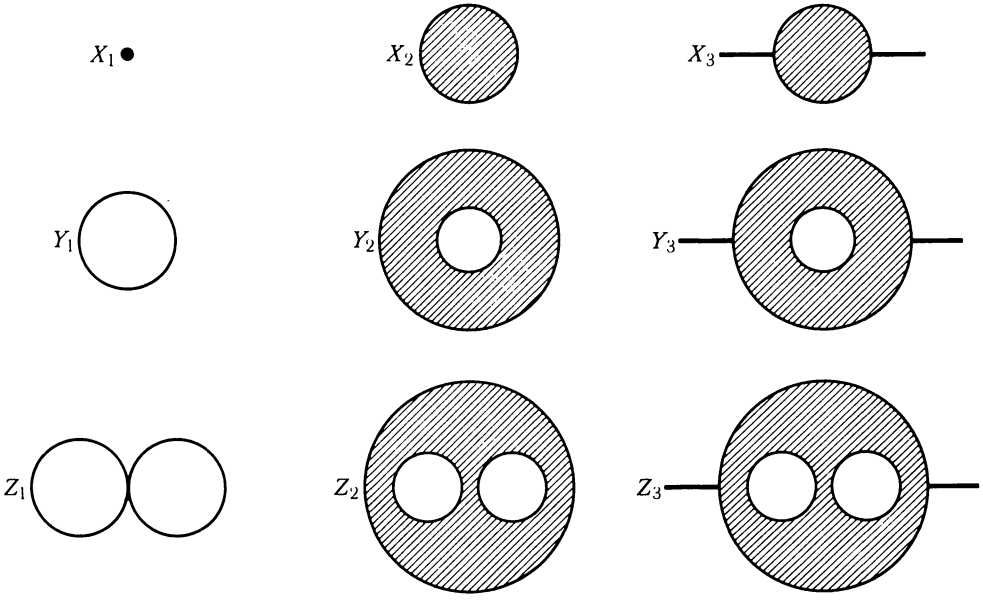


Fig.71. The spaces X_1, X_2, X_3 have the same homotopy type T_X; the spaces Y_1, Y_2, Y_3 have the same homotopy type T_Y; the spaces Z_1, Z_2, Z_3 have the same homotopy type T_Z. The types T_X, T_Y, T_Z are pairwise distinct.

3.2.21. THEOREM. *The relation of having the same homotopy type is an equivalence in the family of metric spaces.*

PROOF. Reflexivity and symmetry of the relation are both obvious. To prove it is transitive, assume $X \simeq Y$ and $Y \simeq Z$ with $f: X \to Y$, $g_1: Y \to X$, $fg_1 \simeq \mathrm{id}_Y$, $g_1 f \simeq \mathrm{id}_X$ and $g_2: Y \to Z$, $h: Z \to Y$, $g_2 h \simeq \mathrm{id}_Z$, $hg_2 \simeq \mathrm{id}_Y$. Then $g_2 f: X \to Z$ and $g_1 h: Z \to X$ and by the last lemma we have $(g_2 f)(g_1 h) \simeq g_2(\mathrm{id}_Y)h = g_2 h \simeq \mathrm{id}_Z$ and $(g_1 h)(g_2 f) \simeq g_1(\mathrm{id}_Y)f = g_1 f \simeq \mathrm{id}_X$, that is $X \simeq Z$. ■

The class of all metric spaces which have the same homotopy type as the space X is called the *homotopy type* of the space X. A property of metric spaces which, if enjoyed by one space is enjoyed by all spaces of the same homotopy type, is called a *homotopy type invariant*. It follows from Example 3.2.19 that every topological type is contained in some homotopy type. The theory of homotopy type invariants is thus in some sense a generalization of topology. How far reaching is the generalization may be gauged from the following theorem.

3.2.22. THEOREM. *For a metric space to be contractible it is necessary and sufficient that it have the homotopy type of a singleton space.*

PROOF. Necessity of the condition follows from Assertion 3.2.16 and Example 3.2.18. To show sufficiency suppose that $X \simeq \{y_0\}$; say $f: X \to \{y_0\}$, $g: \{y_0\} \to X$, $fg \simeq \mathrm{id}_{\{y_0\}}$ and $gf \simeq \mathrm{id}_X$. Taking $x_0 = g(y_0) \in X$, we observe that the composition gf satisfies the condition $gf(X) = \{x_0\}$. The relation $gf \simeq \mathrm{id}_X$ implies then that the space X is contractible to the point x_0. ■

3.2.23. COROLLARY. *Contractibility is a homotopy type invariant.* ■

From Theorem 3.2.21 in view of Example 3.2.11 and Theorem 3.2.12 we obtain the following.

3.2.24. EXAMPLE. *Closed unit balls of all dimensions have the same homotopy type. The type is distinct from the homotopy type of the unit sphere of any dimension.* ■

We close this section with some remarks about the homotopy theory of pairs of spaces. If A is a subspace of a metric space X and B is a subspace of a metric space Y, then every map $f: X \to Y$ satisfying $f(A) \subset B$ is called a *pair map* of (X, A) into (Y, B) and we write $f: (X, A) \to (Y, B)$. If $f_0, f_1: (X, A) \to (Y, B)$ and there exists a continuous pair map $H: (X \times I, A \times I) \to (Y, B)$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for $x \in X$, then the pair map f_0 is said to be *homotopic* to the pair map f_1, the map H is called a *homotopy* from f_0 to f_1 and we write $f_0 \underset{H}{\simeq} f_1$ or $f_0 \simeq f_1$. It is obvious that the notions of pair map and homotopy between such maps reduce to the usual notions of map and homotopy when the distinguished subspaces are empty. The proof of Theorem 3.2.3 easily carries over to the case of pairs of spaces and we obtain the following.

3.2.25. THEOREM. *The relation of homotopy is an equivalence in the set of continuous pair maps of one pair into another.* ■

Also several other notions and results of this section may be carried over to the case of pairs of spaces. Consider for instance the identity map between pairs $\mathrm{id}_{(X,A)}: (X, A) \to (X, A)$ defined by the formula $\mathrm{id}_{(X,A)}(x) = x$ for each $x \in X$. We say that the pairs (X, A) and (Y, B) where $A \subset X$ and $B \subset Y$ *have the same homotopy type*, if there are continuous pair maps $f: (X, A) \to (Y, B)$ and $g: (Y, B) \to (X, A)$ such that $fg \simeq \mathrm{id}_{(Y,B)}$ and $gf \simeq \mathrm{id}_{(X,A)}$. An easily proved analogue of Theorem 3.2.21 holds.

3.2.26. THEOREM. *The relation of having the same homotopy type is an equivalence in the family of pairs (X, A) with $A \subset X$.* ■

We do not develop here the homotopy theory of pairs of spaces through lack of space; in subsequent paragraphs we will only discuss those fragments of this generalization which turn out to be indispensable to other constructions (see also Supplement 3.S.1).

Exercises

a) Show that any two continuous maps f, g of a space X into the sphere S^{m-1} with the property that $f(x) \neq -g(x)$ for $x \in X$ are homotopic.

b) Deduce Theorem 3.2.12 from Theorems 3.2.14 and 3.1.15.

c) Check that the retraction described in Example 3.1.12 is a deformation retraction.

d) Show that homotopic domination is an ordering in the family of homotopy types of metric spaces.

e) Give an example of a compact and connected set $X \subset \mathbf{R}^2$ such that the complements $\mathbf{R}^2 \setminus X$ and $\mathbf{R}^2 \setminus S^1$ are homeomorphic but X and S^1 have different homotopy types.

3.3. Fibrations and coverings

The notion of a fibration constitutes an important generalization of the notion of the metric product of two spaces. Let E, B, W be metric spaces and let p be a continuous map of the space E onto the space B. The system (E, B, W, p) is called a *fibration system* or simply a *fibration* if there exists an open covering $\{U_t\}_{t \in T}$ of the space B and a family of homeomorphisms $\varphi_t \colon U_t \times W \to p^{-1}(U_t)$ for $t \in T$ such that $p\varphi_t(u, w) = u$ for every $u \in U_t$, $w \in W$ and $t \in T$. We shall call E the *fibre space*, B the *base space*, W the *fibre* and p the *fibre map* of the fibration system (E, B, W, p). When mention of W can be suppressed, we shall identify the fibration system (E, B, W, p) with the fibre map alone and denote it more simply by $p \colon E \to B$ (see also Supplement 3.S.5).

3.3.1. EXAMPLE. *Metric product.* Let $E = B \times W$ and let the map $p \colon E \to B$ be the projection onto the first factor, that is $p(b, w) = b$ for $b \in B$ and $w \in W$. Consider the trivial covering of the space B consisting of the one open set $U_0 = B$. Let φ_0 be the identity map of the set $U_0 \times W = B \times W = E$ onto $p^{-1}(B) = E$. Then $p\varphi_0(u, w) = p(u, w) = u$ for every $u \in U_0$ and $w \in W$. Thus $p \colon B \times W \to B$ is a fibration with fibre W. ■

3.3.2. EXAMPLE. *The Möbius band.* Let $B = \{(x^1, x^2, x^3) \in \mathbf{R}^3 : (x^1, x^2) \in S^1, x^3 = 0\}$ and take the parametrization $b \colon [0, 2\pi) \to B$ defined by the formula $b(\alpha) = (\cos\alpha, \sin\alpha, 0)$. Let

$$a'(\alpha) = ((1 + \sin\frac{\alpha}{2})\cos\alpha,\ (1 + \sin\frac{\alpha}{2})\sin\alpha,\ \cos\frac{\alpha}{2}),$$

and

$$a''(\alpha) = ((1 - \sin\frac{\alpha}{2})\cos\alpha,\ (1 - \sin\frac{\alpha}{2})\sin\alpha,\ -\cos\frac{\alpha}{2}),$$

for $0 \leq \alpha < 2\pi$. It is easy to see that the interval $I(\alpha) = \overline{a'(\alpha)a''(\alpha)}$ with centre $b(\alpha)$ and length 2 lies in the plane passing through the x^3-axis and the point $b(\alpha)$; hence $I(\alpha_1) \cap I(\alpha_2) = \emptyset$ for $\alpha_1 \neq \alpha_2$. The union $E = \bigcup_{0 \leq \alpha < 2\pi} I(\alpha)$ is called the *Möbius band*; B is its *equator* and the union $\bigcup_{0 \leq \alpha < 2\pi} \{a'(\alpha)\} \cup \{a''(\alpha)\}$ is its *edge*.

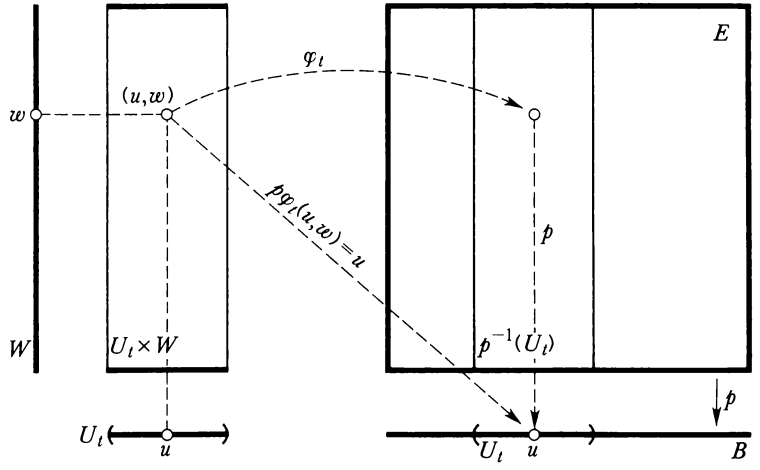


Fig.72. The map $p: E \to B$ is a fibration with fibre W if there is an open covering $\{U_t\}_{t \in T}$ of the space B and a family of homeomorphisms $\varphi_t: U_t \times W \to p^{-1}(U_t)$ such that $p\varphi_t(u, w) = u$ for $u \in U_t$, $w \in W$, $t \in T$.

It is easy to verify that the map $p: E \to B$ which sends each point $x \in I(\alpha)$ to the point $b(\alpha)$ is continuous. Consider the open covering of the space B by the two open sets $U_1 = B \backslash b(0)$, $U_2 = B \backslash b(\pi)$. Let $W = [-1, 1]$ and define a map $\varphi: [0, 2\pi) \times W \to E$ by the formula:

$$\varphi(\alpha, w) = \left(\left(1 + w \sin\frac{\alpha}{2}\right)\cos\alpha,\ \left(1 + w \sin\frac{\alpha}{2}\right)\sin\alpha,\ w\cos\frac{\alpha}{2}\right).$$

Next define for $i = 1, 2$ the map $\varphi_i: U_i \times W \to p^{-1}(U_i)$ by taking:

$$\varphi_1(u, w) = \varphi(b^{-1}(u), w) \quad \text{for} \quad u \in U_1,\ w \in W,$$

$$\varphi_2(u, w) = \begin{cases} \varphi(b^{-1}(u), w) & \text{if } 0 < b^{-1}(u) < \pi, \quad w \in W, \\ \varphi(b^{-1}(u), -w) & \text{if } \pi < b^{-1}(u) < 2\pi, \quad w \in W. \end{cases}$$

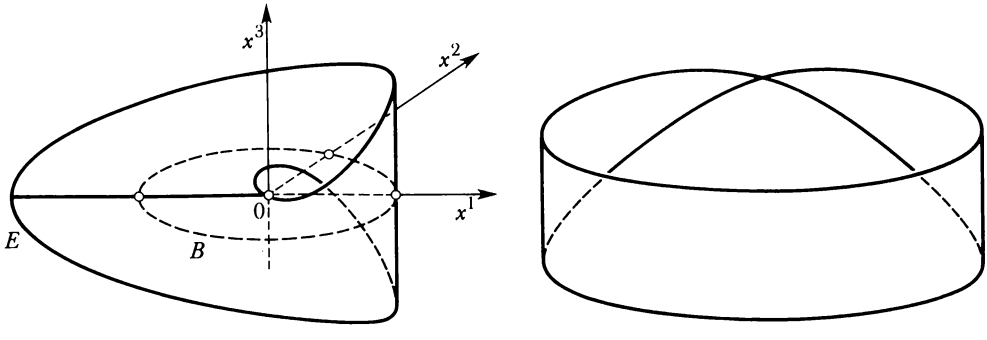


Fig.73. The Möbius band constructed in Example 3.3.2, side by side with a more perspicuous model.

It is readily checked that φ_i is a homeomorphism of the product $U_i \times W$ onto the inverse image $p^{-1}(U_i)$ for $i = 1,2$. Since also $p\varphi_i(u,w) = u$ for all $u \in U_i$, $w \in W$, $i = 1,2$, the map $p: E \to B$ is a fibration with fibre W. ■

3.3.3. EXAMPLE. *The Hopf fibration.* This is an example of a fibration in which the fibre space is the sphere S^3, the base space is S^2 and the fibre is the circle S^1. For ease of description it is more convenient to replace the spheres S^3, S^2, S^1 by certain isometric copies.

Let $\mathbf{C}$ be the set of complex numbers; let $\mathbf{C}^2$ denote the set of pairs (z^1, z^2) with $z^1, z^2 \in \mathbf{C}$ equipped with the metric ρ defined by the formula:

$$\rho((z_1^1, z_1^2), (z_2^1, z_2^2)) = \sqrt{|z_1^1 - z_2^1|^2 + |z_1^2 - z_2^2|^2} \quad \text{for} \quad (z_1^1, z_1^2), (z_2^1, z_2^2) \in \mathbf{C}^2.$$

The space $\mathbf{C}^2$ is thus obviously isometric with Euclidean space $\mathbf{R}^4$. Under the isometry the sphere $S^3 \subset \mathbf{R}^4$ corresponds to the set of pairs $(z^1, z^2) \in \mathbf{C}^2$ for which $z^1\bar{z}^1 + z^2\bar{z}^2 = 1$ (here $\bar{z}$ denotes the conjugate of z). Identifying the isometric spaces we shall write $S^3 = \{(z^1, z^2) \in \mathbf{C}^2 : z^1\bar{z}^1 + z^2\bar{z}^2 = 1\}$.

Consider now the set $P_{\mathbf{C}}^1$ of equivalence classes of the relation of proportionality (with complex coefficients) on the set of pairs $(z^1, z^2) \in \mathbf{C}^2$ where $(z^1, z^2) \neq (0,0)$. Denote the equivalence class of the pair (z^1, z^2) by $[z^1, z^2]$. Let $b \in S^2$ and let g be a fixed homeomorphism of $S^2 \backslash \{b\}$ onto the plane $\mathbf{R}^2$ (compare Corollary 1.3.30), where we identify $\mathbf{R}^2$ with the set $\mathbf{C}$.

Define a map $h: S^2 \to P_{\mathbf{C}}^1$ by the formulas $h(x) = [1, g(x)]$ for $x \in S^2$, $x \neq b$ and $h(b) = [0,1]$. The map h is one-to-one and we may use it to carry across the metric of the sphere S^2 onto the space $P_{\mathbf{C}}^1$. Identifying the isometric spaces we will write $S^2 = P_{\mathbf{C}}^1$.

Finally, we shall treat the circle S^1 as being the set of complex numbers $z \in \mathbf{C}$ such that $|z| = 1$.

Now define a map $p: S^3 \to S^2$ by the formula $p(z^1, z^2) = [z^1, z^2]$ for $(z^1, z^2) \in S^3$. The map is well-defined since if $(z^1, z^2) \in S^3$ the complex numbers z^1, z^2 do not vanish simultaneously. It takes the sphere S^3 onto the sphere S^2 since division of any pair (z^1, z^2), with z^1, z^2 not both zero, by $\sqrt{z^1\bar{z}^1 + z^2\bar{z}^2}$ yields a representative of the same class $[z^1, z^2]$ which lies on S^3. Finally, the map p is continuous. For, if $z^1 \neq 0$ then $p(z^1, z^2) = [1, z^2/z^1] = hg^{-1}(z^2/z^1)$, whereas if $z^1 = 0$ then $p(z^1, z^2) = [0, z^2] = [0,1] = h(b)$. It suffices to make use of the fact that g is a homeomorphism and that if $\lim_n |z_n| = \infty$ where $z_n \in \mathbf{C}$ for $n = 1, 2, \ldots$, then $\lim_n g^{-1}(z_n) = b$.

Now write $a_1 = [1,0]$, $a_2 = [0,1] \in P_{\mathbf{C}}^1 = S^2$ and let $U_1 = S^2 \backslash \{a_1\}$, $U_2 = S^2 \backslash \{a_2\}$. Every point of U_1 can be expressed as $[z,1]$ and every point of U_2 as $[1,z]$ where $z \in \mathbf{C}$. We define the maps $\varphi_i: U_i \times S^1 \to \mathbf{C}^2$ for $i = 1,2$ by the formulas:

$$\varphi_1([z,1], c) = c/\sqrt{z\bar{z} + 1}(z,1) \quad \text{for} \quad [z,1] \in U_1, \quad c \in S^1,$$
$$\varphi_2([1,z], c) = c/\sqrt{z\bar{z} + 1}(1,z) \quad \text{for} \quad [1,z] \in U_2, \quad c \in S^1.$$

The values of both maps φ_i for $i = 1,2$ lie in the sphere S^3 and since

$$p^{-1}(U_1) = \{(z^1, z^2) \in S^3 : z^2 \neq 0\},$$
$$p^{-1}(U_2) = \{(z^1, z^2) \in S^3 : z^1 \neq 0\},$$

we have $\varphi_i: U_i \times S^1 \to p^{-1}(U_i)$ for $i = 1,2$.

To show that the map φ_i is a homeomorphism of the set $U_i \times S^1$ onto $p^{-1}(U_i)$ for $i = 1, 2$ we observe that it has an inverse map φ_i^{-1} defined respectively by the formulas:

$$\varphi_1^{-1}((z^1, z^2)) = ([z^1/z^2, 1],\ z^2/|z^2|) \quad \text{if} \quad (z^1, z^2) \in S^3, \quad z^2 \neq 0,$$

$$\varphi_2^{-1}((z^1, z^2)) = ([1, z^2/z^1],\ z^1/|z^1|) \quad \text{if} \quad (z^1, z^2) \in S^3, \quad z^1 \neq 0.$$

The continuity of each of the maps φ_i, φ_i^{-1} for $i = 1, 2$ is obvious.

It remains to notice that

$$p\varphi_1([z, 1], c) = p(c/\sqrt{z\bar{z} + 1}(z, 1)) = [z, 1] \quad \text{for} \quad [z, 1] \in U_1, \quad c \in S^1,$$
$$p\varphi_2([1, z], c) = p(c/\sqrt{z\bar{z} + 1}(1, z)) = [1, z] \quad \text{for} \quad [1, z] \in U_2, \quad c \in S^1.$$

We have thus shown that $p\colon S^3 \to S^2$ is a fibration with fibre S^1 which we call the *Hopf fibration* or the *Hopf map* (see Supplement 3.S.4). ■

Let $\tilde{Y}$ and Y be metric spaces and let p be a continuous map of $\tilde{Y}$ onto Y. A system $(\tilde{Y}, Y, p)$ is called a *covering system* or simply a *covering* if there exists an open covering $\{U_t\}_{t \in T}$ of the space Y such that for each $t \in T$ the set $p^{-1}(U_t)$ is the union of k pairwise disjoint open sets $\tilde{U}_{t,j}$ where $k \leq \aleph_0$ and $j = 1, 2, \ldots, k$ or $j = 1, 2, \ldots$ and $p|\tilde{U}_{t,j}$ is a homeomorphism of the set $\tilde{U}_{t,j}$ onto U_t for each j. $\tilde{Y}$ is said to be a *covering space*, Y the *base space*, p the *covering map* and k the *multiplicity* of the covering system $(\tilde{Y}, Y, p)$. We shall also identify the covering system $(\tilde{Y}, Y, p)$ with the covering map alone and denote it in the alternative form $p\colon \tilde{Y} \to Y$.

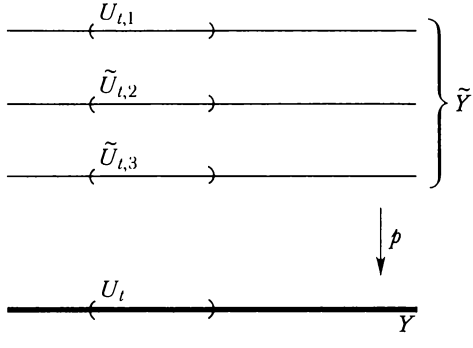


Fig.74. The map $p\colon \tilde{Y} \to Y$ is a covering map when there is an open covering $\{U_t\}_{t \in T}$ of the space Y such that $p^{-1}(U_t) = \bigcup_j \tilde{U}_{t,j}$ for each $t \in T$ where the sets $\tilde{U}_{t,j}$ are open and pairwise disjoint and $p|\tilde{U}_{t,j}$ is a homeomorphism of $\tilde{U}_{t,j}$ onto U_t for each j.

The following result holds.

3.3.4. THEOREM. *Covering systems are identical with fibration systems whose fibre is a discrete countable metric space.*

PROOF. Let $p\colon \tilde{Y} \to Y$ be a covering with multiplicity k and let $W = \{1, 2, \ldots, k\}$ if $k < \aleph_0$ and $W = \{1, 2, \ldots\}$ if $k = \aleph_0$. Equip W with the discrete metric and then, using the notation introduced in the definition of a covering, define for each $t \in T$ a map $\varphi_t\colon U_t \times W \to p^{-1}(U_t) = \bigcup_j \tilde{U}_{t,j}$ by the formula $\varphi_t(u, j) = (p|\tilde{U}_{t,j})^{-1}(u)$ where $u \in U_t$,

$j \in W$. This map is of course a homeomorphism of the set $U_t \times W$ onto $p^{-1}(U_t)$ and since $p\varphi_t(u,j) = p(p|\tilde{U}_{t,j})^{-1}(u) = u$ for $u \in U_t$, $j \in W$, $t \in T$ we have that $p: \tilde{Y} \to Y$ is a fibration with fibre W.

Conversely, let $p: E \to B$ be a fibration whose fibre W is a discrete countable metric space. In the notation of the definition of a fibration, take $\tilde{U}_{t,j} = \varphi_t(U_t \times \{j\})$ for $t \in T$, $j \in W$. Since for each $t \in T$ the map φ_t is a homeomorphism, the sets $\tilde{U}_{t,j}$ are pairwise disjoint and open and their union is $p^{-1}(U_t)$. Moreover, denoting by $p_{t,j}: U_t \times \{j\} \to U_t$ the homeomorphism defined by the formula $p_{t,j}(u_t, j) = u_t$, where $u_t \in U_t$, $j \in W$, we have $p|\tilde{U}_{t,j} = p_{t,j}(\varphi_t|U_t \times \{j\})^{-1}$ so that $p|\tilde{U}_{t,j}$ is a homeomorphism of $\tilde{U}_{t,j}$ onto U_t for each $j \in W$. Thus the fibration $p: E \to B$ is a covering. ■

We now give some examples of covering systems.

3.3.5. EXAMPLE. Consider the map $p: \mathbf{R}^1 \to S^1$ defined by the formula $p(x) = (\cos 2\pi x, \sin 2\pi x)$ for $x \in \mathbf{R}^1$. For each point $y \in S^1$ let $U_y = S^1 \backslash \{y\}$ and let $\tilde{U}_{y,j} = (x + 2\pi(j-1), x + 2\pi j)$ where x is a fixed arbitrary point of the inverse image $p^{-1}(y)$ and $j = 0, \pm 1, \ldots$ Then $p^{-1}(U_y) = \mathbf{R}^1 \backslash p^{-1}(y) = \bigcup_j \tilde{U}_{y,j}$. Since for every point $y \in S^1$ the sets $\tilde{U}_{y,j}$ for $j = 0, \pm 1, \ldots$ are open in $\mathbf{R}^1$ and pairwise disjoint, while $p|\tilde{U}_{y,j}$ is a homeomorphism of $\tilde{U}_{y,j}$ onto U_y, we see that $p: \mathbf{R}^1 \to S^1$ is a covering map with multiplicity $\aleph_0$. ■

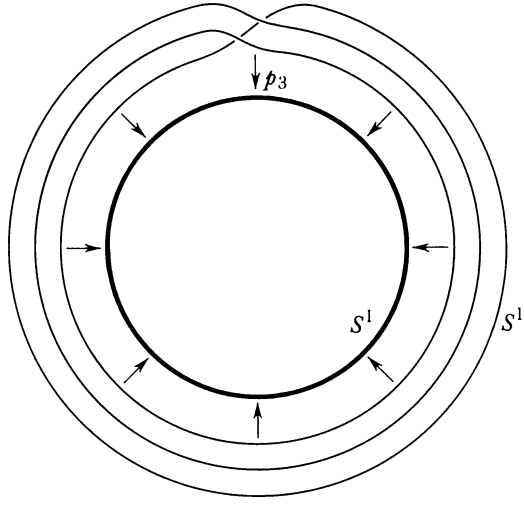


Fig.75. The covering map $p_n: S^1 \to S^1$ described in Example 3.3.6 consists of wrapping the circle S^1 around itself n times; the figure illustrates the case $n = 3$.

3.3.6. EXAMPLE. Treat the circle S^1 as being the set of complex numbers $z \in \mathbf{C}$ satisfying $|z| = 1$. For $n = 1, 2, \ldots$ consider the maps $p_n: S^1 \to S^1$ defined by the formula $p_n(z) = z^n$, where z^n denotes the n^{th}-power of the complex number z. For each complex number $z = \cos\zeta + i\sin\zeta \in S^1$ let $U_z = S^1 \backslash \{z\}$ and let

$$\tilde{U}_{z,j} = \left\{ \cos\vartheta + i\sin\vartheta \in S^1 : \frac{\zeta + 2\pi(j-1)}{n} < \vartheta < \frac{\zeta + 2\pi j}{n} \right\},$$

where $j = 1, 2, \ldots, n$. Thus $p^{-1}(U_z) = S^1 \setminus p_n^{-1}(z) = \bigcup_{j=1}^{n} \tilde{U}_{z,j}$. Since for every number $z \in S^1$ the sets $\tilde{U}_{z,j}$ for $j = 1, 2, \ldots, n$ are open in S^1 and pairwise disjoint and $p_n | \tilde{U}_{z,j}$ is a homeomorphism of $\tilde{U}_{z,j}$ onto U_z, we have that $p_n: S^1 \to S^1$ is a covering map with multiplicity n where $n = 1, 2, \ldots$ ■

3.3.7. EXAMPLE. Let P^m be the m-dimensional projective space and let $p: S^m \to P^m$ be a map which sends the point $x \in S^m$ to the equivalence class $[x] \in P^m$. Regarding P^m as a metric space with the metric introduced in Example 1.1.7 we claim that the map p is continuous (Example 1.3.5). For each element $[x] \in P^m$ let $U_{[x]}$ consists of those $[y] \in P^m$ with $x \cdot y \neq 0$; it is easy to check that this definition does not depend on the choice of representatives. We define $\tilde{U}_{[x],1} = \{y \in S^m : x \cdot y > 0\}$ and $\tilde{U}_{[x],2} = \{y \in S^m : x \cdot y < 0\}$. Then $p^{-1}(U_{[x]}) = \tilde{U}_{[x],1} \cup \tilde{U}_{[x],2}$ and since the sets $\tilde{U}_{[x],j}$ for $j = 1, 2$ are open in S^m and disjoint, and $p | \tilde{U}_{[x],j}$ is a homeomorphism of $\tilde{U}_{[x],j}$ onto $\tilde{U}_{[x]}$, we have that $p: S^m \to P^m$ is a covering map with multiplicity 2. ■

Let $p: E \to B$ be a fixed continuous map of the space E onto B and let $f: X \to B$. Any continuous map $f: X \to E$ such that $p\tilde{f} = f$ is called a continuous *lifting* of the map f. A comparison of the definitions of extensions and liftings of maps yields the observation that there is a kind of duality between the concepts: if $i: A \to X$ is the inclusion map then $f^*: X \to Y$ is an extension of the map $f: A \to Y$ when $f^* i = f$; if $p: E \to B$ takes E onto B then $\tilde{f}: X \to E$ is a lifting of the map $f: X \to B$ when $p\tilde{f} = f$. An analogous problem to the one treated in Theorems 3.2.7 and 3.2.8 thus naturally arises, namely the problem of lifting homotopic maps with the simultaneous lifting of the homotopy which connects them. Just as with extensions we shall show that in certain situations the possibility of lifting is a property not only of the map but also of its homotopy class.

Let $p: E \to B$ be a continuous map of the space E onto B. We say that the map has the *homotopy lifting property relative to the space* X if, for each map $f: X \to B$ which has a continuous lifting $\tilde{f}: X \to E$, any homotopy F of f has a lifting $\tilde{F}$ which is a homotopy of $\tilde{f}$.

The next theorem on homotopy lifting establishes an important property of fibrations.

3.3.8. THEOREM. *Every fibration has the homotopy lifting property relative to any polyhedron* X.

PROOF. Let $p: E \to B$ be a fibration with fibre W, let the sets U_t for $t \in T$ form an open covering of the base space B and let the homeomorphisms $\varphi_t: U_t \times W \to p^{-1}(U_t)$ satisfy the equation $p\varphi_t(u, w) = u$ for $u \in U_t$, $w \in W$, $t \in T$. For each $t \in T$ let the continuous map $q_t: U_t \times W \to W$ be defined by the formula $q_t(u, w) = w$ for $u \in U_t$, $w \in W$.

Let X be a polyhedron, let $\tilde{f}: X \to E$ be a lifting and $F: X \times I \to B$ be a homotopy of $f: X \to B$.

Since the sets $F^{-1}(U_t)$ for $t \in T$ form an open covering of the compact space $X \times I$, there is, by Lemma 1.8.13 and Corollary 2.4.9, a triangulation K of the polyhedron X and a sequence of numbers $0 = r_0 < r_1 < \ldots < r_k = 1$ such that for every simplex

$\Delta \in K$ and every number $j = 0, 1, \ldots, k-1$ there exists an index $t \in T$ for which $F(\Delta \times [r_j, r_{j+1}]) \subset U_t$.

We will now construct the required homotopy, i.e. a continuous map $\tilde{F}: X \times E \to E$ such that

$$\tilde{F}(x, 0) = \tilde{f}(x) \quad \text{for} \quad x \in X \tag{1}$$

and

$$p\tilde{F} = F. \tag{2}$$

The construction of the map $\tilde{F}$ consists of gradually extending the map $\tilde{f}$ from the base $X \times \{0\}$ to the whole cylinder $X \times I$, all the while preserving the condition that there exists a lifting of the homotopy F restricted to the set for which the extension has already been constructed. The construction will proceed by a triple induction, but for greater clarity we shall present it in three successive steps.

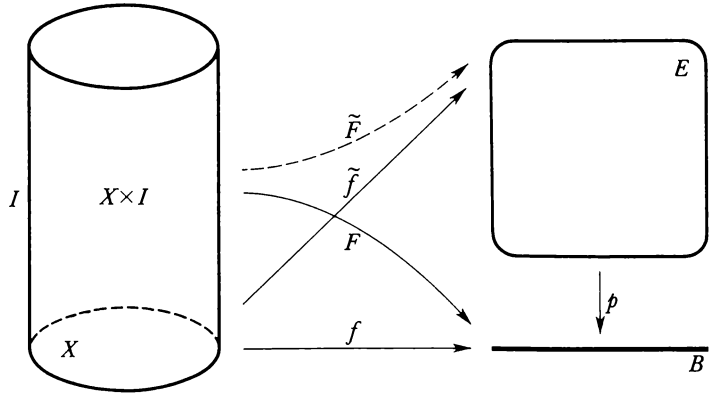


Fig.76. The map $p: E \to B$ has the homotopy lifting property relative to the space X, when for every map $f: X \to B$, for any lifting $\tilde{f}: X \to E$ and any homotopy F of f there is a homotopy $\tilde{F}$ of $\tilde{f}$ which is a lifting of the homotopy F.

S t e p 1. We take $\tilde{F}_0 = \tilde{f}: X \times \{0\} \to E$. Then of course $p\tilde{F}_0 = p\tilde{f} = f = F|X \times \{0\}$. Suppose that for some j with $0 \leq j \leq k-1$ there is a continuous map $\tilde{F}_j: X \times [0, r] \to E$, such that

$$\tilde{F}_j(x, 0) = \tilde{f}(x) \quad \text{for} \quad x \in X \tag{3}_j$$

and

$$p\tilde{F}_j = F|X \times [0, r_j]. \tag{4}_j$$

To obtain a continuous map $\tilde{F}_{j+1}: X \times [0, r_{j+1}] \to E$ satisfying $(3)_{j+1}$ and $(4)_{j+1}$, it suffices to find a continuous map $\tilde{F}'_j: X \times I_j \to E$ where $I_j = [r_j, r_{j+1}]$ such that

$$\tilde{F}'_j|X \times \{r_j\} = \tilde{F}_j|X \times \{r_j\} \tag{5}$$

and

$$p\tilde{F}'_j = F|X \times I_j. \tag{6}$$

We do this in the next step.

S t e p 2. Let $X^n = |K^{[n]}|$, where $K^{[n]}$ denotes the n-dimensional skeleton of the complex K for $n = 0, 1, \ldots, \dim K$. Define a continuous map $\tilde{F}^0_j: X^0 \times I_j \to E$ by the formula:

$$\tilde{F}^0_j(x,r) = \varphi_t(F(x,r), q_t\varphi_t^{-1}\tilde{F}_j(x,r_j)) \quad \text{for} \quad x \in X^0,\ r \in I_j,$$

where the index $t \in T$ satisfies the condition $F(\{x\} \times I_j) \subset U_t$. Then evidently $\tilde{F}^0_j|X^0 \times \{r_j\} = \tilde{F}_j|X^0 \times \{r_j\}$ and $p\tilde{F}^0_j = F|X^0 \times I_j$. Assume that for some n, where $0 \leq n \leq \dim K - 1$, a map $\tilde{F}^n_j: X^n \times I_j \to E$ is given so that

$$\tilde{F}^n_j|X^n \times \{r_j\} = \tilde{F}_j|X^n \times \{r_j\} \tag{7}$$

and

$$p\tilde{F}^n_j = F|X^n \times I_j. \tag{8}$$

In order to obtain the continuous map $\tilde{F}^{n+1}_j$ satisfying $(7)^{n+1}_j$ and $(8)^{n+1}_j$ it suffices to find for each $(n+1)$-dimensional simplex $\Delta \in K$ a continuous map $\tilde{F}^{n+1,\Delta}_j: \Delta \times I_j \to E$ such that

$$\tilde{F}^{n+1,\Delta}_j|\Delta \times \{r_j\} = \tilde{F}_j|\Delta \times \{r_j\}, \tag{9}$$

$$\tilde{F}^{n+1,\Delta}_j|(\mathrm{bd}\,\Delta) \times I_j = \tilde{F}^n_j|(\mathrm{bd}\,\Delta) \times I_j \tag{10}$$

and

$$p\tilde{F}^{n+1,\Delta}_j = F|\Delta \times I_j. \tag{11}$$

This will be done in the next step.

S t e p 3. Suppose $F(\Delta \times I_j) \subset U_t$. Take $G = \Delta \times \{r_j\} \cup (\mathrm{bd}\,\Delta) \times I_j$ and let the continuous map $g: G \to E$ be defined by the condition $g|\Delta \times \{r_j\} = \tilde{F}_j|\Delta \times \{r_j\}$, and $g|(\mathrm{bd}\,\Delta) \times I_j = \tilde{F}^n_j|(\mathrm{bd}\,\Delta) \times I_j$. By Example 3.1.12 there exists a retraction $d: \Delta \times I_j \to G$. Define the continuous map $\tilde{F}^{n+1,\Delta}_j: \Delta \times I_j \to E$ by the formula

$$\tilde{F}^{n+1,\Delta}_j(x,r) = \varphi_t(F(x,r), q_t\varphi_t^{-1}gd(x,r)) \quad \text{for} \quad x \in \Delta, \quad r \in I_j.$$

We then have $\tilde{F}^{n+1,\Delta}_j(x,r_j) = \varphi_t(F(x,r_j), q_t\varphi_t^{-1}\tilde{F}_j(x,r_j))$ for $x \in \Delta$ and so (9) does indeed hold. Furthermore, we have $\tilde{F}^{n+1,\Delta}_j(x,r) = \varphi_t(F(x,r), q_t\varphi_t^{-1}\tilde{F}^n_j(x,r))$ for $x \in \mathrm{bd}\,\Delta$, $r \in I_j$ and so (10) also holds. Finally, condition (11) is satisfied in an obvious way.

The proof has thus been completed. ■

3.3.9. COROLLARY. *Let $p: E \to B$ be a fibration and suppose $p(e_0) = b_0$ where $e_0 \in E$. For every path d beginning at b_0 in the space B there is a path $\tilde{d}$ beginning at e_0 in the space E such that $p\tilde{d} = d$.*

PROOF. Consider the singleton space $\{x_0\}$. The path $d: I \to B$ may be regarded as a homotopy of the constant map $c_0: \{x_0\} \to B$ with $c(x_0) = b_0 = d(0)$. Since this map has a continuous lifting $\tilde{c}_0: \{x_0\} \to E$ with $\tilde{c}_0(x_0) = e_0$ there exits a continuous lifting $\tilde{d}: I \to E$ of the homotopy d such that $\tilde{d}(0) = e$. ■

Using Theorem 3.3.8 we prove a generalization of that same theorem. In fact the proof could have been given straight off in general, but we feared that the technical details might obscure the simple idea of the proof. Let $p: E \to B$ be a continuous map of E onto B. We say that the map p has the *homotopy lifting property relative to the pair* (X, A) where $A \subset X$, if, for every map $f: X \to B$ which has a continuous lifting $\tilde{f}: X \to E$, any homotopy F of the map f with the property that the restriction $F_0 = F|A \times I$ has a lifting $\tilde{F}_0$ which is a homotopy of the restriction $\tilde{f}|A$, possesses a lifting $\tilde{F}$ which is a homotopy of $\tilde{f}$ and is an extension of the homotopy $\tilde{F}_0$. Evidently the map $p: E \to B$ has the homotopy lifting property relative to the pair $(X, \emptyset)$ if and only if it has the homotopy lifting property relative to the space X.

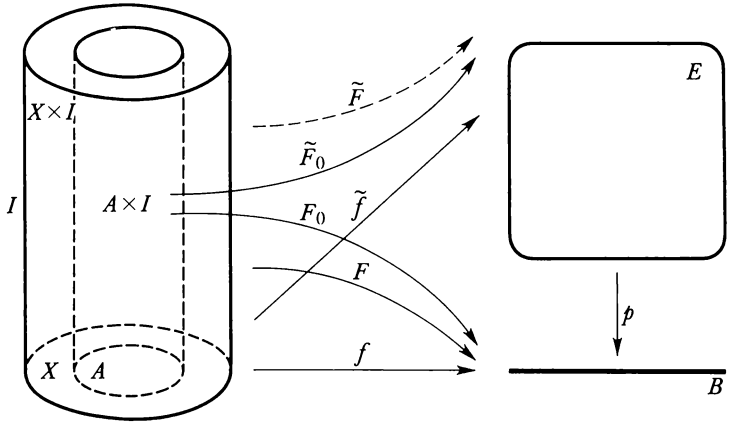


Fig.77. The map $p: E \to B$ has the homotopy lifting property relative to the pair (X, A) if for any map $f: X \to B$, any lifting $\tilde{f}: X \to E$ and any homotopy $F: X \times I \to B$ whose restriction F_0 to $A \times I$ has a lifting $\tilde{F}_0$ which is a homotopy of $\tilde{f}|A$, there exists a homotopy $\tilde{F}$ of the map $\tilde{f}$ which is a lifting of F and agrees with $\tilde{F}_0$ on $A \times I$.

First we prove the following.

3.3.10. LEMMA. *Every fibration has the homotopy lifting property relative to the pair* $(\Delta, \operatorname{bd}\Delta)$ *where* Δ *is any simplex.*

PROOF. Let $G = (\Delta \times \{0\}) \cup ((\operatorname{bd}\Delta) \times I)$. It is easy to see that there exists a homeomorphism h of the product $\Delta \times I$ onto itself such that $h(G) = \Delta \times \{0\}$ (see Exercise (e)). Let $\tilde{f}: \Delta \to E$ be a continuous lifting and let F be a homotopy of the map $f: \Delta \to B$. Let $\tilde{F}_0$ be a lifting of the restriction $F_0 = F|(\operatorname{bd}\Delta) \times I$ which is a homotopy of the restriction $\tilde{f}|\operatorname{bd}\Delta$. Define a continuous map $H: G \to E$ by the formula $H(x,0) = \tilde{f}(x)$ for $x \in \Delta$ and $H(x,r) = \tilde{F}_0(x,r)$ for $x \in \operatorname{bd}\Delta$, $r \in I$. Let $\Phi = Fh^{-1}: \Delta \times I \to B$ and $\tilde{\Psi} = Hh^{-1}|\Delta \times \{0\}: \Delta \times \{0\} \to E$. Since $p\tilde{\Psi} =$

$pHh^{-1}|\Delta\times\{0\} = Fh^{-1}|\Delta\times\{0\} = \Phi|\Delta\times\{0\}$ there exists by Theorem 3.3.8 a continuous map $\tilde{\Phi}\colon \Delta \times I \to E$ such that $\tilde{\Phi}|\Delta \times \{0\} = \tilde{\Psi}$ and $p\tilde{\Phi} = \Phi$. Taking $\tilde{F} = \tilde{\Phi}h$ we have $p\tilde{F} = p\tilde{\Phi}h = \Phi h = F$, $\tilde{F}(x,0) = \tilde{\Phi}h(x,0) = \tilde{\Psi}h(x,0) = H(x,0) = \tilde{f}(x)$ for $x \in \Delta$ and $\tilde{F}|(\mathrm{bd}\,\Delta) \times I = \tilde{\Phi}h|(\mathrm{bd}\,\Delta) \times I = \tilde{\Psi}h|(\mathrm{bd}\,\Delta) \times I = H|(\mathrm{bd}\,\Delta) \times I = \tilde{F}_0$. ■

We can now turn to the promised generalization of Theorem 3.3.8.

3.3.11. THEOREM. *Every fibration has the homotopy lifting property relative to any pair (X, A), where X and A are polyhedra.*

PROOF. By Theorem 2.6.15 we may assume that $X = |K|$ and $A = |\mathcal{L}|$, where $\mathcal{L}$ is a simplicial subcomplex of the simplicial complex K. Let $i_0\colon X \times \{0\} \to X$ be defined as in Example 3.2.2. Let $\tilde{f}\colon X \to E$, $p\tilde{f} = f$, $F\colon X \times I \to B$, $F|X \times \{0\} = fi_0$, $F_0 = F|A \times I$, $\tilde{F}_0\colon A \times I \to E$, $p\tilde{F}_0 = F_0$, and $F_0|A \times \{0\} = \tilde{f}i_0|A \times \{0\}$. We need to construct a map $\tilde{F}\colon X \times I \to E$ such that $p\tilde{F} = F$, $\tilde{F}|X \times \{0\} = \tilde{f}i_0$ and $\tilde{F}|A \times I = \tilde{F}_0$.

Let $K^{[m]}$ denote the m-dimensional skeleton of the complex K and let $X_m = |\mathcal{L} \cup K^{[m-1]}|$ for $m = 0, 1, \ldots, \dim K$. Using Lemma 3.3.10 and applying a double induction on m and on the number of m-dimensional simplices in $K^{[m-1]} \setminus \mathcal{L}$ we may construct for $m = 0, 1, \ldots, \dim K$ a map $\tilde{F}_m\colon X_m \times I \to E$ such that $p\tilde{F}_m = F|X_m \times I$, $\tilde{F}_m|X_m \times \{0\} = \tilde{f}i_0|X_m \times \{0\}$, $\tilde{F}_m|A \times I = \tilde{F}_0$ and $\tilde{F}_{m+1}|X_m \times I = \tilde{F}_m$. The map $\tilde{F} = \tilde{F}_{\dim K}$ has all the required properties. ■

We now consider the problem of the uniqueness of the lifting. We prove the following.

3.3.12. THEOREM. *Let $p\colon \tilde{Y} \to Y$ be a covering map and X a connected space. If $\tilde{f}', \tilde{f}''\colon X \to \tilde{Y}$ are continuous liftings of the map $f\colon X \to Y$ and there exists a point $x_0 \in X$ such that $\tilde{f}'(x_0) = \tilde{f}''(x_0)$, then $\tilde{f}' = \tilde{f}''$.*

PROOF. Let the sets U_t for $t \in T$ form an open covering of the space Y such that for each $t \in T$ the inverse image $p^{-1}(U_t)$ is a union of pairwise disjoint open sets $\tilde{U}_{t,j}$, with $p|\tilde{U}_{t,j}$ a homeomorphism of $\tilde{U}_{t,j}$ onto U_t for each j. Let $A = \{x \in X : \tilde{f}'(x) = \tilde{f}''(x)\}$; evidently A is a closed set and $x_0 \in A$.

We shall show that the set A is open in X. For let $a \in A$, $f(a) = p\tilde{f}'(a) \in U_t$, and $\tilde{f}'(a) \in \tilde{U}_{t,j}$ for some $t \in T$ and some j. Put $V_a = \tilde{f}'^{-1}(\tilde{U}_{t,j}) \cap \tilde{f}''^{-1}(\tilde{U}_{t,j})$; this is an open set in X and since $\tilde{f}''(a) = \tilde{f}'(a) \in U_{t,j}$, we have $a \in V_a$. Moreover, if $x \in V_a$ then $\tilde{f}'(x) \in \tilde{U}_{t,j}$, $\tilde{f}''(x) \in \tilde{U}_{t,j}$ and $p\tilde{f}'(x) = p\tilde{f}''(x)$; and, because $p|\tilde{U}_{t,j}$ is a homeomorphism, we have $\tilde{f}'(x) = \tilde{f}''(x)$, so $x \in A$; thus $V_a \subset A$.

Now X is connected so $A = X$, that is $\tilde{f}' = \tilde{f}''$. ■

By Corollary 3.3.9 and Theorem 3.3.12 we obtain the following corollary (see also Supplement 3.S.7).

3.3.13. COROLLARY. *Let $p\colon \tilde{Y} \to Y$ be a covering map and let $p(\tilde{y}_0) = y_0$, where $\tilde{y}_0 \in \tilde{Y}$. For every path d beginning at y_0 in the space Y there exists exactly one path $\tilde{d}$ beginning at $\tilde{y}_0$ in the space $\tilde{Y}$ such that $p\tilde{d} = d$.* ■

Exercises

a) Let $\tilde{Y} = \{(y^1, y^2) \in \mathbf{R}^2 : y^2 = \sin y^1\}$ and let $Y = \{(y^1, y^2) \in \mathbf{R}^2 : y^1 = 0 \text{ and } -1 \leq y^2 \leq 1\}$. Decide whether the map defined by the formula $p(y^1, y^2) = (0, y^2)$ for $(y^1, y^2) \in \tilde{Y}$ is a covering map.

b) Give an example of a map $p: E \to B$ which does not have the homotopy lifting property relative to a singleton space.

c) Suppose that $p_i: \tilde{Y}_i \to Y_i$ for $i = 1, 2$ are covering maps. Check whether $p: \tilde{Y} \to Y$, where $\tilde{Y} = \tilde{Y}_1 \times \tilde{Y}_2$, $Y = Y_1 \times Y_2$ and $p = p_1 \times p_2$, is a covering map. Examine the analogous question for fibrations.

d) Let $p: E \to B$ be a fibration with fibre W. Show that if the spaces B and W are compact, then the space E is also compact.

e) Show that for every simplex Δ there exists a homeomorphism h of the product $\Delta \times I$ onto itself such that $h(G) = \Delta \times \{0\}$, where $G = \Delta \times \{0\} \cup (\operatorname{bd} \Delta) \times I$ (see the proof of Lemma 3.3.10).

3.4. The fundamental group

In this section we shall – loosely speaking – associate with every metric space a group known as its *fundamental group* in such a way that the groups corresponding to homeomorphic spaces shall be isomorphic. This process is characteristic of algebraic topology and supplies an important tool for the study of the topological properties of various spaces. We shall make use of the concepts and methods of group theory within the scope embraced by the book [12].

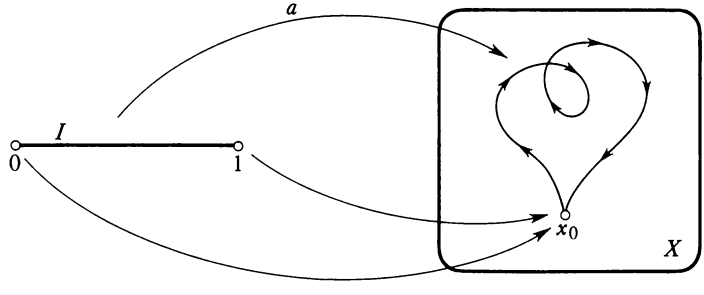


Fig.78. A loop a in the space X based at x_0.

Let X be a metric space and let $x_0 \in X$ be a fixed point which we shall call the *base point*. Every path in X from the point x_0 to the same point x_0 is known as a *loop in X based at x_0*. A loop a in X based at x_0 is thus a continuous map $a: I \to X$ such that $a(0) = a(1) = x_0$. We can therefore regard the loop a as a pair map $a: (I, \operatorname{bd} I) \to (X, x_0)$ where $\operatorname{bd} I = \{0, 1\}$. We denote the set of all loops in the space X based at x_0 by $L(X, x_0)$.

We now define two operations and distinguish a certain element of the set $L(X, x_0)$. For any two loops $a, b \in L(X, x_0)$ their *composition* $a * b$ is defined by the formula:

$$(a * b)(r) = \begin{cases} a(2r), & \text{if } 0 \leq r \leq \frac{1}{2}, \\ b(2r-1), & \text{if } \frac{1}{2} \leq r \leq 1. \end{cases}$$

Evidently $(a * b)(0) = a(0) = x_0 = b(1) = (a * b)(1)$ and $a(2 \cdot \frac{1}{2}) = a(1) = x_0 = b(0) = b(2 \cdot \frac{1}{2} - 1)$, so $a * b \in L(X, x_0)$. The constant loop $e \in L(X, x_0)$ defined by the condition $e(I) = \{x_0\}$ is called the *trivial* loop. Finally, for any loop $a \in L(X, x_0)$, we define the *inverse* loop $\bar{a}$ by taking $\bar{a}(r) = a(1-r)$ for $r \in I$; evidently $\bar{a} \in L(X, x_0)$.

Whenever we consider the relation of homotopy between two loops $a, b \in L(X, x_0)$ we will always regard them as pair maps $a, b: (I, \operatorname{bd} I) \to (X, x_0)$. We now prove the following.

3.4.1. THEOREM. *Let $a, a', b, b' \in L(X, x_0)$. If $a \simeq a'$ and $b \simeq b'$, then $a * b \simeq a' * b'$ and $\bar{a} \simeq \bar{a}'$.*

PROOF. Let $a \underset{H}{\simeq} a'$ and $b \underset{G}{\simeq} b'$. Define the map $H * G: I \times I \to x$ by the formula:

$$(H * G)(r, s) = \begin{cases} H(2r, s), & \text{if } 0 \leq r \leq \frac{1}{2},\ s \in I, \\ G(2r-1, s), & \text{if } \frac{1}{2} \leq r \leq 1,\ s \in I. \end{cases}$$

Then $(H * G)(0, s) = H(0, s) = x_0$ and $(H * G)(1, s) = G(1, s) = x_0$ for $s \in I$, so $H * G: (I \times I, (\operatorname{bd} I) \times I) \to (X, x_0)$. Moreover $H(2 \cdot \frac{1}{2}, s) = H(1, s) = x_0 = G(0, s) = G(2 \cdot \frac{1}{2} - 1, s)$ for $s \in I$, so the map $H * G$ is continuous. Finally $(H * G)(r, 0) = (a * a')(r)$ and $(H * G)(r, 1) = (b * b')(r)$ for $r \in I$, so $a * b \underset{H*G}{\simeq} a' * b'$.

Now define the map $\bar{H}: I \times I \to X$ by the formula $\bar{H}(r, s) = H(1-r, s)$ for $r \in I$, $s \in I$; this is of course continuous. Moreover $\bar{H}(0, s) = H(1, s) = x_0$ and $\bar{H}(1, s) = H(0, s) = x_0$ for $s \in I$, so $\bar{H}: (I \times I, (\operatorname{bd} I) \times I) \to (X, x_0)$. We also have $\bar{H}(r, 0) = H(1-r, 0) = a(1-r) = \bar{a}(r)$ and $\bar{H}(r, 1) = H(1-r, 1) = a'(1-r) = \bar{a}'(r)$ for $r \in I$, so $\bar{a} \underset{\bar{H}}{\simeq} \bar{a}'$. ■

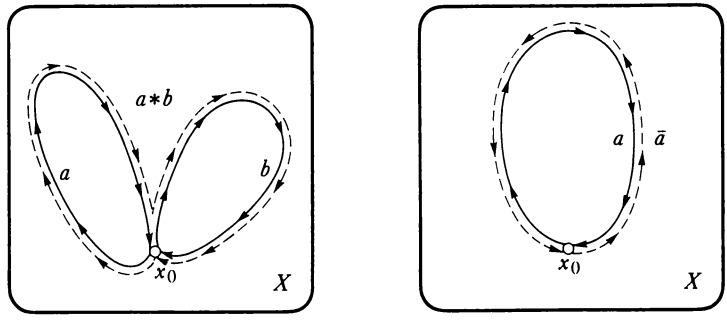


Fig.79. The composition $a * b$ of the loops $a, b \in L(X, x_0)$. The inverse loop $\bar{a}$ of $a \in L(X, x_0)$.

The homotopy class (in the sense of pair maps) of a loop $a \in L(X, x_0)$ will be denoted $[a]$. The set of equivalence classes of loops belonging to $L(X, x_0)$ will be denoted

by $\pi_1(X, x_0)$. In view of Theorem 3.4.1 the following operations on the set $\pi_1(X, x_0)$ are well defined.

Let $\alpha = [a]$ and $\beta = [b]$ with $a, b \in L(X, x_0)$. The class $[a * b]$ is called the *product* of the equivalence classes α and β and is denoted $\alpha\beta$; the operation which sends the pair $\alpha, \beta \in \pi_1(X, x_0)$ to the product $\alpha\beta \in \pi_1(X, x_0)$ is called *multiplication*. The class $\epsilon = [e]$ where $e \in L(X, x_0)$ is the trivial loop is called the *unit* class or briefly the *unity*. Let $\alpha = [a]$ where $a \in L(X, x_0)$. The class $[\bar{a}]$ is called the *inverse* class of the class α and is denoted by α^{-1}.

The following holds.

3.4.2. THEOREM. *If* $\alpha, \beta, \gamma \in \pi_1(X, x_0)$, *then*

(1) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;

(2) $\alpha\epsilon = \alpha$; *and*

(3) $\alpha\alpha^{-1} = \epsilon$.

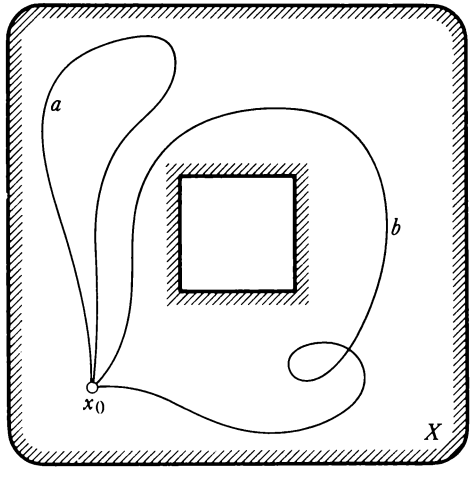


Fig.80. The loop a is homotopic to the trivial loop, so it represents the unity of the group $\pi_1(X, x_0)$. The loop b is not homotopic to the trivial loop, so it represents an element of the group $\pi_1(X, x_0)$ distinct from unity.

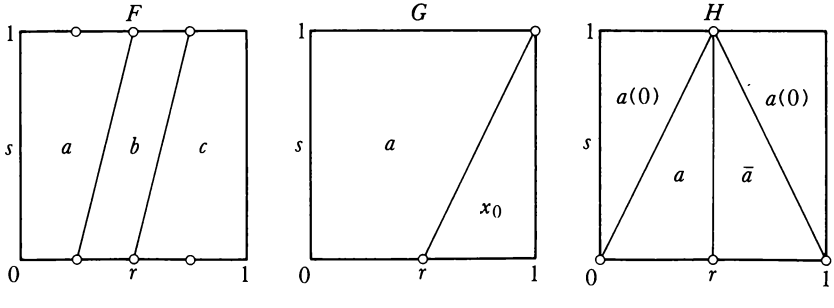


Fig.81. Diagrams illustrating the definition of the homotopies F, G, H in the proof of Theorem 3.4.2.

PROOF. Let $\alpha = [a]$, $\beta = [b]$, and $\gamma = [c]$ where $a, b, c \in L(X, x_0)$. To prove (1) we

need to show $(a * b) * c \simeq a * (b * c)$. Define a map $F: I \times I \to X$ by the formula:

$$F(r,s) = \begin{cases} a(4r/(s+1)), & \text{if } 0 \le r \le \frac{1}{4}(s+1), \ s \in I, \\ b(4r-s-1), & \text{if } \frac{1}{4}(s+1) \le r \le \frac{1}{4}(s+2), \ s \in I, \\ c((4r-s-2)/(2-s)), & \text{if } \frac{1}{4}(s+2) \le r \le 1, \ s \in I. \end{cases}$$

It may readily be checked that $F: (I \times I, (\operatorname{bd} I) \times I) \to (X, x_0)$, that F is a continuous map and that $F(r,0) = ((a * b) * c)(r)$ while $F(r,1) = (a * (b * c))(r)$ for $r \in I$. Thus

$$(a * b) * c \underset{F}{\simeq} a * (b * c).$$

To prove the equation (2) we need to show that $a * e \simeq a$. Define a map $G: I \times I \to X$ by the formula:

$$G(r,s) = \begin{cases} a(2r/(s+1)), & \text{if } 0 \le r \le \frac{1}{2}(s+1), \ s \in I, \\ x_0, & \text{if } \frac{1}{2}(s+1) \le r \le 1, \ s \in I. \end{cases}$$

It is easy to check that $G: (I \times I, (\operatorname{bd} I) \times I) \to (X, x_0)$, that G is a continuous map and that $G(r,0) = (a * e)(r)$ while $G(r,1) = a(r)$ for $r \in I$. Thus $a * e \underset{G}{\simeq} a$.

Finally to prove equation (3) we need to show that $a * \bar{a} \simeq e$. Define the map $H: I \times I \to X$ by the formula:

$$H(r,s) = \begin{cases} a(0), & \text{if } 0 \le r \le \frac{1}{2}s, \ s \in I, \\ a(2r-s), & \text{if } \frac{1}{2}s \le r \le \frac{1}{2}, \ s \in I, \\ a(2-2r-s), & \text{if } \frac{1}{2} \le r \le 1 - \frac{1}{2}s, \ s \in I, \\ a(0), & \text{if } 1 - \frac{1}{2}s \le r \le 1, \ s \in I. \end{cases}$$

It is easily checked that $H: (I \times I, (\operatorname{bd} I) \times I) \to (X, x_0)$, that H is a continuous map and that $H(r,0) = (a * \bar{a})(r)$ and $H(r,1) = e(r) = x_0$ for $r \in I$. Thus $a * \bar{a} \underset{H}{\simeq} e$. ■

It follows from Theorem 3.4.2, that the set $\pi_1(X, x_0)$ is a group under the operation of multiplication of equivalence classes, with the unity class acting as the unity of the group, and the inverse class acting as the group inverse; the group is known as the *fundamental group* of the space X relative to the base point x_0.

3.4.3. EXAMPLE. *Fundamental group of the circle.* Consider the covering map $p: \mathbf{R}^1 \to S^1$ given by the formula $p(x) = (\cos 2\pi x, \sin 2\pi x)$ for $x \in \mathbf{R}^1$, which was studied in Example 3.3.5. Let $y_0 = (1,0) = p(0)$. By Corollary 3.3.13, for each loop $a \in L(S^1, y_0)$ there is exactly one path $\tilde{a}$ on the real line $\mathbf{R}^1$ beginning at 0 such that $p\tilde{a} = a$. If moreover $a, a' \in L(S^1, y_0)$ and $a \underset{H}{\simeq} a'$ (with a and a' being regarded as pairs), then by Theorem 3.3.8 and by the uniqueness of the path $\tilde{a}'$ (compare Theorem 3.3.12) we have $\tilde{a} \underset{\tilde{H}}{\simeq} \tilde{a}'$, where $p\tilde{H} = H$. It follows that the integer $\tilde{a}(1)$ depends only on the homotopy class α of the loop a; denote it by $\varphi(\alpha)$.

The correspondence $\varphi: \pi_1(S^1, y_0) \to \mathbf{Z}$, where $\mathbf{Z}$ denotes the group of integers, is a homomorphism. For, let $\alpha = [a]$ and $\beta = [b]$, where $a, b \in L(S^1, y_0)$ and let $p\tilde{a} = a$

and $p\tilde{b} = b$ where the paths $\tilde{a}, \tilde{b}$ begin at 0. Define a path $\tilde{c}$ on $\mathbf{R}^1$ beginning at 0 by means of the formula:

$$\tilde{c}(r) = \begin{cases} \tilde{a}(2r), & \text{if } 0 \leq r \leq \frac{1}{2}, \\ \tilde{a}(1) + \tilde{b}(2r-1), & \text{if } \frac{1}{2} \leq r \leq 1. \end{cases}$$

Now $p\tilde{c} = a * b$ so by the uniqueness of the lifting we infer that $\varphi(\alpha\beta) = \varphi([a * b]) = \tilde{c}(1) = \tilde{a}(1) + \tilde{b}(1) = \varphi(\alpha) + \varphi(\beta)$.

The homomorphism φ is a monomorphism, for if we have $\varphi([a]) = 0$, that is $p\tilde{a} = a$ and $\tilde{a}(0) = \tilde{a}(1) = 0$, then in view of the contractibility of the real line $\mathbf{R}^1$ we have $\tilde{a} \simeq e_0$ and hence $a = p\tilde{a} \simeq pe_0 = e$ (where e_0 denotes the trivial loop in $\mathbf{R}^1$ based at 0); thus $[a] = \epsilon$. The homomorphism φ is also an epimorphism since for every integer n if we define the path $\tilde{a}: I \to \mathbf{R}^1$ by the formula $\tilde{a}(r) = nr$ for $r \in I$ and take $a = p\tilde{a}$, then we have $p([a]) = n$. We have shown that the fundamental group of the circle $\pi_1(S^1, y_0)$ is isomorphic with the group $\mathbf{Z}$ of integers. ■

We now examine how the fundamental group of a space depends on the choice of a base point. Let $x_0, x_1 \in X$ and suppose d is a path in X from x_0 to x_1. To each loop $a \in L(X, x_1)$ let us associate the loop $d_\#(a) \in L(X, x_0)$ defined by the formula:

$$d_\#(a)(r) = \begin{cases} d(3r), & \text{if } 0 \leq r \leq \frac{1}{3}, \\ a(3r-1), & \text{if } \frac{1}{3} \leq r \leq \frac{2}{3}, \\ d(3-3r), & \text{if } \frac{2}{3} \leq r \leq 1. \end{cases}$$

Let us also denote by $\bar{d}$ the path in X from x_1 to x_0 defined by the formula $\bar{d}(r) = d(1-r)$ for $r \in I$. Then the following holds.

3.4.4. LEMMA. (1) *If $a, a' \in L(X, x_1)$ and $a \simeq a'$, then $d_\#(a) \simeq d_\#(a')$;*
(2) *if $a, b \in L(X, x_1)$, then $d_\#(a * b) \simeq d_\#(a) * d_\#(b)$; and*
(3) *if $a \in L(X, x_1)$, then $\bar{d}_\# d_\#(a) \simeq a$.*

PROOF. (1) Suppose that $a \underset{F}{\simeq} a'$ and define the map $F': (I \times I, (\operatorname{bd} I) \times I) \to (X, x_0)$ by the formula:

$$F'(r,s) = \begin{cases} d(3r), & \text{if } 0 \leq r \leq \frac{1}{3}, \\ F(3r-1, s), & \text{if } \frac{1}{3} \leq r \leq \frac{2}{3}, \\ d(3-3r), & \text{if } \frac{2}{3} \leq r \leq 1, \end{cases}$$

for $s \in I$. Then $d_\#(a) \underset{F'}{\simeq} d_\#(a')$.

(2) Define the map $G: (I \times I, (\operatorname{bd} I) \times I) \to (X, x_0)$ by the formula:

$$G(r,s) = \begin{cases} d(6r/(2-s)), & \text{if } 0 \leq r \leq \frac{1}{6}(2-s) \\ a(6r+s-2), & \text{if } \frac{1}{6}(2-s) \leq r \leq \frac{1}{6}(3-s) \\ d(-6r-s+4), & \text{if } \frac{1}{6}(3-s) \leq r \leq \frac{1}{2} \\ d(6r-s-2), & \text{if } \frac{1}{2} \leq r \leq \frac{1}{6}(s+3) \\ b(6r-s-3), & \text{if } \frac{1}{6}(s+3) \leq r \leq \frac{1}{6}(s+4) \\ d(6r-6)/(s-2)), & \text{if } \frac{1}{6}(s+4) \leq r \leq 1, \end{cases}$$

for $s \in I$. Then $d_\#(a * b) \underset{G}{\simeq} d_\#(a) * d_\#(b)$.

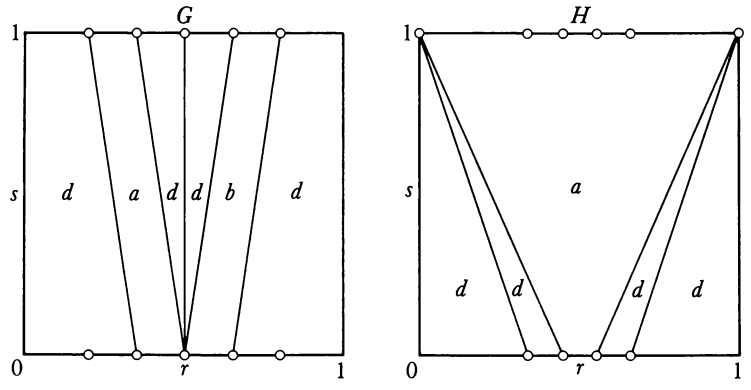


Fig.82. Diagrams illustrating the definitions of the homotopies G and H in the proof of Lemma 3.4.4.

(3) Define a map $H\colon (I \times I, (\mathrm{bd}\, I) \times I) \to (X, x_1)$ by the formula:

$$H(r,s) = \begin{cases} d(1-3r), & \text{if } 0 \le r \le \frac{1}{3}(1-s), \\ d(9r+4s-3), & \text{if } \frac{1}{3}(1-s) \le r \le \frac{1}{9}(4-4s), \\ a((9r+4s-4)/(8s+1)), & \text{if } \frac{1}{9}(4-4s) \le r \le \frac{1}{9}(4s+5), \\ d(-9r+4s+6), & \text{if } \frac{1}{9}(4s+5) \le r \le \frac{1}{3}(s+2), \\ d(3r-2), & \text{if } \frac{1}{3}(s+2) \le r \le 1, \end{cases}$$

for $s \in I$. Then $\bar{d}_{\#} d_{\#}(a) \underset{H}{\simeq} a$. ■

Property (1) of Lemma 3.4.4 implies that the map $d_{\#}\colon L(X, x_1) \to L_1(X, x_0)$ induces a map $d_*\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ defined by the formula $d_*(\alpha) = [d_{\#}(a)]$, where $\alpha = [a] \in \pi_1(X, x_1)$. Moreover the following holds.

3.4.5. THEOREM. *The map $d_*\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.*

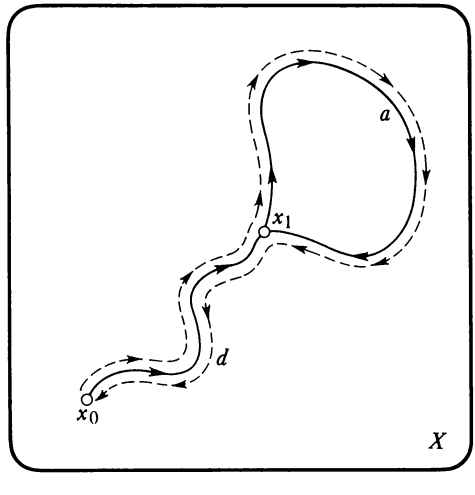


Fig.83. The path d from x_0 to x_1 determines an isomorphism $d_*\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ (Theorem 3.4.5).

PROOF. It follows from property (2) of Lemma 3.4.4 that d_* is a homomorphism. We deduce from property (3), taking advantage of the symmetry of the argument, that $\bar{d}_* d_* = \mathrm{id}$ and $d_* \bar{d}_* = \mathrm{id}$; this implies that d_* is an isomorphism with $\bar{d}_*$ as its inverse isomorphism. ■

It follows from Theorem 3.4.5 that if a space X is pathwise connected, then, up to isomorphism, the group $\pi_1(X, x_0)$ does not depend on the choice of the base point $x_0 \in X$. In this case we may therefore suppress the distinguished point in the notation and in place of $\pi_1(X, x_0)$ we may simply write $\pi_1(X)$ with the tacit assumption that the space X is non-empty (see also Supplement 3.S.6).

Now we show that continuous maps between spaces induce homomorphisms of the fundamental groups of the spaces. Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous pair map. To each loop $a \in L(X, x_0)$ there corresponds a loop $f_\#(a) \in L(Y, y_0)$ defined by the formula $f_\#(a)(r) = fa(r)$ for $r \in I$. The following obviously holds.

3.4.6. ASSERTION. (1) *If* $a, a' \in L(X, x_0)$ *and* $a \simeq a'$, *then* $f_\#(a) \simeq f_\#(a')$; *and*
(2) *if* $a, b \in L(X, x_0)$, *then* $f_\#(a * b) = f_\#(a) * f_\#(b)$. ■

It follows from property (1) of the assertion that the map $f_\#\colon L(X, x_0) \to L(Y, y_0)$ determines a map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by the formula $f_*(\alpha) = [f_\#(a)]$ where $\alpha = [a] \in \pi_1(X, x_0)$. By property (2) we infer that the map f_* is a homomorphism, we call it the homomorphism *induced* by the map f.

We note that the following obviously holds.

3.4.7. ASSERTION. (1) *If* $f, g\colon (X, x_0) \to (Y, y_0)$ *and* $f \simeq g$, *then* $f_\#(a) \simeq g_\#(a)$ *for each loop* $a \in L(X, x_0)$;
(2) *if* $f\colon (X, x_0) \to (Y, y_0)$ *and* $g\colon (Y, y_0) \to (Z, z_0)$, *then* $(gf)_\# = g_\# f_\#$; *and*
(3) $\mathrm{id}_\# = \mathrm{id}$, *where on the left-hand side is the identity map of the pair* (X, x_0) *and the right-hand side is the identity map of the set* $L(X, x_0)$. ■

The following is an immediate consequence of the assertion above.

3.4.8. THEOREM. (1) *If* $f, g\colon (X, x_0) \to (Y, y_0)$ *and* $f \simeq g$, *then* $f_* = g_*$;
(2) *if* $f\colon (X, x_0) \to (Y, y_0)$ *and* $g\colon (Y, y_0) \to (Z, z_0)$, *then* $(gf)_* = g_* f_*$; *and*
(3) $\mathrm{id}_* = 1$, *where* id *denotes the identity map of the pair* (X, x_0) *and* 1 *is the identity isomorphism of the group* $\pi_1(X, x_0)$. ■

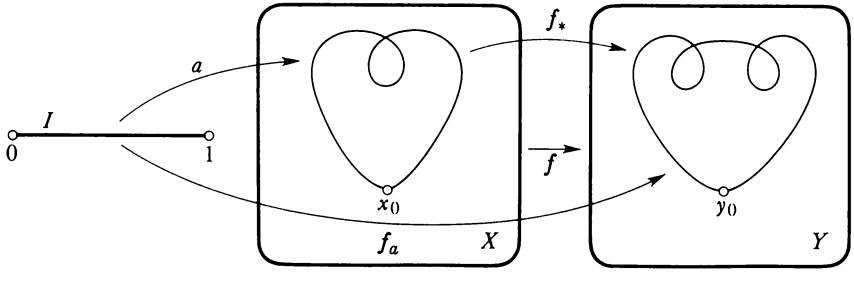


Fig.84. The continuous map $f\colon (X, x_0) \to (Y, y_0)$ induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

We now generalize property (1) of the theorem above as follows.

3.4.9. THEOREM. *Let $f, g: X \to Y$ and $x_0 \in X$. Consider the corresponding pair maps $f': (X, x_0) \to (Y, f(x_0))$ and $g': (X, x_0) \to (Y, g(x_0))$. If $f \underset{H}{\simeq} g$ and the path d in the space Y from $f(x_0)$ to $g(x_0)$ is defined by the formula $d(r) = H(x_0, r)$ for $r \in I$, then $f'_* = d_* g'_*$.*

PROOF. Suppose $a \in L(X, x_0)$ and define a continuous map $h: I \times I \to Y$ by the formula $h(r,s) = H(a(r), s)$ for $r \in I$, $s \in I$ and a loop $b \in L(I \times I, (0,0))$ by the formula:

$$b(t) = \begin{cases} (0, 6t), & \text{if } 0 \leq t \leq \frac{1}{6}, \\ (6t-1, 1), & \text{if } \frac{1}{6} \leq t \leq \frac{1}{3}, \\ (1, 3-6t), & \text{if } \frac{1}{3} \leq t \leq \frac{1}{2}, \\ (2-2t, 0), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is easy to see that $h_\#(b) = d_\#(g'_\#(a)) * f'_\#(\bar{a})$. Now the loop b is homotopic to the trivial loop in $L(I \times I, (0,0))$, hence $\epsilon = d_* g'_*([a]) f'_*([a]^{-1})$, that is $f'_*([a]) = d_* g'_*([a])$. ■

Using the theorems above we prove the following.

3.4.10. THEOREM. *If pairs (X, x_0) and (Y, y_0) have the same homotopy type, then the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic. If pathwise connected spaces X and Y have the same homotopy type, then the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.*

PROOF. If the pair maps $f: (X, x_0) \to (Y, y_0)$ and $g: (Y, y_0) \to (X, x_0)$ satisfy the condition $fg \simeq \mathrm{id}_{(Y, y_0)}$ and $gf \simeq \mathrm{id}_{(X, x_0)}$, then in view of Theorem 3.4.8 we have $f_* g_* = (fg)_* = (\mathrm{id}_{(Y, y_0)})_* = 1$ and $g_* f_* = (gf)_* = (\mathrm{id}_{(X, x_0)})_* = 1$. Thus f_* and g_* are inverse isomorphisms.

Passing to the proof of the second part of the theorem, suppose that the maps $f: X \to Y$ and $g: Y \to X$ satisfy the conditions $fg \underset{H'}{\simeq} \mathrm{id}_Y$ and $gf \underset{H''}{\simeq} \mathrm{id}_X$. Consider any point $x_0 \in X$ and let $f': (X, x_0) \to (Y, f(x_0))$, $g': (Y, f(x_0)) \to (X, x_0)$ and $f'': (X, gf(x_0)) \to (Y, fgf(x_0))$ be the pair maps determined in the obvious way by the maps f and g. Then $f''g'$ maps $(Y, f(x_0))$ to $(Y, fgf(x_0))$ and $g'f'$ maps (X, x_0) to $(X, gf(x_0))$. Consider the path d' in the space Y from $fgf(x_0)$ to $f(x_0)$ defined by the formula $d'(r) = H'(f(x_0), r)$ for $r \in I$ and the path d'' in the space X from $gf(x_0)$ to x_0 defined by the formula $d''(r) = H''(x_0, r)$ for $r \in I$. Applying Theorem 3.4.9 we deduce that $(f''g')_* = d'_*(\mathrm{id}_{(Y, f(x_0))})_*$ and $(g'f')_* = d''_*(\mathrm{id}_{(X, x_0)})_*$ and so by Theorem 3.4.8 we have $f''_* g'_* = d'_*$ and $g'_* f'_* = d''_*$. But d'_* and d''_* are isomorphisms (see Theorem 3.4.5), so g'_* is both a monomorphism and an epimorphism, hence is an isomorphism. By the pathwise connectedness of the space, the actual choice of base points is inessential, hence the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic. ■

By Example 3.2.19 we obtain the following.

3.4.11. COROLLARY. *If pathwise connected spaces X and Y are homeomorphic, then the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.* ■

Assertion 3.2.10 and Theorems 3.2.22 and 3.4.10 imply the following.

3.4.12. COROLLARY. *The fundamental group of any contractible space is trivial.* ■

A metric space which is pathwise connected and whose fundamental group is trivial is called *simply connected.* Corollary 3.4.12 can therefore be alternatively expressed by saying that every contractible space is simply connected.

We now determine the fundamental group of the metric product of two spaces. To be specific, we prove the following.

3.4.13. THEOREM. *Let $x_1 \in X_1$ and $x_2 \in X_2$. Then the fundamental group $\pi_1(X_1 \times X_2, (x_1, x_2))$ is isomorphic to the direct product of the groups $\pi_1(X_1, x_1)$ and $\pi_1(X_2, x_2)$.*

PROOF. Consider the projection $p_i: X_1 \times X_2 \to X_i$ where $p_i(x_1, x_2) = x_i$ for $(x_1, x_2) \in X_1 \times X_2$, $i = 1, 2$. Let the homomorphism $\varphi: \pi_1(X_1 \times X_2, (x_1, x_2)) \to \pi_1(X_1, x_1) \times \pi_1(X_2, x_2)$ be defined by the formula $\varphi(\alpha) = (p_{1*}(\alpha), p_{2*}(\alpha))$ for $\alpha \in \pi_1(X_1 \times X_2, (x_1, x_2))$. We show that φ is an isomorphism.

To show that φ is a monomorphism, assume that $\varphi(\alpha) = (\epsilon_1, \epsilon_2)$ where $\alpha = [a] \in \pi_1(X_1 \times X_2, (x_1, x_2))$ and ϵ_i is the unity of the group $\pi_1(X_i, x_i)$ for $i = 1, 2$. Thus $p_i a \simeq e_i$, where $e_i \in L(X_i, x_i)$ is the trivial loop for $i = 1, 2$. Let H_i be a homotopy from $p_i a$ to e_i for $i = 1, 2$. Then, taking $H(r, s) = (H_1(r, s), H_2(r, s))$ for $(r, s) \in I \times I$, we have $a \underset{H}{\simeq} e$, where $e \in L(X_1 \times X_2, (x_1, x_2))$ is the trivial loop. Thus α is the unity in the group $\pi_1(X_1 \times X_2, (x_1, x_2))$.

To show that φ is an epimorphim suppose that $\alpha_i = [a_i] \in \pi_1(X_i, x_i)$ for $i = 1, 2$. Taking $a(r) = (a_1(r), a_2(r))$ for $r \in I$ we obtain a loop $a \in L(X_1 \times X_2, (x_1, x_2))$ such that $p_i a = a_i$ for $i = 1, 2$. Hence taking $\alpha = [a] \in \pi_1(X_1 \times X_2, (x_1, x_2))$ we obtain $\varphi(\alpha) = (\alpha_1, \alpha_2)$. ■

3.4.14. EXAMPLE. *The fundamental group of the torus.* Let $T = S^1 \times S^1$. Using Example 3.4.3 and Theorem 3.4.13 we infer that the group $\pi_1(T)$ is isomorphic to the group $\mathbf{Z} \times \mathbf{Z}$, that is, it is an abelian group on two generators. ■

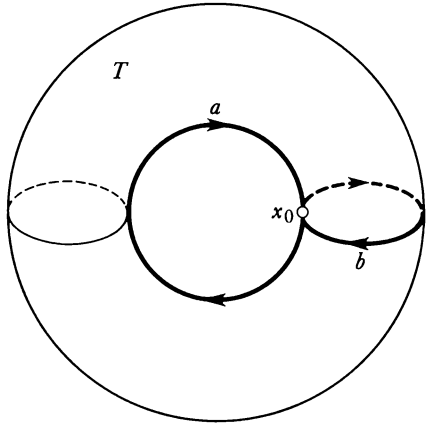


Fig.85. The generators $[a]$ and $[b]$ of the fundamental group of the torus T.

The definition of the fundamental group, though conceptually simple and intuitively acceptable, is inadequate for practical purposes. Already in the case of the

circle, simplest of all barring the trivial case, finding the group required an appeal to results in the theory of covering maps (see Example 3.4.3). Even in the case of polyhedra, despite their particularly simple structure, the definition itself does not directly provide an algorithm for determining in a finite number of steps the generators and relations of the fundamental group; it is not even evident whether the number of generators is finite. We now proceed to the presentation of a narrower theory which yields just such an algorithm for polyhedra.

Let K be a simplicial complex. A sequence (possibly including repetitions) of vertices $v_0, v_1, \ldots, v_k$ of the complex is called an *edge-path* from the vertex v_0 to the vertex v_k in the complex K if $\Delta\{v_{j-1}, v_j\} \in K$ for $j = 1, 2, \ldots, k$; we denote the path by $v_0 v_1 \ldots v_k$ and call the vertex v_0 its *beginning* and v_k its *end*. We shall apply the following operations to edge-paths:

Reduction. The path $v_0 v_1 \ldots v_k$ for which $\Delta\{v_{j-1}, v_j, v_{j+1}\} \in K$ for some j with $1 \leq j \leq k-1$ may be replaced by the path $v_0 v_1 \ldots v_{j-1} v_{j+1} \ldots v_k$; the path $v_0 v_0$ for $v_0 \in K$ my be replaced by the path v_0.

Expansion. The path $v_0 v_1 \ldots v_{j-1} v_{j+1} \ldots v_k$ may be replaced by the path $v_0 v_1 \ldots v_k$ if $\Delta\{v_{j-1}, v_j, v_{j+1}\} \in K$; the path v_0 for $v_0 \in K$ may be replaced by the path $v_0 v_0$.

Observe that if two consecutive vertices of an edge-path are equal then by applying a reduction we may drop one of the them. Similarly an expansion allows us to write in a vertex v of an edge-path one more time immediately before or after its occurrence.

We say that two edge-paths $\hat{d}$ and $\hat{d}'$ of a complex K are *edge-homotopic*, which we write as $\hat{d} \simeq \hat{d}'$, if the path $\hat{d}'$ may be obtained from the path $\hat{d}$ by a finite number of reductions and expansions. Evidently if $\hat{d} \simeq \hat{d}'$, then the edge-paths $\hat{d}$ and $\hat{d}'$ have identical beginning and end. It is equally clear that the relation of edge-homotopy is an equivalence on the set of all edge-paths with a fixed beginning and end. The equivalence class of this relation containing an edge-path $\hat{d}$ will be denoted by $[\hat{d}]$.

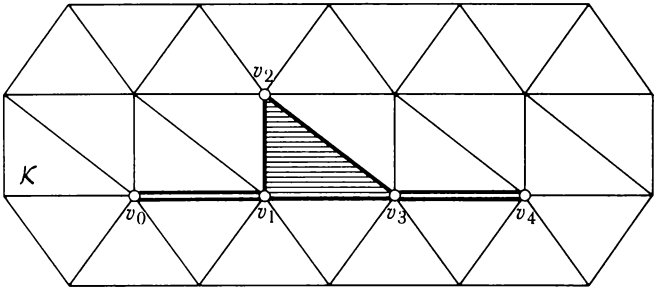


Fig.86. The edge-paths $v_0 v_1 v_2 v_3 v_4$ and $v_0 v_1 v_3 v_4$ of the complex K are edge-homotopic since $\Delta\{v_1 v_2 v_3\} \in K$.

An edge-path from a vertex $v_0 \in K$ to the same vertex v_0 is called an *edge-loop based at* v_0. The set of all edge-loops based at a vertex v_0 in a complex K will be denoted by $\hat{L}(K, v_0)$. For any two edge-loops $\hat{a} = a_0 a_1 \ldots a_k$ and $\hat{b} = b_0 b_1 \ldots b_l$ where $a_0 = v_0 = a_k$ and $b_0 = v_0 = b_l$ we define their *composition* $\hat{a} * \hat{b}$ to be $a_0 a_1 \ldots a_k b_0 b_1 \ldots b_l$; this defines an operation on the set $\hat{L}(K, v_0)$. We call the edge-loop $\hat{e} \in \hat{L}(K, v_0)$ consisting of the one vertex v_0 the *trivial* edge-loop. For each loop $\hat{a} = a_0 a_1 \ldots a_k$, where

$a_0 = v_0 = a_k$ we define its *inverse* loop to be $\bar{\hat{a}} = a_k a_{k-1} \dots a_2 a_1$.

Evidently if $\hat{a}, \hat{a}', \hat{b}, \hat{b}' \in \hat{L}(K, v_0)$, $\hat{a} \frown \hat{a}'$, and $\hat{b} \frown \hat{b}'$ then $\hat{a} * \hat{b} \frown \hat{a}' * \hat{b}'$ and $\bar{\hat{a}} \frown \bar{\hat{a}}'$. We can therefore introduce into the set $\hat{\pi}_1(K, v_0)$ of equivalence classes of the edge-homotopy an operation of *multiplication* which sends the classes $\hat{\alpha} = [\hat{a}]$ and $\hat{\beta} = [\hat{b}] \in \hat{\pi}_1(K, v_0)$ to their *product* $\hat{\alpha}\hat{\beta} = [\hat{a} * \hat{b}] \in \hat{\pi}_1(K, v_0)$. The class $\hat{\varepsilon} = [\hat{e}]$, where $\hat{e} \in \hat{L}(K, v_0)$ is the trivial edge-loop, will be called the *unit* class or briefly the *unity*. Finally the class $\hat{\alpha}^{-1} = [\bar{\hat{a}}]$ is the *inverse* class of the equivalence class $\hat{\alpha} = [\hat{a}] \in \hat{\pi}_1(K, v_0)$. If $\hat{\alpha} = [\hat{a}]$, $\hat{\beta} = [\hat{b}]$ and $\hat{\gamma} = [\hat{c}]$, then since $(\hat{a} * \hat{b}) * \hat{c} = \hat{a} * (\hat{b} * \hat{c})$ and $\hat{a} * \hat{e} = \hat{a}$, we have $(\hat{\alpha}\hat{\beta})\hat{\gamma} = \hat{\alpha}(\hat{b}\hat{\gamma})$ and $\hat{\alpha}\hat{\varepsilon} = \hat{a}$; furthermore, it is easy to see that $\hat{a} * \bar{\hat{a}} \frown \hat{e}$ and so $\hat{\alpha}\hat{\alpha}^{-1} = \hat{\varepsilon}$. It follows that the set $\hat{\pi}_1(K, v_0)$ forms a group under class multiplication as defined above, with the unit class as the unity of the group and the inverse class as the group inverse. We call this the *edge-group* of the complex K based at v_0.

If φ is a simplicial map of a simplicial complex K into the simplicial complex $\mathcal{L}$, then to each edge-path $\hat{d} = v_0, v_1 \dots, v_k \in K$ there corresponds an edge-path $\varphi_\#(\hat{d}) = \varphi(v_0)\varphi(v_1)\dots\varphi(v_k) \in \mathcal{L}$. Of course if $\hat{d} \frown \hat{d}'$ then $\varphi_\#(\hat{d}) \frown \varphi_\#(\hat{d}')$. It follows that a simplicial map $\varphi(K, v_0) \to (K, w_0)$ induces a map $\varphi_*: \hat{\pi}_1(K, v_0) \to \hat{\pi}_1(\mathcal{L}, w_0)$ defined by the formula $\varphi_*(\hat{\alpha}) = [\varphi_\#(\hat{a})]$ where $\hat{\alpha} = [\hat{a}] \in \hat{\pi}_1(K, v_0)$. It is also easy to see that φ_* is a homomorphism of the group $\hat{\pi}_1(K, v_0)$ into the group $\hat{\pi}_1(\mathcal{L}, w_0)$ and further $(\psi\varphi)_* = \psi_*\varphi_*$ and $\mathrm{id}_* = 1$.

The construction of the edge-group was introduced using terminology similar to that appearing in the definition of the fundamental group. We now show that the similarity is not just formal. Let $\hat{d} = v_0 v_1 \dots v_k$ be an edge-path from the vertex v_0 to the vertex v_k in a simplicial complex K. Associate with it the ordinary path $\kappa_\#(\hat{d}) = d: I \to |K|$ from the point v_0 to the point v_k in the polyhedron $|K|$ defined by the formula $d(r) = (j - kr)v_{j-1} + (1 - j + kr)v_j$ for $(j-1)/k \leq r \leq j/k$ and $j = 1, 2, \dots, k$ (if $k = 0$ we understand d to be the path with constant value v_0). It is easy to see that if $\hat{d} \frown \hat{d}'$ then $\kappa_\#(\hat{d}) \simeq \kappa_\#(\hat{d}')$; we thus obtain a map $\kappa_*: \hat{\pi}_1(K, v_0) \to \pi_1(|K|, v_0)$ via the formula $\kappa_*(\hat{\alpha}) = [\kappa_\#(\hat{a})]$ for $\hat{\alpha} = [\hat{a}] \in \hat{\pi}_1(K, v_0)$. It follows immediately from the definitions of $\kappa_\#$, of the operations of the edge-group, and of the fundamental group, that κ_* is a homomorphism of the group $\hat{\pi}_1(K, v_0)$ into the group $\pi_1(|K|, v_0)$. The homomorphism is natural in the sense that for any simplicial map $\varphi: (K, v_0) \to (\mathcal{L}, w_0)$ we have $|\varphi|_* \kappa_* = \kappa_* \varphi_*$.

We now prove a basic theorem which justifies the introduction of the edge-group.

3.4.15. THEOREM. *The homomorphism $\kappa_*: \hat{\pi}_1(K, v_0) \to \pi_1(|K|, v_0)$ is an isomorphism.*

PROOF. To prove that κ_* is a monomorphism, we need to check that if $\hat{a} \in \hat{L}(K, v_0)$ and $a = \kappa_\#(\hat{a}) \simeq e \in L(|K|, v_0)$, then $\hat{a} \frown \hat{e} \in \hat{L}(K, v_0)$. Let $H: (I \times I, (\mathrm{bd}\, I) \times I) \to (|K|, v_0)$ be a homotopy from a to e. Consider a triangulation $\mathcal{M}$ of the square $I \times I$ for which, as in Theorem 2.5.10, there is a simplicial approximation $\Phi: \mathcal{M} \to K$ to the map H. Furthermore we may assume that if $\hat{a}$ contains $k + 1$ vertices, then the points $(j/k, 0)$ where $j = 0, 1, \dots, k$ appear among the vertices of the simplicial complex $\mathcal{M}$. Let $\hat{b} = b_0 b_1 \dots b_p$ be the edge-path in $\mathcal{M}$ composed of all the vertices of $\mathcal{M}$ which lie on the segment $B = I \times \{0\}$ in their natural order so that $b_0 = (0, 0)$ and $b_p = (1, 0)$. Let $\mathcal{T}$ denote the triangulation of the segment B defined by the vertices $b_0, b_1, \dots, b_p$.

Since $\Phi|\mathcal{T}$ is a simplicial approximation of the map $H|B$ with $H(r,0) = a(r)$ or $r \in I$, we have $a(j/k) = \Phi(j/k,0)$ for $j = 0,1,\ldots,k$, hence $\hat{a} \frown \Phi_{\#}(\hat{b})$. Let $\hat{c} = c_0c_1\ldots c_q$ be an edge-path in $\mathcal{M}$ consisting of all the vertices of $\mathcal{M}$ which lie in the set $C = \{0\} \times I \cup I \times \{1\} \cup \{1\} \times I$ in their natural order with $c_0 = (0,0)$ an $c_q = (1,0)$. It is easy to check that $\hat{b} \frown \hat{c}$ so that $\Phi_{\#}(\hat{b}) = \Phi_{\#}(\hat{c})$. Since $H(C) = \{v_0\}$, also $|\Phi|(C) = \{v_0\}$ and so $\Phi_{\#}(\hat{c}) = \hat{e}$. Thus we do indeed have $\hat{a} \frown \Phi_{\#}(\hat{b}) \frown \Phi_{\#}(\hat{c}) \frown \hat{e}$.

To show that κ_* is an epimorphism consider an arbitrary element $\alpha = [a]$ of $\pi_1(|K|,v_0)$. By Theorem 2.5.10 there exists a simplicial subdivision $\mathcal{N}$ of the interval I and a simplicial approximation $\varphi\colon \mathcal{N} \to K$ of the map $a\colon I \to |K|$. By the definition of a simplicial approximation and since $a(0) = v_0 = a(1)$ we obtain $\varphi(0) = v_0 = \varphi(1)$. By Example 3.2.5 we have $|\varphi| \simeq a$; moreover from the form of the homotopy constructed in this example we infer that it is a homotopy of the pair maps $|\varphi|, a\colon (I, \operatorname{bd} I) \to (|K|,v_0)$, so we have $\alpha = [|\varphi|]$. Suppose $0 = r_0 < r_1 < \ldots < r_{k-1} < r_k = 1$ are all the vertices of the triangulation $\mathcal{N}$ of the interval I. Denote by $\overline{\mathcal{N}}$ the triangulation of the interval which has vertices j/k for $j = 0,1,\ldots$. It is easy to find a simplicial isomorphism $\psi\colon \overline{\mathcal{N}} \to \mathcal{N}$ such that $\psi(j/k) = r_j$ for $j = 0,1,\ldots,k$. Then the pair maps $|\varphi|$, $|\varphi\psi|\colon (I, \operatorname{bd} I) \to (|K|,v_0)$ are homotopic and so $\alpha = [|\varphi\psi|]$. On the other hand putting $\hat{a} = v_0v_1\ldots v_k$, where $v_j = \varphi\psi(j/k)$ for $j = 0,1,\ldots,k$, we obtain $\kappa_{\#}(\hat{a}) = |\varphi\psi|$. Thus taking $\hat{\alpha} = [\hat{a}]$ we arrive at $\kappa_*(\hat{\alpha}) = [\kappa_{\#}(\hat{a})] = [|\varphi\psi|] = \alpha$. ■

By Theorem 3.4.15 and in view of Theorems 3.4.5, 3.1.24 and 2.3.6 it follows that if a simplicial complex is connected, then the edge-group $\hat{\pi}_1(K,v_0)$, up to isomorphism, does not depend on the choice of the vertex v_0. In such a case we may suppress the mention of the distinguished vertex in the notation and in place of $\hat{\pi}_1(K,v_0)$ we will write simply $\hat{\pi}_1(K)$ with the tacit assumption that the complex K is non-empty. If K is a connected complex and the group $\hat{\pi}_1(K)$ is trivial, then the complex K is said to be *simply connected*. A graph (that is, a simplicial complex of dimension ≤ 1) which is simply connected is called a *tree*. The following simple result is worth noting.

3.4.16. ASSERTION. *For a graph to be a tree it is necessary and sufficient that it does not contain an edge-loop $v_0v_1\ldots v_{k-1}v_0$ with $k \geq 3$ and $v_i \neq v_j$ for $i \neq j$.* ■

Observe that in the definition of the edge-group we made use of simplices of dimension 2 at most. The following is therefore a consequence of Theorem 3.4.15.

3.4.17. COROLLARY. *If $K^{[2]}$ denotes the 2-dimensional skeleton of the simplicial complex K, then the groups $\pi_1(|K|)$ and $\pi_1(|K^{[2]}|)$ are isomorphic.* ■

Using Theorem 3.4.15 we may now give an effective algorithm which will permit the computation of all the generators and relations of the fundamental group of a polyhedron. We begin with a proof of the following lemma.

3.4.18. LEMMA. *For every simply connected subcomplex $\mathcal{L}$ of a connected simplicial complex K there exists a simply connected subcomplex K_* such that $\mathcal{L} \subset K_*$, $\dim K_* \leq \max(1, \dim \mathcal{L})$, and K_* contains all the vertices of the complex K.*

PROOF. Since the complex K contains a finite number of simplexes there exists a simply connected subcomplex K_* such that $\mathcal{L} \subset K_*$, $\dim K_* \leq \max(1, \dim \mathcal{L})$ and K_*

is maximal with respect to this property. If the complement $K \setminus K_*$ contained even one vertex, then in view of the connectedness of the complex K and the fact that subcomplex K_* is non-empty there would exist vertices $a \in K_*$ and $b \in K \setminus K_*$ with $\Delta(a,b) \in K$. Taking $K_*' = K_* \cup \{\Delta(a,b)\} \cup \{b\}$ we would obtain a simply connected subcomplex satisfying the conditions $\mathcal{L} \subset K_*'$ and $\dim K_*' \leq \max(1, \dim \mathcal{L})$, despite the maximality of K_*. ■

For the applications at hand the following corollary of Lemma 3.4.18 is more adequate.

3.4.19. COROLLARY. *For every non-empty, connected complex K there exists a simply connected subcomplex K_* containing all the vertices of the complex K.* ■

For any simplicial complex K and any subcomplex K_* denote by $G(K, K_*)$ the group generated by all the edge-paths in K of the form vv' with the relations:

(1) $$vv' = 1, \quad \text{if} \quad \Delta\{v, v'\} \in K_*, \quad \text{and}$$

(2) $$(vv')(v'v'') = vv'', \quad \text{if} \quad \Delta\{v, v', v''\} \in K.$$

Note that from relation (2) it follows that $vv = 1$ and $(vv')^{-1} = v'v$.

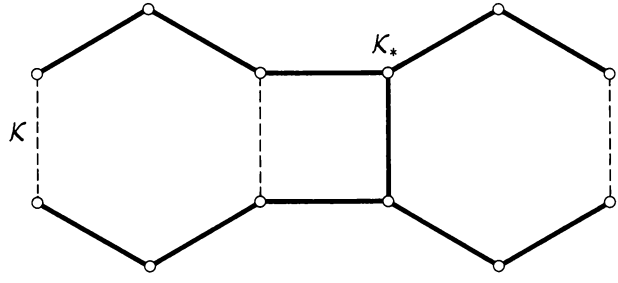


Fig.87. For every non-empty, connected simplicial complex K there exists a simply connected subcomplex K_* which contains all the vertices of K (Corollary 3.4.19).

We now prove the following.

3.4.20. THEOREM. *If a simply connected subcomplex K_* of a connected simplicial complex contains all the vertices of K, then the group $\hat{\pi}_1(K)$ is isomorphic with the group $G(K, K_*)$.*

PROOF. Let v_0 be a fixed vertex of the complex K and let v', v'' be arbitrary vertices of the complex such that $\Delta\{v', v''\} \in K$. Now the subcomplex K_* is connected and contains all the vertices of the complex K, so there are in K_* edge-paths $\hat{d}' = v_0 v_1' \dots v_{k'-1}' v'$ and $\hat{d}'' = v_0 v_1'' \dots v_{k''-1}'' v''$ from v_0 to v' and from v_0 to v''. Put $f(v'v'') = v_0 v_1' \dots v_{k'-1}' v' v'' v_{k''-1}'' \dots v_1'' v_0 \in \hat{L}(K, v_0)$ and $\varphi(v'v'') = [f(v'v'')] \in \hat{\pi}_1(K, v_0)$. From the simple connectedness of K_* it easily follows that $\varphi(v'v'')$ depends only on v' and v''. Observe that each element of the group $\hat{\pi}_1(K, v_0)$ is a product of elements of this form. In fact for any edge-loop $\hat{a} = v_0 v^1 v^2 \dots v^{k-1} v_0 \in \hat{L}(K, v_0)$ we have

$$\hat{a} \frown f(v_0 v^1) * f(v^1 v^2) * \dots * f(v^{k-2} v^{k-1}) * f(v^{k-1} v_0).$$

We shall now prove that the map φ takes relations of the group $G(K, K_*)$ to identities in the group $\hat{\pi}_1(K, v_0)$. Certainly, if $vv' = 1$ where $\Delta\{v, v'\} \in K_*$, then by the simple connectivity of the complex K_* we have $f(vv') \hat{} v_0$ in K_*, hence also in K; thus $\varphi(vv') = [f(vv')] = \epsilon$. If however $(vv')(v'v'') = vv''$, where $\Delta\{v, v', v''\} \in K$ then, as may easily be seen, $f(vv') * f(v'v'') \hat{} f(vv'')$, hence $\varphi(vv')\varphi(v'v'') = [f(vv') * f(v'v'')] = [f(vv'')] = \varphi(vv'')$.

The map φ thus extends to an epimorphism of the group $G(K, K_*)$ onto the group $\hat{\pi}_1(K, v_0)$, which we shall also denote by φ. In order to prove that φ is an isomorphism we shall construct a homomorphism $\psi: \hat{\pi}_1(K, v_0) \to G(K, K_*)$ such that $\psi\varphi = 1$. To each edge-loop $\hat{a} = v_0 v_1 \dots v_{k-1} v_0 \in \hat{L}(K, v_0)$ associate the element $g(\hat{a}) = (v_0 v_1)(v_1 v_2) \dots (v_{k-1} v_0) \in G(K, K_*)$. If $\hat{a} \hat{} \hat{a}'$, then by relation (2) of the group $G(K, K_*)$ we have $g(\hat{a}) = g(\hat{a}')$. Moreover of course $g(\hat{a} * \hat{b}) = g(\hat{a}) g(\hat{b})$ for $\hat{a}, \hat{b} \in \hat{L}(K, v_0)$. Thus the map g determines a homomorphism $\psi: \hat{\pi}_1(K, v_0) \to G(K, K_*)$ defined by the formula $\psi(\hat{\alpha}) = g(\hat{a})$ for $\hat{\alpha} = [\hat{a}] \in \hat{\pi}_1(K, v_0)$.

Using the notation introduced during the construction of f and appealing to relation (1) in the group $G(K, K_*)$ we have

$$\begin{aligned} gf(vv') &= g(v_0 v_1' \dots v_{k'-1}' v' v'' v_{k''-1}'' \dots v_1'' v_0) \\ &= (v_0 v_1')(v_1' v_2') \dots (v_{k'-1}' v')(v' v'')(v'' v_{k''-1}'') \dots (v_1'' v_0) = (v' v''). \end{aligned}$$

Thus

$$\psi\varphi(vv') = \psi[f(vv')] = gf(vv') = vv'.$$

Since $\psi\varphi$ is the identity homomorphism on the generators of the group $G(K, K_*)$, we have $\psi\varphi = 1$, which completes the proof. ■

3.4.21. COROLLARY. *The fundamental group of any polyhedron has a finite number of generators and relations.* ■

The next theorem facilitates the determination of the group $G(K, K_*)$ and is effectively an application of Theorem 3.4.20.

3.4.22. THEOREM. *Let $v_0, v_1, \dots, v_m$ be the vertices of a simplicial complex K and let K_* be a subcomplex of K. The group $G(K, K_*)$ is generated by the edge-paths of the form $v_i v_j$, with $i < j$ and $\Delta\{v_i, v_j\} \in K \setminus K_*$, and has relations of the form $(v_i v_j)(v_j v_k) = v_i v_k$ where $i < j < k$, $\Delta\{v_i, v_j, v_k\} \in K \setminus K_*$, and each of the generators $v_i v_j$, $v_j v_k$ and $v_i v_k$ will be equivalent to 1 whenever its vertices span a simplex in K_*.*

PROOF. First we show that the generators of the group $G(K, K_*)$ may be expressed by means of the generators mentioned in the theorem. Certainly, if $\Delta\{v_i, v_j\} \in K_*$ then $v_i v_j = 1$; if, however, $i > j$ and $\Delta\{v_i, v_j\} \in K \setminus K_*$ then $v_i v_j = (v_j v_i)^{-1}$ where $v_j v_i$ is a generator mentioned in the theorem.

It remains to show that all of the relations (2) of the group $G(K, K_*)$ follow from the relations given in the theorem. If $\Delta\{v_i, v_j, v_k\} \in K_*$ then the corresponding relation is $1 \cdot 1 = 1$. If the simplex $\Delta\{v_i, v_j, v_k\} \in K \setminus K_*$ is of dimension < 2 then the corresponding relations are fulfilled in any group. If finally the simplex $\Delta\{v_i, v_j, v_k\} \in K \setminus K_*$ is of dimension 2 but the inequalities $i < j < k$ do not hold, then it suffices to apply the appropriate permutation to the vertices, to write down the corresponding relation in

the statement of the theorem and then to observe that it is equivalent to a relation (2) in the group $G(K, K_*)$. ■

3.4.23. COROLLARY. *The fundamental group of any connected 1-dimensional polyhedron is free. The number of generators of the group equals the number of 1-dimensional simplices in $K \setminus K_*$, where K is any triangulation of the polyhedron and K_* is any tree in K containing all the vertices of K.*

PROOF. By Corollary 3.4.19 there exists a simply connected subcomplex K_* which contains all the vertices of the complex K. By Theorems 3.4.15, 3.4.20 and 3.4.22 the group $\pi_1(|K|)$ is isomorphic to the group generated by all the 1-dimensional simplices in $K \setminus K_*$ and there are no relations to consider since K contains no simplices of dimension 2. ■

3.4.24. EXAMPLE. *Fundamental group of a bouquet of circles.* Let $B_k = \bigcup_{j=1}^{k} S_j$, where the subspace S_j is homeomorphic with the unit circle S^1 for $j = 1, 2, \ldots, k$ and there is a point p such that $S_j \cap S_{j'} = \{p\}$ for $j \neq j'$. The space B_k is known as a *bouquet* of k circles. We shall show that $\pi_1(B_k)$ is a free group on k generators. We may of course assume that S_j is the union of three line segments $\overline{pa_j} \cup \overline{a_j b_j} \cup \overline{b_j p}$ for $j = 1, 2, \ldots, k$. Taking K to consist of the points p, a_j, b_j and the line segments $\overline{pa_j}$, $\overline{pb_j}$, $\overline{a_j b_j}$ for $j = 1, 2, \ldots, k$ and K_* to consist of the points p, a_j, b_j and the line segments $\overline{pa_j}$, $\overline{pb_j}$ for $j = 1, 2, \ldots, k$ we remark that $\overline{a_j b_j}$ for $j = 1, 2, \ldots, k$ are all the 1-dimensional simplices lying in $K \setminus K_*$. The proposition follows therefore from Corollary 3.4.23.

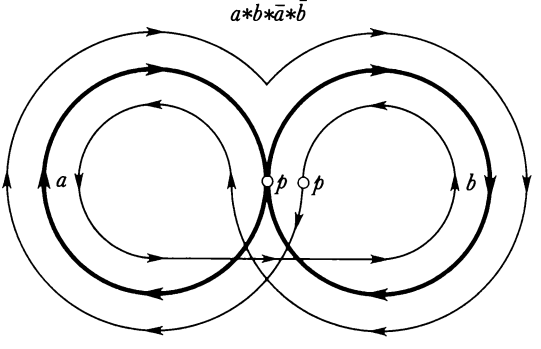


Fig.88. The fundamental group $\pi_1(B_2, p)$ of the bouquet of two circles is non-commutative (Example 3.4.24). The loops a and b represent the generators of the group. The loop $a * b * \bar{a} * \bar{b}$ is nonhomotopic to the trivial loop, hence $[a][b][a]^{-1}[b]^{-1} \neq 1$, or $[a][b] \neq [b][a]$.

We note that the bouquet of k circles for $k \geq 2$ is a space whose fundamental group is non-commutative. ■

3.4.25. EXAMPLE. Let D denote the disc from whose interior k interiors of pairwise disjoint discs T_j have been removed for $j = 1, 2, \ldots, k$ and let S_j denote the circumference of T_j for $j = 1, 2, \ldots, k$. Let $p_j \in S_j$ for $j = 1, 2, \ldots, k$ and let $p_0 \in D \setminus \bigcup_{j=1}^{k} S_j$. For $j = 1, 2, \ldots, k$ there exists in D a broken line L_j joining p_0 and p_j which is homeomorphic to the unit interval I and such that $L_{j'} \setminus \{p_{j'}\} \subset D \setminus \bigcup_{j=1}^{k} S_j$ for $j' = 1, 2, \ldots, k$ and

$L_j \cap L_{j'} = \{p_0\}$ for $j \neq j'$. Then, as may easily be seen, the union $\bigcup_{j=1}^{k}(S_j \cup L_j)$ is a deformation retract of the set D and has the homotopy type of the bouquet B_k. Hence the fundamental group $\pi_1(D, p_0)$ is free and has k generators. Representatives of these generators may also easily be identified. With a fixed orientation of the disc, which fixes an orientation of the circumferences S_j in an obvious way, these are the loops a_j for $j = 1, 2, \ldots, k$ defined as follows: for $0 \leq r \leq \frac{1}{3}$ the point $a_j(r)$ describes the broken line L_j from p_0 to p_j; for $\frac{1}{3} \leq r \leq \frac{2}{3}$ the point $a_j(r)$ describes the circle S_j from p_j to p_j according to the fixed orientation; for $\frac{2}{3} \leq r \leq 1$ the point $a_j(r)$ describes the broken line L_j from the point p_j back to p_0.

Applying Theorem 3.4.10 it is readily inferred that also the fundamental group of the plane with k points removed is free and has k generators. ■

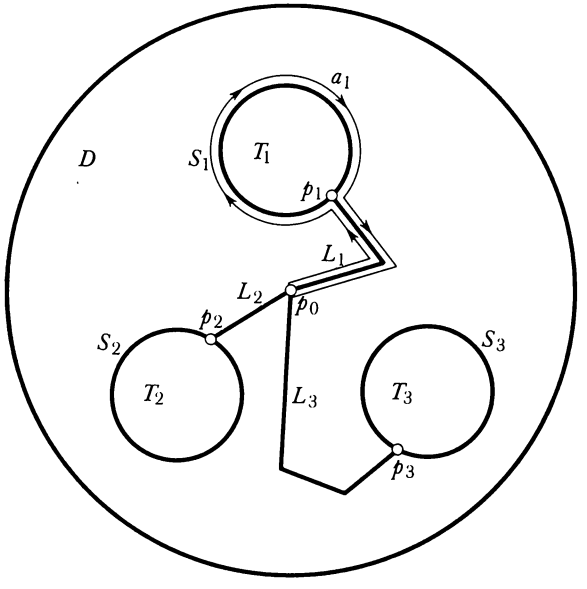


Fig.89. The fundamental group of a disc punctured by three holes T_1, T_2, T_3 is free and has three generators (Example 3.4.25); the loop a_1 represents one of them.

3.4.26. EXAMPLE. *The fundamental group of the sphere* S^{m-1}. In Example 3.4.3 we showed that $\pi_1(S^1) \approx \mathbf{Z}$. Observe now that the group $\pi_1(S^{m-1})$ is trivial when $m > 2$. In fact the sphere S^{m-1} may be replaced by a homeomorphic image of the polyhedron $\operatorname{bd}\Delta^m$ with its natural triangulation K as described in Example 2.3.2. Removing from K any $(m-1)$-dimensional face of the simplex Δ^m we obtain a subcomplex K_* which is simply connected since the polyhedron $|K_*|$ is contractible. The complement $K \setminus K_*$ however does not contain any 1-dimensional simplex and so the fundamental group of the space $\operatorname{bd}\Delta^m$ is trivial. ■

We will now prove a useful theorem concerning the fundamental group of a union of polyhedra. We begin with a case that is rather special. This is the following result which is an easy consequence of Assertion 3.4.16.

3.4.27. LEMMA. *If a simplicial complex is the union of two trees K_1 and K_2 and the intersection $K_1 \cap K_2$ is connected, then K is also a tree.* ■

The following theorem holds (see Supplement 3.S.9).

3.4.28. THEOREM (Van Kampen). *Let a polyhedron X be the union of connected polyhedra X_1, X_2 which have a connected intersection $X_0 = X_1 \cap X_2$ and let $x_0 \in X_0$. The fundamental group $\pi_1(X, x_0)$ is isomorphic to the free product of the groups $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ with the additional relations $i_{1*}(\alpha) = i_{2*}(\alpha)$, where α runs through the generators of the group $\pi_1(X_0, x_0)$ and $i_j\colon (X_0, x_0) \to (X_j, x_0)$ is the inclusion map for $j = 1, 2$.*

PROOF. By Theorem 2.6.15 there is a triangulation K of the polyhedron X and a triangulation K_0 of the polyhedron X_0 such that K_0 is a subcomplex of the simplicial complex K. Denote by K_j the set of simplices of the complex K which are contained in X_j, for $j = 1, 2$; this is a simplicial subcomplex of K which is a triangulation of X_j. Without loss of generality we may assume that x_0 is a vertex of K_0.

By Lemma 3.4.18 there is a tree $K_{0*} \subset K_0$ which contains all the vertices of the complex K_0. Applying the same lemma again we obtain a tree $K_{1*} \subset K_1$ such that $K_{0*} \subset K_{1*}$ and K_{1*} contains all the vertices of the complex K_1. Similarly there exists a tree $K_{2*} \subset K_2$ such that $K_{0*} \subset K_{2*}$ and K_{2*} contains all the vertices of the complex K_2. It follows from these properties that $K_{j*} \cap K_0 = K_{0*}$ for $j = 1, 2$ and so $K_{0*} = K_{1*} \cap K_{2*}$; hence, taking $K_* = K_{1*} \cup K_{2*}$, we obtain by Lemma 3.4.27 a tree K_* which contains all the vertices of the complex K.

It follows furthermore from the equation $K_{j*} \cap K_0 = K_{0*}$ for $j = 1, 2$ that $(K_1 \setminus K_{1*}) \cup (K_2 \setminus K_{2*}) = K \setminus K_*$ and $(K_1 \setminus K_{1*}) \cap (K_2 \setminus K_{2*}) = K_0 \setminus K_{0*}$. Applying Theorem 3.4.22 we infer that the generators of the group $G(K_1, K_{1*})$ together with the generators of the group $G(K_2, K_{2*})$ comprise the generators of the group $G(K, K_*)$; those that are generators of the group $G(K_0, K_{0*})$ will appear twice. What is therefore needed is a set of relations which will appropriately identify them. The remaining relations of the group $G(K, K_*)$ are the relations of the groups $G(K_1, K_{1*})$ and $G(K_2, K_{2*})$. To complete the proof, it is enough to apply Theorem 3.4.20. ■

3.4.29. COROLLARY. *If a polyhedron X is the union of connected polyhedra X_1 and X_2 whose intersection $X_0 = X_1 \cap X_2$ is simply connected, then the fundamental group $\pi_1(X, x_0)$, where $x_0 \in X_0$, is isomorphic to the free product of the groups $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$.* ■

3.4.30. COROLLARY. *If a polyhedron X is the union of a connected polyhedron X_1 and a simply connected polyhedron X_2 which have a connected intersection $X_0 = X_1 \cap X_2$, then the fundamental group $\pi_1(X, x_0)$, where $x_0 \in X_0$, is isomorphic with the group obtained from $\pi_1(X_1, x_0)$ by the addition of the relation $i_*(\alpha) = 1$, where α runs through the generators of the group $\pi_1(X_0, x_0)$ and $i\colon (X_0, x_0) \to (X_1, x_0)$ denotes the inclusion map.* ■

It is worth noting that Van Kampen's Theorem (3.4.28) may be used to identify easily the fundamental groups of the torus (cf. Example 3.4.14), the bouquet of circles

(cf. Example 3.4.24) or the $(m-1)$-dimensional sphere for $m > 2$ (cf. Example 3.4.26); we suggest this as an exercise for the reader.

The concepts of fundamental group and covering map are connected in an interesting way. In view of the elementary character of this book we limit ourselves to a presentation of the simplest results, listing in Supplements 3.S.6–3.S.8 more detailed information. We first prove the following theorem (see also Supplement 3.S.7):

3.4.31. THEOREM. *If $p\colon \tilde{Y} \to Y$ is a covering map and $p(\tilde{y}_0) = y_0$ where $\tilde{y}_0 \in \tilde{Y}$, then the homomorphism $p_*\colon \pi_1(\tilde{Y}, \tilde{y}_0) \to \pi_1(Y, y_0)$ is a monomorphism.*

PROOF. Let $e \in L(Y, y_0)$ and $\tilde{e} \in L(\tilde{Y}, \tilde{y}_0)$ denote the trivial loops; $\tilde{e}$ is thus a lifting of e. Let $F\colon (I \times I, (\operatorname{bd} I) \times I) \to (Y, y_0)$ be any homotopy of the map e. Being a constant map into y_0, the restriction $F_0 = F|(\operatorname{bd} I) \times I$ obviously has a continuous lifting $\tilde{F}_0$ where $\tilde{F}_0((\operatorname{bd} I) \times I) = \{\tilde{y}_0\}$. Applying Theorem 3.3.11 to the case when $X = I$ and $A = \operatorname{bd} I$ we deduce that there is a homotopy $\tilde{F}$ of the map $\tilde{e}$ such that $p\tilde{F} = F$ and $\tilde{F}|(\operatorname{bd} I) \times I = \tilde{F}_0$.

If we therefore assume that $\tilde{a} \in L(\tilde{Y}, \tilde{y}_0)$ and $e \underset{F}{\simeq} p\tilde{a} = p_{\#}(\tilde{a})$, then, taking $\tilde{a}'(r) = \tilde{F}(r, 1)$ for $r \in I$, we have $\tilde{a}' \in L(\tilde{Y}_0, \tilde{y}_0)$, $p\tilde{a}' = p\tilde{a}$ and $\tilde{e} \underset{\tilde{F}}{\simeq} \tilde{a}'$. In view of Corollary 3.3.13 we have $\tilde{a}' = \tilde{a}$, so $\tilde{e} \simeq \tilde{a}$, which completes the proof. ■

We say that a metric space X is *locally pathwise connected at the point $x \in X$* if for every neighbourhood U of the point x there is a neighbourhood V of the point which is contained in U such that for any pair of points $x', x'' \in V$ there is a path from x' to x'' lying in U. If the space is locally pathwise connected at every point then we say that the space is *locally pathwise connected* (see Supplements 3.S.3 and 6.S.8). It is easy to check that local pathwise connectedness is a topological property. A more complete account of the notion is deferred to Section 6.5. Here we note only the obvious fact that every polytope is a locally pathwise connected space.

We now prove a fundamental theorem on the existence of continuous liftings.

3.4.32. THEOREM. *Let $p\colon \tilde{Y} \to Y$ be a covering map and let $p(\tilde{y}_0) = y_0$ where $\tilde{y}_0 \in \tilde{Y}$. The continuous map $f\colon (X, x_0) \to (Y, y_0)$, where X is both pathwise connected and locally pathwise connected, has a continuous lifting $\tilde{f}\colon (X, x_0) \to (\tilde{Y}, \tilde{y}_0)$ if and only if $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$.*

PROOF. If $p\tilde{f} = f$, then by Theorem 3.4.8 we have $p_*\tilde{f}_* = f_*$, hence the given condition is necessary.

To prove its sufficiency assume that $f\colon (X, x_0) \to (Y, y_0)$. For any point $x \in X$ let d_x be a path from x_0 to x. Since fd_x is a path in the space Y beginning at y_0, by Corollary 3.3.13 there is exactly one path $\tilde{d}_x$ in the space $\tilde{Y}$ beginning at $\tilde{y}_0$ such that $p\tilde{d}_x = fd_x$. Put $\tilde{f}(x) = \tilde{d}_x(1)$.

We show that this definition is valid; that is, it does not depend on the choice of path d_x from x_0 to x. Now if d'_x is also a path from x_0 to x, then defining a loop $a \in L(X, x_0)$ by the formula:

$$a(r) = \begin{cases} d_x(2r), & \text{if } 0 \le r \le \frac{1}{2}, \\ d'_x(2-2r), & \text{if } \frac{1}{2} \le r < 1, \end{cases}$$

we conclude that $f_*([a]) \in f_*(\pi_1(X,x_0)) \subset p_*(\pi_1(\tilde{Y},\tilde{y}_0))$ and so there exists a loop $\tilde{b} \in L(\tilde{Y},\tilde{y}_0)$ such that $p\tilde{b} = fa$. Thus if $p\tilde{d}_x = fd_x$ and $p\tilde{d}'_x = fd'_x$, then $\tilde{d}_x(1) = \tilde{b}(\frac{1}{2}) = \tilde{d}'_x(1)$.

The definition of the map $\tilde{f}: X \to \tilde{Y}$ implies immediately that $\tilde{f}(x_0) = \tilde{y}_0$ and that $p\tilde{f}(x) = p\tilde{d}_x(1) = fd_x(1) = f(x)$ for $x \in X$. So it remains to prove that the map $\tilde{f}$ is continuous.

Let $x \in X$, $y = f(x)$ and $\tilde{y} = \tilde{f}(x)$. Since the map $p: \tilde{Y} \to Y$ is a covering there exists a neighbourhood U of the point y in the space Y and a neighbourhood $\tilde{U}$ of the point $\tilde{y}$ in the space $\tilde{Y}$ such that $p|\tilde{U}$ is a homeomorphism of $\tilde{U}$ onto U.

Now the space X is locally pathwise connected at the point x, so there is a neighbourhood V of the point such that if $x', x'' \in V$, then there is a path from x' to x'' in $f^{-1}(U)$. We shall show that $\tilde{f}|V = (p|\tilde{U})^{-1}f|V$. The map on the right hand side of the equation is continuous, whence the continuity of $\tilde{f}$ at the point x would follow.

Let d_x be a path from x_0 to x in the space X. If $x' \in V$, there exists a path $d_{x,x'}$ from x to x' in $f^{-1}(U)$. Taking

$$d_{x'}(r) = \begin{cases} d_x(2r), & \text{if } 0 \le r \le \frac{1}{2}, \\ d_{x,x'}(2r-1), & \text{if } \frac{1}{2} \le r \le 1, \end{cases}$$

we obtain a path $d_{x'}$ from x_0 to x' in the space X. Let the path $\tilde{d}_{x'}$ in the space $\tilde{Y}$ beginning at $\tilde{y}_0$ satisfy the equation $p\tilde{d}_{x'} = fd_{x'}$. Since $fd_{x,x'}$ is a path in U from $f(x)$ to $f(x')$ the path $(p|\tilde{U})^{-1}fd_{x,x'}$ is a continuous lifting of it. But then again the path $\tilde{d}_{x,x'}$ defined by the formula $\tilde{d}_{x,x'}(r) = \tilde{d}_{x'}(\frac{1}{2}r + \frac{1}{2})$ for $r \in I$ has the same property and moreover $\tilde{d}_{x,x'}(0) = (p|\tilde{U})^{-1}fd_{x,x'}(0)$. Applying Theorem 3.3.13 we obtain $\tilde{f}(x') = \tilde{d}_{x'}(1) = \tilde{d}_{x,x'}(1) = (p|\tilde{U})^{-1}fd_{x,x'}(1) = (p|\tilde{U})^{-1}f(x')$, which completes the proof. ■

From Theorems 3.3.12 and 3.4.32 we obtain the following.

3.4.33. COROLLARY. *Let $p: \tilde{Y} \to Y$ be a covering map and let $p(\tilde{y}_0) = y_0$, where $\tilde{y}_0 \in \tilde{Y}$. If a space X is locally pathwise connected and simply connected, then every continuous map $f: (X,x_0) \to (Y,y_0)$ has exactly one continuous lifting $\tilde{f}: (X,x_0) \to (\tilde{Y},\tilde{y}_0)$.* ■

From Theorem 3.4.32 we also have the following consequence (see also Supplement 3.S.8):

3.4.34. COROLLARY. *Let $p: \tilde{Y} \to Y$ and $p': \tilde{Y}' \to Y'$ be covering maps and let $p(\tilde{y}_0) = y_0$ and $p'(\tilde{y}'_0) = y'_0$, where $\tilde{y}_0 \in \tilde{Y}$ and $\tilde{y}'_0 \in \tilde{Y}'$. Suppose we are given a continuous pair map $f: (Y,y_0) \to (Y',y'_0)$. If the space $\tilde{Y}$ is both pathwise connected and locally pathwise connected, then a necessary and sufficient condition for the existence of a continuous pair map $\tilde{f}: (\tilde{Y},\tilde{y}_0) \to (\tilde{Y}',\tilde{y}'_0)$ with $p'\tilde{f} = fp$ is $f_*p_*(\pi_1(\tilde{Y},\tilde{y}_0)) \subset p'_*(\pi_1(\tilde{Y}',\tilde{y}'_0))$.*

PROOF. It suffices to apply Theorem 3.4.32 to the composition $fp: (\tilde{Y},\tilde{y}_0) \to (Y',y'_0)$. ■

Exercises

a) Determine the fundamental group of a torus from which the interior of a set homeomorphic to the disc has been removed.

b) Determine the fundamental group of the Möbius band (see Example 3.3.2).

c) Determine the fundamental group of the 1-dimensional skeleton of the 3-dimensional simplex. Generalize the argument to the case of a simplex of arbitrary dimension.

d) Give an example of a compact connected space $X \subset \mathbf{R}^2$ whose complement $\mathbf{R}^2 \setminus X$ is disconnected, such that the fundamental group $\pi_1(X)$ is trivial.

3.S. Supplements

3.S.1. If $f_0, f_1 : X \to Y$ and for some $A \subset X$ we have $f_0|A = f_1|A$, then a homotopy H from f_0 to f_1 which satisfies the condition $H(a,r) = f_0(a)$ for $a \in A$, $r \in I$ is known as a *homotopy relative to the set* A and we write $f_0 \underset{H}{\simeq} f_1 \operatorname{rel} A$. This notion evidently reduces to ordinary homotopy when $A = \emptyset$. Many of the properties of ordinary homotopy carry over to relative homotopy (see Problems 3.P.23 and [6], p. 15,16).

3.S.2. Alongside the notion of a deformation retract defined in Section 3.2 certain related notions are also studied. We say that the set $A \subset X$ is a *weak deformation retract* of the space X if the inclusion map $i : A \to X$ is a homotopic equivalence. The set A is on the other hand a *strong deformation retract* of the space X, if there is a retraction $r : X \to A$, known as a *strong deformation retraction*, such that $ir \simeq \mathrm{id}_X \operatorname{rel} A$. Obviously every strong deformation retraction is a deformation retraction and every deformation retract is a weak deformation retract. (See also Problems 3.P.26–3.P.29 and [6], p. 32,33.)

The notions of homotopic domination and homotopy type were introduced by J. H. C. Whitehead (1936). If $X \underset{h}{\leq} Y$ and $Y \underset{h}{\leq} X$ we say that the spaces X and Y are *h-equivalent* and we write $X \underset{h}{=} Y$. Obviously if $X \simeq Y$, then $X \underset{h}{=} Y$ but not conversely in general (see Problem 3.P.24).

A finer classification of spaces was introduced by K. Borsuk (see e.g. [2], p. 7–20). A map $f : X \to Y$ is called an *r-map* if there exists a map $g : Y \to X$ such that $fg = \mathrm{id}_Y$; this is a natural generalisation of a retraction to which it reduces in the case when g is the inclusion map of Y into X. It may be shown that the r-maps coincide with maps which are compositions of retractions and homeomorphisms. (See Problem 3.P.30 and also [2], p. 10).

If there exists an r-map $f : X \to Y$, then we say that the space X *r-dominates* the space Y or that the space Y is *r-dominated* by the space X and write $Y \underset{r}{\leq} X$. If $X \underset{r}{\leq} Y$ and $Y \underset{r}{\leq} X$, then we say that the spaces X and Y are *r-equivalent* and we write $X \underset{r}{=} Y$.

3.S.3. Some authors use the terms *arcwise connected* instead of *pathwise connected* and *locally arcwise connected* instead of *locally pathwise connected.* This incompatibility in terminology will further widen in the next chapter where the term 'arc' will be applied to any homeomorphic image of the unit interval, while it is the habit of some authors of calling any continuous image of the unit interval a 'continuous arc'. The distinction between an arc and a continuous arc turns out to be inessential in regard

to the definition of the terms 'arcwise connected space' and 'locally arcwise connected space' (see Assertions 6.5.21 and 6.5.22).

In any metric space X an equivalence relation $\equiv$ may be introduced by taking $x \equiv y$ if and only if there is a path from x to y in X. The equivalence classes of this relation are called the *path components* of the space X. Evidently every path component of the space X is contained in a component of the space; path components are not in general closed sets of X.

The notion of pathwise connectedness introduced in Section 3.1 is a particular case of connectedness in dimension n. We say that a metric space is *connected in dimension* m if every continuous map $f: S^m \to X$ has a continuous extension $f^*: \bar{B}^{m+1} \to X$. Similarly the notion of local pathwise connectedness introduced in Section 3.4 is a particular example of local connectedness in dimension m. We say that a metric space X is *locally connected in dimension* m *at a point* $x \in X$, if for each neighbourhood U of the point x there is a neighbourhood V of the point, contained in U, such that every continuous map $f: S^m \to X$ satisfying the condition $f(S^m) \subset V$ has a continuous extension $f^*: \bar{B}^{m+1} \to X$ satisfying $f^*(\bar{B}^{m+1}) \subset U$. The space X is *locally connected* in dimension m if it is locally connected in dimension m at any point. (See Supplement 6.S.16.)

3.S.4. The Hopf fibration (Example 3.3.3) was constructed in 1931; it was the first example of a map of a sphere into a sphere of lower dimension which is not homotopic to a constant. Replacing the complex numbers used in this construction by quaternions allows an analogous construction of a fibration $p: S^7 \to S^4$ with fibre S^3. Similarly, using Cayley numbers, we obtain a fibration $p: S^{15} \to S^8$ with fibre S^7. In 1935 Hopf defined for each map $f: S^{2n-1} \to S^n$ a certain integer (known now as the *Hopf invariant*) which depends only on the homotopy class of the map f; the examples of fibrations mentioned above all have Hopf invariant equal to 1 (cf. e.g. [5], p. 379–387).

3.S.5. Initially the theory of fibrations was just a collection of examples and many authors independently produced various ideas for a general presentation of the notion. To this day the terminology of this branch of topology is not fully agreed upon. In 1950 J. P. Serre suggested an axiomatic definition of fibration as an arbitrary map $p: E \to B$ which has the homotopy lifting property relative to any polyhedron (equivalent conditions to this definition are given in Problem 3.P.22). Fibrations in the sense in which we have taken them bear in Serre's terminology the name of *locally trivial fibrations.* W. Hurewicz (1955) and M. L. Curtis (1956) considered maps $p: E \to B$ which have the property stated in Corollary 3.3.9 with the additional hypothesis that the path $\tilde{d}$ depends, after a fashion, continuously on the path d and the point e_0; the property is known under the name of the *path lifting axiom* and it turns out that spaces with this property and only they have the homotopy lifting property relative to all spaces (see [6], p. 82,83).

Fibrations $p: E \to B$ are also studied where the fibre W has a group of homeomorphisms G acting on it and for each point $b \in B$ there is a family of homeomorphisms H_b of the fibre W onto the preimage $p^{-1}(b)$ such that

(1) if $g \in G$, $h \in H_b$ then $hg \in H_b$, and

(2) for every $h', h'' \in H_b$ there is a homeomorphism $g \in G$ such that $h'' = h'g$.

Moreover if $b \in U_t$ then the homeomorphism $h: W \to p^{-1}(b)$ defined by the formula $h(w) = \varphi_t(b, w)$ for $w \in W$, where $\varphi_t: U_t \times W \to p^{-1}(U_t)$ denotes the homeomorphism in the definition of the fibration $p: E \to B$, belongs to H_b. It is easy to see that, for example, in the case of the Möbius band (Example 3.3.2) the two element group can be made to act on the fibre W in a natural way.

Fibrations whose fibres are vector spaces are called *vector bundles*; some of them (for instance the tangent vector bundle to a smooth manifold) play a major role in differential topology.

3.S.6. Let $d_*: \pi_1(X, x_1) \to \pi_1(X, x_0)$ be the isomorphism corresponding in Theorem 3.4.5 to the path d from x_0 to x_1. It may easily be shown that if $d \simeq d' \operatorname{rel}\{0,1\}$ then $d_* = d'_*$. Moreover if $x_0 = x_1$ and d is a loop based at x_0 then $d_*(\alpha) = \delta\alpha\delta^{-1}$ where $\delta = [d]$; in other words, in this case d_* is an inner automorphism of the group $\pi_1(X, x_0)$ determined by the element δ (see Problem 3.P.15 and [6], p. 41).

3.S.7. Let $p: \tilde{Y} \to Y$ be a covering map, $y_0 \in Y$ and $\tilde{y}_0 \in p^{-1}(y_0)$. If d is a path beginning at y_0 in Y, then by Corollary 3.3.13 there is in $\tilde{Y}$ precisely one path $\tilde{d}$ beginning at $\tilde{y}_0$ which is a lifting of the path d. It turns out (see Problem 3.P.34 and [6], p. 87) that the endpoint $\tilde{d}(1)$ of the path $\tilde{d}$ depends only on the point $\tilde{y}_0$ and the homotopy class $[d] \operatorname{rel}\{0,1\}$.

By Theorem 3.4.31 the homomorphism $p_*: \pi_1(\tilde{Y}, \tilde{y}_0) \to \pi_1(Y, y_0)$ is a monomorphism. It may be shown (see Problem 3.P.35 and [6], p. 88) that if the space $\tilde{Y}$ is pathwise connected, then for a fixed $y_0 \in Y$ the images $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$, where $\tilde{y}_0 \in p^{-1}(y_0)$, form a family of conjugate subgroups of the group $\pi_1(Y, y_0)$. Moreover there exists a one-to-one correspondence between points of the preimage $p^{-1}(y_0)$ and the right cosets of the group $\pi_1(Y, y_0)$ relative to the subgroup $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$. Under this correspondence the point $\tilde{y}_0$ corresponds to the subgroup $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ and if $\tilde{y}'_0 \in p^{-1}(y_0)$ and d is the path from $\tilde{y}'_0$ to $\tilde{y}_0$ in $\tilde{Y}$, then the composition pd is a loop in Y based at y_0 which represents the element of the group $\pi_1(Y, y_0)$ lying in the coset corresponding to the point $\tilde{y}'_0$.

We call the family $\{p_*(\pi_1(\tilde{Y}, \tilde{y}_0)) : \tilde{y}_0 \in p^{-1}(y_0)\}$ the *characteristic class* of the covering map $p: \tilde{Y} \to Y$ at the point y_0. The class consists of exactly one subgroup if and only if the subgroup $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ is a normal divisor of the group $\pi_1(Y, y_0)$ for some (and hence for every) point $\tilde{y}_0 \in p^{-1}(y_0)$. We then say that the covering map $p: \tilde{Y} \to Y$ is *regular*. It may be proved (see Problem 3.P.36 and [6], p. 88) that regularity of a covering map does not depend on the choice of the point $y_0 \in Y$.

3.S.8. Using Corollary 3.4.34 it is easy to prove that if $p: \tilde{Y} \to Y$ and $p': \tilde{Y}' \to Y$ are covering maps for which the spaces $\tilde{Y}, \tilde{Y}'$ are both pathwise connected and locally pathwise connected, $p(\tilde{y}_0) = y_0 = p'(\tilde{y}'_0)$ and $p_*(\pi_1(\tilde{Y}, \tilde{y}_0)) \subset p'_*(\pi_1(\tilde{Y}', \tilde{y}'_0))$, then there exists a covering map $q: \tilde{Y} \to \tilde{Y}'$ such that $p'q = p$ and $q(\tilde{y}_0) = \tilde{y}'_0$. The assumption that $p_*(\pi_1(\tilde{Y}, \tilde{y}_0)) \subset p'_*(\pi_1(\tilde{Y}', \tilde{y}'_0))$ is of course satisfied when the space $\tilde{Y}$ is simply connected. Hence also the covering map $p: \tilde{Y} \to Y$ is then called a *universal* covering map of the space Y. It can be proved, under some local assumptions about the space

Y, which we shall not state, that for every subgroup G of the group $\pi_1(Y, y_0)$ there is a covering map $p: \tilde{Y} \to Y$ such that $G = p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ for some point $\tilde{y}_0 \in p^{-1}(y_0)$. Taking for G the trivial subgroup we obtain a universal covering map of the space Y (see [6], p. 93).

3.S.9. E. R. Van Kampen proved Theorem 3.4.28 in 1933. It is worth remarking that the theorem ceases to be true on dropping the hypothesis that X_1 and X_2 are polyhedra. Let A_1 be the union of circles in $\mathbf{R}^3$ which lie in the plane $x^3 = 0$, have centres at the points $(1/n, 0, 0)$ and radii $1/n$ respectively for $n = 1, 2, \ldots$ Denote by X_1 the union of all line segments in $\mathbf{R}^3$ which have endpoints $(0, 0, 1)$ and $a_1 \in A_1$. Let X_2 be the set symmetric to X_1 relative to the point $(0, 0, 0)$. Then X_1 and X_2 are compact, contractible (hence simply connected) and $X_1 \cap X_2 = \{(0, 0, 0)\}$ and on the other hand as can easily be shown (see Problem 3.P.21) the union $X = X_1 \cup X_2$ is not simply connected.

However what does hold is the analogue of Van Kampen's Theorem in which the assumptions are that the sets X_1 and X_2 are open in X, pathwise connected and their intersection $X_1 \cap X_2$ is non-empty and pathwise connected (see Problem 3.P.20).

3.S.10. The fundamental group is the first in a sequence of homotopy groups $\pi_n(X, x_0)$ of the space X based at the point x_0 for $n = 1, 2, \ldots$ The definition of the group $\pi_n(X, x_0)$ is a generalization of the definition of the fundamental group; we present only a sketch of it. Let $L_n(X, x_0)$ denote the set of pair maps $a: (I^n, \operatorname{bd} I^n) \to (X, x_0)$. On this set we define the composition operation by the formula:

$$(a * b)(x^1, x^2, \ldots, x^n) = \begin{cases} a(2x^1, x^2, \ldots, x^n), & \text{if } 0 \leq x^1 \leq \frac{1}{2}, \\ b(2x^1 - 1, x^2, \ldots, x^n), & \text{if } \frac{1}{2} \leq x^1 \leq 1. \end{cases}$$

We distinguish a constant map e into the point x_0 and to each element a we assign the element $\bar{a}$ defined by the formula $\bar{a}(x^1, x^2, \ldots, x^n) = a(1 - x^1, x^2, \ldots, x^n)$. Let $\pi_n(X, x_0)$ denote the set of homotopy classes, in the sense of pair maps belonging to $L_n(X, x_0)$. It transpires that taking $[a][b] = [a * b]$, $[a]^{-1} = [\bar{a}]$ and $\epsilon = [e]$ gives the set $\pi_n(X, x_0)$ a group structure; we call it the *n-th homotopy group* of the space X based at x_0.

It may be proved that for $n \geq 2$ the group $\pi_n(X, x_0)$ is commutative; for this reason the fundamental group has an exceptional status in the sequence of homotopy groups. Nevertheless there is not the least difficulty in carrying across proofs to the general case for the theorems that the fundamental group of a pathwise connected space, up to isomorphism, does not depend on the choice of the base point and that the fundamental group is a homotopy type invariant. The theory of homotopy groups is currently one of the most important areas of algebraic topology (see for instance [5] or [6]).

3.S.11. Let $p_0 = (1, 0) \in S^1$. It is easy to establish a one-to-one correspondence between the loops of a space X based at a point x_0 and continuous pair maps $f: (S^1, p_0) \to (X, x_0)$. The theory of the fundamental group can thus be based on maps of this kind instead of loops; this method is closer in spirit to geometric intuition but less convenient in defining various homotopies. Similarly the n-th homotopy group can be based on continuous pair maps $f: (S^n, p_0) \to (X, x_0)$ where $p_0 = (1, 0, \ldots, 0)$.

Consider now the set $L^1(X, x_0)$ of continuous pair maps $f: (X, x_0) \to (S^1, p_0)$. We can equip this set in an obvious way with the commutative group structure making use of the multiplication of complex numbers on the circle S^1. Passing to the set $\pi^1(X, x_0)$ of homotopy classes of maps of $L^1(X, x_0)$ we obtain a group known as the *Bruschlinsky group* of the space X based at x_0; the notion is in some sense dual to that of the fundamental group. A direct transfer of the construction to the case of the sets $L^n(X, x_0)$ of pair maps of (X, x_0) into (S^n, p_0) is not possible in view of the absence of any appropriate group action on the sphere S^n when $n > 1$. However, it turns out that under additional hypotheses on the space X (e.g. if X is a polyhedron of dimension $< 2n$) the set $\pi^n(X, x_0)$ of homotopy classes of maps of $L^n(X, x_0)$ (but not the set $L^n(X, x_0)$ itself) can be equipped with a natural commutative group structure; the group is known as the *n-th cohomotopy group* of the space X based at x_0. The cohomotopy groups were defined by K. Borsuk (1936) and their properties were studied by E. H. Spanier (1949); they are for this reason also known as the *Borsuk-Spanier groups* (see e.g. [2] or [6]).

3.P. Problems

3.P.1. Prove that the boundary of the Möbius band (see Example 3.3.2) is not a retract of the space.

3.P.2. Let $K_\Delta^{[m]}$ be the m-dimensional skeleton of the simplicial complex K_Δ consisting of the n-dimensional simplex Δ and all its faces (cf. Examples 2.2.1 and 2.2.3). Show that if $m < n$ then $|K_\Delta^{[m]}|$ is not a retract of Δ.

3.P.3. Let $\bar{B}^\omega = \{x = \{x^1, x^2, \ldots\} \in \mathbf{R}^\omega : \sum_{i=1}^\infty (x^i)^2 \leq 1\}$ and $S^\omega = \{x = \{x^1, x^2, \ldots\} \in \mathbf{R}^\omega : \sum_{i=1}^\infty (x^i)^2 = 1\}$ (cf. Example 1.1.8). Show that S^ω is a retract of $\bar{B}^\omega$ (cf. [2], p. 13).

3.P.4. Prove that if a continuous map $f: \bar{B}^m \to \mathbf{R}^m$ has the property that $f(S^{m-1}) \subset \bar{B}^m$, then f has a fixed point.

3.P.5. Prove that the space X defined in Example 3.1.22 has the fixed point property.

3.P.6. Show that if $X = X_1 \cup X_2$, where the compact spaces X_1, X_2 have the fixed point property and the intersection $X_1 \cap X_2$ consists of one point, then the space X has the fixed point property. Show by means of an example that the hypothesis that the intersection $X_1 \cap X_2$ is a singleton cannot in general be replaced by the hypothesis that the intersection has the fixed point property. Show by means of an example that the hypotheses of compactness cannot be omitted.

3.P.7. Show that the antipodal map (see Example 1.3.17) of the sphere S^{m-1} is homotopic with the identity when m is even.

3.P.8. Show that a graph $\mathcal{G}$ is a tree if and only if the polyhedron $|\mathcal{G}|$ is contractible.

3.P.9. Give an example of a continuum X with a closed subset $A \subset X$ which has the same homotopy type as X but is not a retract of X.

3.P.10. Show that the metric product of a finite number of pathwise connected spaces is pathwise connected; show that the metric product of finitely many contractible spaces is contractible.

3.P.11. Show that if $X = A \cup B$ where the subspaces A and B are pathwise connected and $A \cap B \neq \emptyset$, then the space X is pathwise connected. Give an example of compact spaces to demonstrate that the analogous property does not hold for contractible spaces even when the intersection $A \cap B$ consists of one point.

3.P.12. Derive the theorem on the non-existence of a retraction of a ball onto its boundary (3.1.15) from the theorem on the non-contractibility of the sphere (3.2.12).

3.P.13. Suppose $f: A \to B$, $g: B \to C$ and $h: C \to D$. Show that the compositions gf and hg are homotopic equivalences if and only if f, g and h are homotopic equivalences.

3.P.14. Show that a pathwise connected space X is simply connected if and only if for any two paths d and d' with common beginning and common end the relation $d \simeq d' \operatorname{rel}\{0,1\}$ holds (see Supplement 3.S.1).

3.P.15. Show that the isomorphism $d_*: \pi_1(X, x_1) \to \pi_1(X, x_0)$ corresponding to the path d from x_0 to x_1 has the following properties:

a) if $d \simeq d' \operatorname{rel}\{0,1\}$ then $d_* = d'_*$, and

b) if $x_0 = x_1$ then $d_*(\alpha) = \delta\alpha\delta^{-1}$, where $\delta = [d]$, $\alpha \in \pi_1(X, x_0)$.

3.P.16. Show that for every continuous map $f: S^1 \to S^1$ there is an integer k such that $f \simeq f_k$, where $f_k: S^1 \to S^1$ is defined by $f_k(z) = z^k$ (where z^k denotes the k-th power of the complex number z). Generalize this result to maps $f: S^m \to S^m$ by constructing for each k an appropriate map $f_k: S^m \to S^m$. (Hint: Define the map f_k by induction on the dimension of the sphere by using the imbedding of the sphere S^{m-1} as the equator of the sphere S^m.)

3.P.17. Determine the fundamental group of the torus by using the Van Kampen Theorem. (Hint: Express the torus as a union of two sets one of which is homeomorphic to a disc, the other having the homotopy type of the bouquet of two circles.)

3.P.18. Determine the fundamental group of the bouquet of k circles using the Van Kampen Theorem.

3.P.19. Using the Van Kampen Theorem, show that for $m > 2$ the group $\pi_1(S^{m-1})$ is trivial. (Hint: Use induction on the dimension of the sphere.)

3.P.20. Prove the Van Kampen Theorem (3.4.28), replacing the hypothesis that X_1 and X_2 are polyhedra by the hypothesis that they are pathwise connected open sets of X and the intersection $X_1 \cap X_2$ is non-empty and pathwise connected.

3.P.21. Show that the space X of Supplement 3.S.9 is not simply connected.

3.P.22. Show that the following properties of a continuous map $p: E \to B$ are equivalent:

(1) the map $p: E \to B$ has the homotopy lifting property relative to any polyhedron,

(2) for $m = 0, 1, \ldots$ the map $p: E \to B$ has the homotopy lifting property relative to the simplex Δ^m,

(3) for $m = 0, 1, \ldots$ the map $p: E \to B$ has the homotopy lifting property relative to the pair $(\Delta^m, \operatorname{bd} \Delta^m)$,

(4) the map $p: E \to B$ has the homotopy lifting property relative to any pair (X, A) where A and X are polyhedra, and

(5) for any pair (X, A) where the polyhedron A is a strong deformation retract of the polyhedron X (see Supplement 3.S.2) and for every continuous map $f: A \to B$ which has a continuous lifting $\tilde{f}: A \to E$ and a continuous extension $f^*: X \to B$, there exists a map $\tilde{f}^*: X \to E$ which is a lifting of the map f^* and an extension of the map $\tilde{f}$ (see [6], p. 63).

3.P.23. Let $A \subset X$ and let $g: A \to Y$ be a continuous map. Show that the relation of homotopy relative to the set A is an equivalence on the set of continuous maps $f: X \to Y$ satisfying $f \mid A = g$ (see Supplement 3.S.1).

3.P.24. Give an example of two compact metric spaces which are h-equivalent (see Supplement 3.S.2), but have distinct homotopy type.

3.P.25. Give an example of two compact metric spaces which are h-equivalent, but are not r-equivalent (see Supplement 3.S.2).

3.P.26. Suppose $A \subset X$ where A and X are polyhedra. Show that A is a strong deformation retract of the space X if and only if it is a deformation retract of X or, equivalently, if and only if it is a weak deformation retract of X (see Supplement 3.S.2).

3.P.27. Give an example of a weak deformation retract which is not a deformation retract. Give an example of a deformation retraction which is not a strong deformation retraction and an example of a deformation retract which is not a strong deformation retract (see Supplement 3.S.2).

3.P.28. Let A be a strong deformation retract of a compact space X and let the pair map $f: (X, A) \to (Y, B)$ restricted to $X \setminus A$ be a homeomorphism of $X \setminus A$ onto $Y \setminus B$. Show that B is a strong deformation retract of the space Y (see Supplement 3.S.2).

3.P.29. Show that if the set A is a strong deformation retract of the metric space X then the set $X \times \{0\} \cup A \times I \cup X \times \{1\}$ is a strong deformation retract of the product $X \times I$ (see Supplement 3.S.2).

3.P.30. Show that in order that a map $f: X \to Y$ be an r-map (see Supplement 3.S.2) it is necessary and sufficient that $f = hr$ where $r: X \to X_0$ is a retraction of X onto a subset X_0 and $h: X_0 \to Y$ is a homeomorphism.

3.P.31. Show that the relation $\equiv$ defined in Supplement 3.S.3 is an equivalence. Determine the number of path components of the space of Example 3.1.22. Give an example of a continuum which has infinitely many path components. Show that a continuous map does not raise the number of path components.

3.P.32. Prove that every polyhedron is locally connected in dimension n, for any n (see Supplement 3.S.3).

3.P.33. Prove that a polyhedron is contractible if and only if it is connected in dimension n for every n (see Supplement 3.S.3).

3.P.34. Show that the end of the path $\tilde{d}$ of Corollary 3.3.13 depends only on the point $\tilde{y}_0$ and on the homotopy class $[d]\,\mathrm{rel}\{0,1\}$.

3.P.35. Show that if a space $\tilde{Y}$ is pathwise connected and $p\colon \tilde{Y} \to Y$ is a covering map, then for any fixed point $y_0 \in Y$ the images $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ for $\tilde{y}_0 \in p^{-1}(y_0)$ form a family of conjugate subgroups in the group $\pi_1(Y, y_0)$ (see Supplement 3.S.7).

3.P.36. Show that the regularity of a covering map does not depend on the chosen point $y_0 \in Y_0$ (see Supplement 3.S.7).

Chapter 4
The topology of Euclidean spaces

In this chapter we shall be applying the methods developed in previous chapters in order to solve a number of fundamental problems concerning Euclidean spaces and their subsets. Section 4.1 is devoted to a closer study of continuous maps into spheres from the point of view of homotopy theory. We elucidate how a map of a sphere into a sphere being essential is related to the existence of a continuous extension of the map onto the ball; then we prove the theorem on the extension of a continuous map of a polyhedron into a sphere. Next we use these results to prove Borsuk's Theorem on the separation of a point from infinity.

In Section 4.2 we prove the topological invariance of some of the notions that are applied to the subsets of Euclidean spaces. We begin with Borsuk's Theorem on the internal characterization of compact sets which separate the sphere. From that we deduce the separation-invariance theorem for spheres and Euclidean spaces; as a particular case we are thereby able to deduce the Jordan Theorem asserting that a plane simple closed curve separates the plane. Next we prove the theorem on the invariance of interior points and from this we conclude the theorem on the invariance of open sets in Euclidean spaces (the 'Invariance of Region' Theorem).

Section 4.3 contains the elements of the theory of position in Euclidean spaces. The theory of knots is represented by an example of a knotted closed curve in $\mathbf{R}^3$. Next we construct an example of a wild arc in $\mathbf{R}^3$ and with its help construct examples of sets homeomorphic with the sphere S^2 which are wildly imbedded. The section also contains a description of the construction of Antoine's necklace.

In Section 4.4 we collect together examples of various subsets of Euclidean spaces and maps defined on them which, though they have no appropriate place else in the book, are regarded as classics and should be known to every mathematician. Thus we describe the construction of the Cantor set, the Sierpiński and the Menger curves, the staircase function and the Peano map, also the construction of a common boundary for three regions in the plane, and finally the construction of an indecomposable space. Additional information on the examples constructed in Section 4.3 and 4.4 and on related examples is given in the Supplements.

4.1. Maps into spheres

Continuous maps of the form $f: X \to S^{m-1}$ frequently appear in the study of various topological properties and so they are worth closer scrutiny. Any continuous map $f: X \to S^{m-1}$ which is non-homotopic to a constant map is called *essential.*

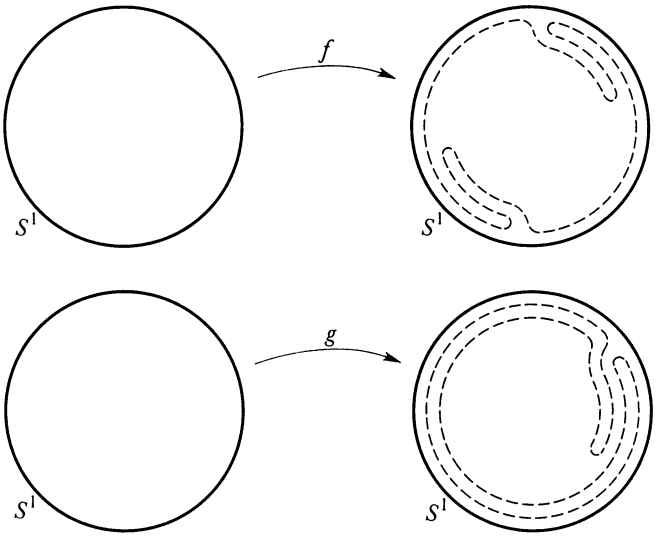


Fig.90. Maps of the circle into the circle: f is essential and g is inessential.

4.1.1. EXAMPLE. It follows from Theorem 3.2.12 that *the identity map* $\mathrm{id}: S^{m-1} \to S^{m-1}$ *is essential.* ■

The next theorem is concerned with extensions of inessential maps.

4.1.2. THEOREM. *If A is a closed subset of a metric space X, then every continuous inessential map $f: A \to S^{m-1}$ has a continuous inessential extension $f^*: X \to S^{m-1}$.*

PROOF. The theorem follows immediately from Corollary 3.2.9 since a constant map of a set A into any point of the sphere S^{m-1} obviously has a continuous extension to the whole space. ■

The inessential nature of a map of a sphere into itself is connected with the possibility of extending it into the whole ball. The following in fact holds.

4.1.3. THEOREM. *For a continuous map $f: S^{m-1} \to S^{m-1}$ to be inessential it is necessary and sufficient that it has a continuous extension $f^*: \bar{B}^m \to S^{m-1}$.*

PROOF. Necessity of the condition follows from Theorem 4.1.2. To prove sufficiency it is enough to make use of the contractibility of the ball $\bar{B}^m$ (Example 3.2.11). ■

We now prove the following theorem which gives a necessary condition for a map to be essential.

4.1.4. THEOREM. *Every essential map $f: X \to S^{m-1}$ takes X onto S^{m-1}.*

PROOF. Suppose that $y \in S^{m-1} \setminus f(X)$ and consider a homeomorphism h of the set $S^{m-1} \setminus \{y\}$ onto the $(m-1)$-dimensional Euclidean space $\mathbf{R}^{m-1}$ (see Corollary 1.3.30).

The composition $hf: X \to \mathbf{R}^{m-1}$ is homotopic to a constant map, since the space $\mathbf{R}^{m-1}$ is contractible. It follows that the map $f = h^{-1}(hf)$ is homotopic to a constant map, contrary to hypothesis. ■

Next we prove the following.

4.1.5. THEOREM. *If X is a polyhedron and $\dim X < m-1$, then every continuous map $f: X \to S^{m-1}$ is inessential.*

PROOF. We may of course replace the sphere S^{m-1} by its homeomorphic copy, the boundary $\operatorname{bd}\Delta^m$ of the m-dimensional unit simplex Δ^m. Let $\mathcal{L}$ denote the natural triangulation of the boundary $\operatorname{bd}\Delta^m$ described in Example 2.3.2. By Theorem 2.5.10 there exists a triangulation K of the polyhedron X and a simplicial approximation $\varphi: K \to \mathcal{L}$ of the map f. It follows from Example 3.2.5 that $f \simeq |\varphi|$. On the other hand since $\dim \varphi(K) \leq \dim K = \dim X < m-1$ and $\dim \mathcal{L} = \dim \operatorname{bd}\Delta^m = m-1$ we have $|\varphi|(X) \subsetneq \operatorname{bd}\Delta^m$. By Theorem 4.1.4 it follows that the map $|\varphi|$ and hence the map f is inessential and that completes the argument. ■

4.1.6. COROLLARY. *If $k < m$, then every continuous map $f: S^{k-1} \to S^{m-1}$ is inessential.* ■

Example 4.1.1 shows that in the case $k = m$ there do exist essential maps of the sphere S^{k-1} onto S^{m-1}. We now consider the case when $k > m$. We first prove the following.

4.1.7. LEMMA. *If $p: S^{m-1} \to B$ is a fibration with non-singleton base space B, then the map p is not homotopic to a constant map.*

PROOF. Suppose $p \simeq c$, where $c: S^{m-1} \to B$ is a constant map and $c(S^{m-1}) = \{b\}$ with $b \in B$. The map p has of course a continuous lifting, namely $\mathrm{id}: S^{m-1} \to S^{m-1}$. From the theorem on homotopy lifting (3.3.8) it follows that there exists a continuous lifting $\tilde{c}: S^{m-1} \to S^{m-1}$ of the map c such that $\tilde{c} \simeq \mathrm{id}$. Hence the map $\tilde{c}$ is essential and according to Theorem 4.1.4 we have $\tilde{c}(S^{m-1}) = S^{m-1}$. Since also $p(S^{m-1}) = B$, we have $c(S^{m-1}) = p\tilde{c}(S^{m-1}) = B$ and hence $B = \{b\}$ contrary to hypothesis. ■

The following important corollary is a consequence of Lemma 4.1.7 (see Example 3.3.3):

4.1.8. COROLLARY. *The Hopf map $p: S^3 \to S^2$ is essential.* ■

Applying the results of this section we may prove the following extension of Theorem 3.1.19.

4.1.9. THEOREM. *Let A be a closed subset of a polyhedron X and let $f: A \to S^{m-1}$ be any continuous map. If $\dim X \leq m-1$, then there is a continuous extension $f^*: X \to S^{m-1}$. If $\dim X \leq m$, then there exists a finite set $B \subset X \setminus A$ and a continuous extension $f^*: X \setminus B \to S^{m-1}$.*

PROOF. By Theorem 3.1.19 there exists an open set U in X containing A and a continuous extension $f'\colon U \to S^{m-1}$ of the map f. By Corollary 2.4.9 there is a triangulation K of the polyhedron X none of whose simplices simultaneously meets both the set A and the set $X\backslash U$. Let $\mathcal{L}$ be a subcomplex of K consisting of all simplices of K which meet A and also their faces; we thus have $|\mathcal{L}| \subset U$. Let $X_i = |\mathcal{L} \cup K^{[i]}|$, where $K^{[i]}$ denotes the i-dimensional skeleton of the complex K for $i = 0, 1, 2, \ldots, \dim K$. Denote by f_0 any extension of the map $f' \mid |\mathcal{L}|$ to the set X_0.

For the proof of the first part of the theorem assume that for some i with $0 \leq i \leq \dim K - 1$ we already have a continuous extension $f_i\colon X_i \to S^{m-1}$ of the map f. We now define an extension $f_{i+1}\colon X_{i+1} \to S^{m-1}$ of the map f_i. For each simplex $\Delta \in K^{[i+1]}\backslash(\mathcal{L} \cup K^{[i]})$ we have $\operatorname{bd}\Delta \subset X_i$. Now $\dim \Delta = i + 1 \leq \dim K \leq m - 1$, so by Corollary 4.1.6 the map $f_i \mid \operatorname{bd}\Delta\colon \operatorname{bd}\Delta \to S^{m-1}$ is inessential and by Theorem 4.1.3 has a continuous extension to the whole of the simplex Δ. These extensions together determine the desired map f_{i+1}. Taking $f^* = f_{\dim K}$ we obtain a continuous extension of the map f to the polyhedron X.

For the proof of the second part of the theorem suppose that $\dim X = m$ and denote by B the set of barycentres of all the m-dimensional simplices $\Delta \in K \setminus \mathcal{L}$. Making use of Example 3.1.11 we deduce that there is a retraction $r\colon X\backslash B \to X_{m-1}$. Applying the first part of the theorem to the polyhedron X_{m-1} we obtain an extension $f_{m-1}\colon X_{m-1} \to S^{m-1}$ of the map f. The composition $f^* = f_{m-1}r\colon X\backslash B \to S^{m-1}$ is the desired continuous extension of the map f. ■

Suppose now that $X \subset \mathbf{R}^m$, $y \in \mathbf{R}^m\backslash X$ and consider the continuous map $p_y\colon X \to S^{m-1}$ defined by the formula $p_y(x) = (x - y)/\|x - y\|$ for $x \in X$. Observe that the map p_y may be either essential or inessential depending on the choice of the point y and the set X. Consider in particular the case when $m = 2$ and $X = S^1$. If $y = (0, 0)$, then $p_y = \mathrm{id}$ and so according to Example 4.1.1 the map p_y is essential. If however $y = (2, 0)$, then $(1, 0) \notin p_y(S^1)$ and so by Theorem 4.1.4 the map p_y is inessential. The next theorem known as the theorem on the *separation of a point from infinity* gives a necessary and sufficient condition for the map p_y to be essential.

4.1.10. THEOREM (Borsuk). *Let $y \in \mathbf{R}^m\backslash X$ where the subspace $X \subset \mathbf{R}^m$ is compact. In order for the continuous map $p_y\colon X \to S^{m-1}$ defined by the formula $p_y(x) = (x - y)/\|x - y\|$ for $x \in X$ to be inessential it is necessary and sufficient for the point y to belong to an unbounded component of the complement $\mathbf{R}^m\backslash X$.*

PROOF. Applying, if necessary, an appropriate similarity we may assume that $y = 0 \in \mathbf{R}^m$ and $X \subset \bar{B}^m$. For the proof that the given condition is necessary suppose that $y \in C$ where C is a bounded component of the complement $\mathbf{R}^m\backslash X$. Since $\operatorname{bd} C \subset X$, the union $X \cup C$ is a closed subset of the ball $\bar{B}^m$. If the map $p_y\colon X \to S^{m-1}$ is inessential, then by Theorem 4.1.2 it has a continuous extension $p_y^*\colon X \cup C \to S^{m-1}$. Define a map $q\colon \bar{B}^m \to S^{m-1}$ by the formula:

$$q(x) = \begin{cases} p_y^*(x), & \text{if } x \in X \cup C, \\ x/\|x\|, & \text{if } x \in B^m\backslash C. \end{cases}$$

Now $(X \cup C) \cap (\bar{B}^m \backslash C) = X$ and $p_y^*(x) = p_y(x) = x/\|x\|$ for $x \in X$ so by Corollary 1.6.29 the map q is continuous. If however $x \in S^{m-1}$, then $q(x) = x/\|x\| = x$, so q would have to be a retraction of $\bar{B}^m$ onto S^{m-1}, contrary to Theorem 3.1.15.

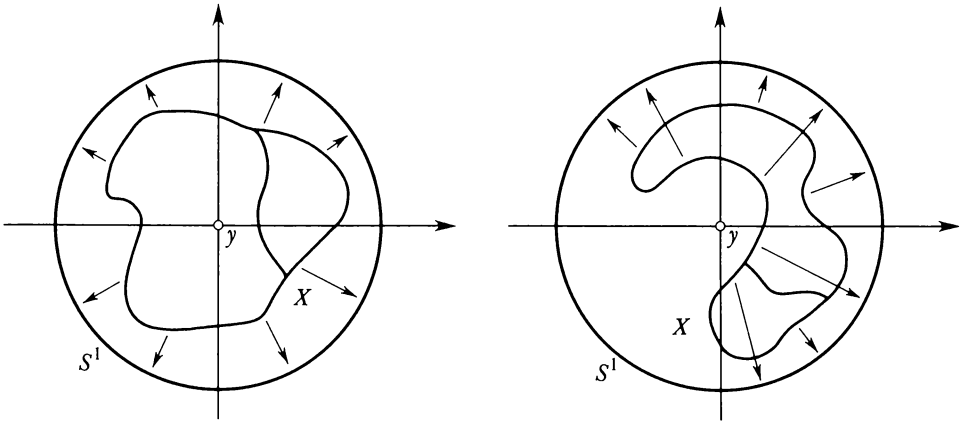


Fig.91. In the figure at left the map p_y is essential since the point y lies in the bounded component of $\mathbf{R}^2 \backslash X$; in the figure at right the map is inessential since the point y belongs to the unbounded component of $\mathbf{R}^2 \backslash X$ (Borsuk's Theorem - 4.1.10).

For the proof that the condition is sufficient, suppose that $y \in C$ where C is an unbounded component of the complement $\mathbf{R}^m \backslash X$ and let $z \in C \backslash \bar{B}^m$. By Theorem 3.1.23 it follows that there is a path d from y to z in C. Define a continuous map $H: X \times I \to S^{m-1}$ by the formula:

$$H(x,r) = (x - d(r))/\|x - d(r)\|$$

for $x \in X$, $r \in I$; the definition is valid since $x \in X$, $d(r) \in C$, for $x \in X$, $r \in I$ and $X \cap C = \emptyset$. Now $d(0) = y$ and $d(1) = z$ so $p_y \underset{H}{\simeq} p_z$. Observe however that $z/\|z\| \notin p_z(X)$; for, if $z/\|z\| = (x - z)/\|x - z\|$ for some $x \in X$, then we would have $x = z(1 + \|x - z\|/\|z\|)$ despite the fact that $x \in \bar{B}^m$ and $z \notin \bar{B}^m$. By Theorem 4.1.4 the map p_z is thus inessential and hence p_y is also inessential. ■

Exercises

a) One of the maps $f, g: S^{m-1} \to S^{m-1}$ is inessential. Can the composition $fg: S^{m-1} \to S^{m-1}$ be essential?

b) Using Corollary 4.1.6 show that the sphere S^{m-1} is simply connected for $m > 2$ (cf. Example 3.4.26).

c) Show that every inessential continuous map $f: S^{m-1} \to S^{m-1}$ has a fixed point. Deduce that the antipodal map is essential.

d) Give an example of an m-dimensional polyhedron X, a closed subset A of X and a continuous map $f: A \to S^{m-1}$ which does not have a continuous extension onto any set $X \backslash B$ where B is a singleton (cf. Theorem 4.1.9).

4.2. Topological invariance of certain properties of sets

Several properties are considered in topology which may hold for subsets of metric spaces. For instance, being closed or open is a property of a set albeit in relation to the space in which it lies. We give another example of a property of this type. We say that a set A lying in a metric space X *separates* the space if the complement $X\setminus A$ is not connected; more precisely we say that A *separates* X *into* n *components* (where n is a cardinal number greater than 1) if the complement $X\setminus A$ has n components.

The properties discussed are topological in the sense that if a set A in a space X has the property P under consideration and h is a homeomorphism of the space X onto a space Y then the set $h(A)$ also has property P in the space Y. This does not imply a positive answer to the more restrictive question: if the set $A \subset X$ has property P and the set $B \subset X$ is homeomorphic to A does the set B also have property P? For instance the singletons $\{0\}$ and $\{1\}$ are obviously homeomorphic but only the latter is open in the space $X = \bigcup_{n=1}^{\infty}\{1/n\} \cup \{0\}$ with the subspace metric of the real line. The singletons $\{0\}$ and $\{\frac{1}{2}\}$ are homeomorphic and yet the former does not separate the unit interval I though the latter separates it into two components. We shall show that if X is the Euclidean space $\mathbf{R}^m$ or the sphere S^{m-1} then certain properties of its subsets are topological in this narrower sense.

We begin with a proof of the following theorem which gives a necessary and sufficient condition for separating a sphere by a compact subset.

4.2.1. THEOREM (Borsuk). *A proper compact subset X of the sphere S^m separates the sphere if and only if there exists an essential map $f: X \to S^{m-1}$.*

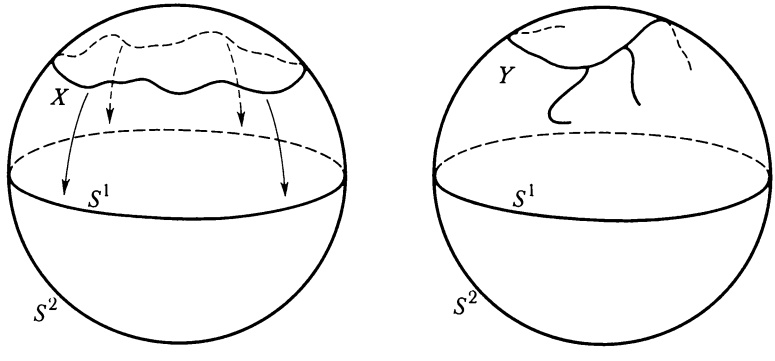


Fig.92. The set X separates the sphere S^2 and hence there exists an essential map of X onto the equator S^1. The set Y does not separate the sphere S^2 and so every map of Y into the equator S^1 is inessential (Borsuk's Theorem – 4.2.1).

PROOF. To prove that the condition is necessary we consider points y, z belonging to different components of the complement $S^m\setminus X$. Let h denote a homeomorphism of the set $S^m\setminus\{y\}$ onto the Euclidean space $\mathbf{R}^m$. Then the point $h(z)$ belongs to a bounded component of $\mathbf{R}^m\setminus h(X)$ and by Theorem 4.1.10 there exists an essential map $p: h(X) \to S^{m-1}$. The map $f = ph|X: X \to S^{m-1}$ is thus also essential.

To prove that the given condition is sufficient suppose that a proper compact set X does not separate the sphere S^m and consider any continuous map $f: X \to S^{m-1}$. By Theorem 4.1.9 there is a finite set $B_k = \{b_0, b_1, \ldots, b_k\} \subset S^m \setminus X$ and a continuous extension $f': S^m \setminus B_k \to S^{m-1}$ of the map f. We shall show that if $k > 0$ then there exists a continuous extension $f'': S^m \setminus B_{k-1} \to S^{m-1}$ of the map f where $B_{k-1} = \{b_0, b_1, \ldots, b_{k-1}\}$.

From Theorem 3.1.23 it follows that there exists a path d from b_k to b_0 in the complement $S^m \setminus (X \cup \{b_1, b_2, \ldots, b_{k-1}\})$. It is easy to see that there exists a finite sequence of real numbers $0 = r_0 < r_1 < \ldots < r_{n-1} < r_n = 1$ and sets $Q_j \subset S^m \setminus (X \cup \{b_1, b_2, \ldots, b_{k-1}\})$ where $j = 1, 2, \ldots, n$ with the following properties:

(1) The set Q_j is homeomorphic to the ball $\bar{B}^m$ and its boundary $\operatorname{bd} Q_j$ is homeomorphic to the sphere S^{m-1} for $j = 1, 2, \ldots, n$, and

(2) $d(r_{j-1}), d(r_j) \in \operatorname{int} Q_j$ for $j = 1, 2, \ldots, n$.

We construct a sequence of continuous extensions $f_j: S^m \setminus (B_{k-1} \cup \{d(r_j)\}) \to S^{m-1}$ of the map f for $j = 0, 1, \ldots, n$ so that $f_0 = f'$ and then we will take $f'' = f_n$. The construction of the sequence $\{f_j\}$ for $j = 0, 1, \ldots, n$ is inductive. Take $f_0 = f'$ and suppose we are given a continuous extension $f_{j-1}: S^m \setminus (B_{k-1} \cup \{d(r_{j-1})\}) \to S^{m-1}$ of the map f where $1 \leq j \leq n$. From properties (1) and (2) it follows that there exists a retraction of the set $Q_j \setminus \{d(r_j)\}$ onto $\operatorname{bd} Q_j$ which in an obvious way determines a retraction $r_j: S^m \setminus \{d(r_j)\} \to S^m \setminus \operatorname{int} Q_j$. Taking $f_j = f_{j-1} r_j | S^m \setminus (B_{k-1} \cup \{d(r_j)\})$, we complete the inductive process on the index j.

Observe that the passage from the extension $f': S^m \setminus B_k \to S^{m-1}$ to the extension $f'': S^m \setminus B_{k-1} \to S^{m-1}$ allows an inductive process on the index k. The end result is a continuous extension $f^*: S^m \setminus \{b_0\} \to S^{m-1}$ of the map f. Since the complement $S^m \setminus \{b_0\}$ is homeomorphic with the Euclidean space $\mathbf{R}^m$, it is contractible and so f^* is an inessential map. It follows that the map f is inessential and that concludes the proof. ■

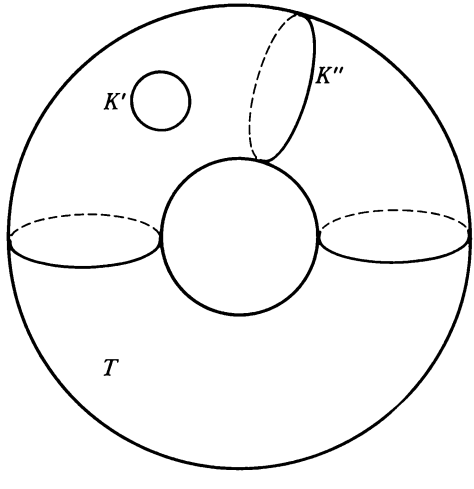


Fig.93. The simple closed curve K' separates the torus, but the simple closed curve K'' does not.

Theorem 4.2.1 gives an internal characterization of compact subsets of the sphere S^{m-1} which separate it. The following theorem known as the *separation invariance theorem*, is a direct consequence.

4.2.2. THEOREM. *Suppose the compact sets $A, B \subset S^m$ are homeomorphic. If A separates the sphere S^m, then so does B.* ■

Let h be a homeomorphism of the space $\mathbf{R}^m$ onto the punctured sphere S^m, say with the point b removed, and let $A \subset \mathbf{R}^m$ be compact. Obviously C is a bounded component of the complement $\mathbf{R}^m \setminus A$ if and only if $h(C)$ is a component of the complement $S^m \setminus h(A)$ which does not contain the point b. Thus the set A separates the space $\mathbf{R}^m$ if and only if the set $h(A)$ separates the sphere S^m. We thus obtain from Theorem 4.2.2 the following.

4.2.3. COROLLARY. *Suppose the compact sets $A, B \subset \mathbf{R}^m$ are homeomorphic. If the set A separates the space $\mathbf{R}^m$, then also B separates $\mathbf{R}^m$.* ■

Observe that the assumption of compactness is essential; for example a line segment without its endpoints and the real line are homeomorphic but only the latter separates the plane. Also the sphere S^m cannot be replaced by an arbitrary space, even by a space as regular as the metric product of two spheres; it is easy, for instance, to pick out on the torus two sets homeomorphic with a circle of which only one separates the torus (see also Supplement 4.S.1).

Since the sphere S^{m-1} cuts Euclidean space $\mathbf{R}^m$ we infer from Corollary 4.2.3 the following.

4.2.4. COROLLARY. *If a set $A \subset \mathbf{R}^m$ is homeomorphic to the sphere S^{m-1}, then A separates the space $\mathbf{R}^m$.* ■

Any set homeomorphic with the unit circle S^1 is called a *simple closed curve*. From Corollary 4.2.4 we obtain in particular the following.

4.2.5. THEOREM (Jordan). *Every simple closed curve lying in the Euclidean plane $\mathbf{R}^2$ separates the plane.* ■

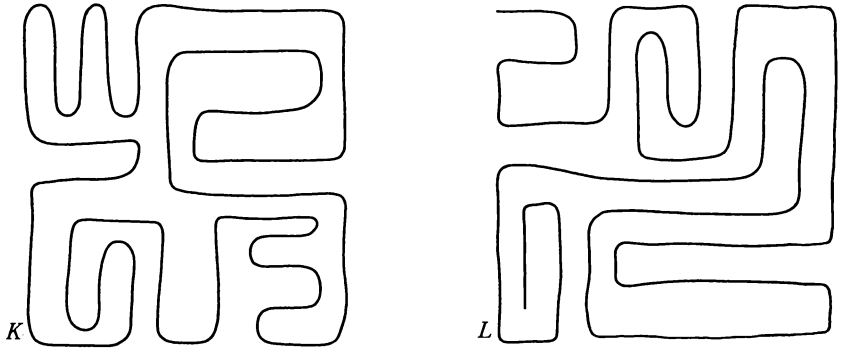


Fig.94. K is a simple closed curve; so it separates the plane (Jordan's Theorem – 4.2.5). L is an arc, so it does not separate the plane (Corollary 4.2.7).

Since for $m > 1$ and $n \leq m$ the set $\bar{B}_0^n = \{(x^1, x^2, \ldots, x^m) \in \bar{B}^m : x^{n+1} = \ldots = x^m = 0\}$ does not separate Euclidean space $\mathbf{R}^m$ and is isometric to the ball $\bar{B}^n$, we obtain by Corollary 4.2.3 the following.

4.2.6. COROLLARY. *If a set $A \subset \mathbf{R}^m$ for $m > 1$ is homeomorphic to the ball $\bar{B}^n$ where $n \leq m$, then the set A does not separate the space $\mathbf{R}^m$.* ■

Any set homeomorphic to the unit interval I is called an *arc*. Every arc has of course exactly two points which do not separate it; we call these the *endpoints* of the arc. From Corollary 4.2.6 we obtain the following.

4.2.7. COROLLARY. *No arc separates Euclidean space $\mathbf{R}^m$ for $m > 1$.* ■

We make use of the separation invariance theorem to prove the following lemma.

4.2.8. LEMMA. *If $A \subset X \subset S^m$ and there is a homeomorphism $h: \bar{B}^m \to X$ such that $h(S^{m-1}) = A$, then the set A separates the sphere S^m into two components: $S^m \backslash X$ and $X \backslash A$.*

PROOF. We have of course $S^m \backslash A = (S^m \backslash X) \cup (X \backslash A)$. Applying Theorem 4.2.2 we infer that the set $S^m \backslash X$ is connected and the set $S^m \backslash A$ is disconnected. The set $X \backslash A = h(\bar{B}^m) \backslash h(S^{m-1}) = h(\bar{B}^m \backslash S^{m-1}) = h(B^m)$ is evidently connected. Thus the sets $S^m \backslash X$ and $X \backslash A$ are components of the set $S^m \backslash A$. ■

4.2.9. COROLLARY. *If $A \subset X \subset S^m$ and there is a homeomorphism $h: \bar{B}^m \to X$ such that $h(S^{m-1}) = A$, then the set $X \backslash A$ is open in S^m.*

PROOF. The sets $S^m \backslash X$ and $X \backslash A$ being components of $S^m \backslash A$ are closed in $S^m \backslash A$. But $S^m \backslash A = (S^m \backslash X) \cup (X \backslash A)$, so they are also both open in $S^m \backslash A$. In particular the set $X \backslash A$ is open in $S^m \backslash A$ and since the set $S^m \backslash A$ is obviously open in S^m, therefore $X \backslash A$ is open in S^m. ■

Regarding Euclidean space $\mathbf{R}^m$ as the sphere S^m with a point removed we obtain from the above the following.

4.2.10. COROLLARY. *If $A \subset X \subset \mathbf{R}^m$ and there is a homeomorphism $h: \bar{B}^m \to X$ such that $h(S^{m-1}) = A$, then the set $X \backslash A$ is open in $\mathbf{R}^m$.* ■

We now prove the following *theorem on the invariance of interior points.*

4.2.11. THEOREM. *Suppose $U, V \subset \mathbf{R}^m$ and let $h: U \to V$ be a homeomorphism. If $u \in \operatorname{int} U$, then $h(u) \in \operatorname{int} V$.*

PROOF. If $u \in \operatorname{int} U$ then there exists $r > 0$ such that $\bar{B}(u; r) \subset U$. Consider the similarity $p: \bar{B}^m \to \bar{B}(u; r)$. Taking $X = hp(\bar{B}^m)$ and $A = hp(S^{m-1})$ we deduce from Corollary 4.2.10 that the set $X \backslash A$ is open in the space $\mathbf{R}^m$. Now $h(u) \in h(B(u; r)) \subset X \backslash A \subset h(U) = V$, so $h(u)$ is an interior point of the set V. ■

From the theorem above we obtain the *theorem on the invariance of open sets*:

4.2.12. THEOREM. *Suppose the sets $U, V \subset \mathbf{R}^m$ are homeomorphic. If U is open in the space $\mathbf{R}^m$, then also V is open in the space $\mathbf{R}^m$.* ■

Now we prove the following.

4.2.13. THEOREM. *If $U, V \subset X$, the set U is homeomorphic to the ball $\bar{B}^n$, the set V is homeomorphic to the ball $\bar{B}^m$ and $\operatorname{int} U \cap \operatorname{int} V \neq \emptyset$, then $n = m$.*

PROOF. Suppose $n < m$, let h_U be a homeomorphism of U onto $\bar{B}^n$ and let h_V be a homeomorphism of V onto $\bar{B}^m$. Consider the natural inclusion map $i\colon \bar{B}^n \to \bar{B}^m$; the set $i(\bar{B}^n)$ has no interior points in $\bar{B}^m$ hence also the closure in $\bar{B}^m$ of the set $ih_U(\operatorname{int} U \cap \operatorname{int} V)$ has empty interior, the same being true therefore relative to $\mathbf{R}^m$. On the other hand, since $\operatorname{int} U \cap \operatorname{int} V$ is open in V, the set $h_V(\operatorname{int} U \cap \operatorname{int} V)$ is open in $\bar{B}^m$ and so has non-empty interior in the space $\mathbf{R}^m$. But the sets $ih_U(\operatorname{int} U \cap \operatorname{int} V)$ and $h_V(\operatorname{int} U \cap \operatorname{int} V)$ are homeomorphic, contrary to Theorem 4.2.11. ■

As a consequence of Theorem 4.2.12 we have the following *theorem on the invariance of the dimension of Euclidean spaces.*

4.2.14. THEOREM. *The Euclidean spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ are not homeomorphic for $n \neq m$.* ■

From the theorem on the invariance of interior points we also obtain the following.

4.2.15. THEOREM. *Any continuous bijective map of the Euclidean space $\mathbf{R}^m$ onto itself is a homeomorphism.*

PROOF. Suppose $f\colon \mathbf{R}^m \to \mathbf{R}^m$ is continuous and bijective. To show that f^{-1} is continuous, it suffices by Theorem 1.6.24 to check that for every open set $U \subset \mathbf{R}^m$ the image $f(U)$ is open in $\mathbf{R}^m$.

Let $y = f(x) \in f(U)$, where $x \in U$ and suppose $\bar{B}(x; r) \subset U$, with $r > 0$. Since the closed ball $\bar{B}(x; r)$ is compact, it follows by Theorem 1.8.15 that the map $f \mid \bar{B}(x; r)$ is a homeomorphism. By Theorem 4.2.11 we thus have $y \in \operatorname{int} f(\bar{B}(x; r)) \subset \operatorname{int} f(U)$, which completes the proof. ■

Exercises

a) Suppose the sets $A, B \subset \mathbf{R}^m$ are homeomorphic and the set A has empty interior in $\mathbf{R}^m$. Does the set B have to have empty interior? If $A, B \subset \mathbf{R}^m$ are homeomorphic and A is dense in $\mathbf{R}^m$, does B have to be dense in $\mathbf{R}^m$?

b) Show that every bouquet of k circles in the plane $\mathbf{R}^2$ (cf. Example 3.4.24) separates the plane.

c) Show that a simple closed curve never separates a Euclidean space $\mathbf{R}^m$ for $m > 2$.

d) Give an example of a connected polyhedron X and two homeomorphic polyhedra $A, B \subset X$ of which only one has interior points in X.

4.3. The theory of position

The discussion in the last section, in which we proved the topological invariance of certain properties of subsets of Euclidean space $\mathbf{R}^m$, would have been unnecessary

if for every pair of homeomorphic sets $A, B \subset \mathbf{R}^m$ there existed a homeomorphism $h: \mathbf{R}^m \to \mathbf{R}^m$ such that $h(A) = B$. However it is easy to see that Euclidean space $\mathbf{R}^m$ is not topologically homogeneous to such a great extent even for $m = 1$. For example the sets $A = \{-1\} \cup I \cup \{2\}$ and $B = I \cup \{2, 3\}$ are homeomorphic and even their complements are homeomorphic, but there is no homeomorphism $h: \mathbf{R}^1 \to \mathbf{R}^1$ such that $h(A) = B$.

We say that two sets $A, B \subset \mathbf{R}^m$ are *equivalently imbedded* if there is a homeomorphism $h: \mathbf{R}^m \to \mathbf{R}^m$ such that $h(A) = B$. Thus any two equivalently imbedded sets are homeomorphic but not, in general, conversely. The relation 'equivalently imbedded' for subsets of a space $\mathbf{R}^m$ is obviously an equivalence (see Supplement 4.S.2).

We will henceforth be concerned with the problem of equivalent imbedding of simple closed curves and arcs in $\mathbf{R}^m$. It may be proved (see e.g. [10], p. 535) that for every simple closed curve C in the plane $\mathbf{R}^2$ there is a homeomorphism $h: \mathbf{R}^2 \to \mathbf{R}^2$ such that $h(C) = S^1$; similarly for every arc $L \subset \mathbf{R}^2$ there is a homeomorphism $h: \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$h(L) = \{(x^1, x^2) \in \mathbf{R}^2 : x^1 \in I,\ x^2 = 0\}.$$

Two simply closed curves in the plane $\mathbf{R}^2$ are thus equivalently imbedded, similarly any two arcs in the plane $\mathbf{R}^2$ are equivalently imbedded. The proof of this theorem, known as *Schönflies Theorem*, is not especially difficult conceptually, but would take too much space to warrant presentation here. However it turns out that similar properties do not apply to simple closed curves and arcs in the Euclidean space $\mathbf{R}^3$.

We proceed to the construction of appropriate examples with the following lemma.

4.3.1. LEMMA. *Let the polyhedron $C \subset \mathbf{R}^3$ be a simple closed curve or an arc. There exists a polyhedron $W \subset \mathbf{R}^3$ homeomorphic to $\bar{B}^2 \times C$ such that $C \subset \operatorname{int} W$ and $\operatorname{bd} W$ is a deformation retract of the difference $W \setminus C$.*

PROOF. Consider the simple curve $C = \bigcup_{j=0}^{n-1} \overline{v_j v_{j+1}}$ where $v_0 = v_n$ and $\overline{v_j v_{j+1}} \cap \overline{v_k v_{k+1}} = \emptyset$ for $1 < |j - k| < n - 1$. We may of course suppose that the vertices v_{j-1}, v_j, v_{j+1} are not collinear for any $j = 0, 1, \ldots, n-1$, where the indices are reduced modulo n when necessary (we shall henceforth always perform such a reduction). Let

$$\epsilon = \frac{1}{3} \inf\{\rho(x, y) : x \in \overline{v_j v_{j+1}},\ y \in \overline{v_k v_{k+1}},\ 1 < |j - k| < n - 1\}.$$

For $j = 0, 1, \ldots, n-1$ take an isometry $f_j: \mathbf{R}^3 \to \mathbf{R}^3$ such that $f_j(v_j) = (0, 0, 0)$, $f_j(v_{j-1})$, $f_j(v_{j+1}) \in \{(x^1, x^2, x^3) \in \mathbf{R}^3 : x^2 = a_j |x^1|,\ x^3 = 0\}$, where $a_j > 0$, and define

$$V_j = f_j^{-1}\left(\left\{(x^1, x^2, x^3) \in \mathbf{R}^3 : a_j |x^1| - \epsilon \leq x^2 \leq -\frac{1}{a_j} |x^1| + \epsilon,\ |x^3| \leq \epsilon\right\}\right).$$

The set V_j is a 3-dimensional cell which contains the vertex v_j in its interior and on whose boundary lie two rectangles P_j', P_j'' intersected perpendicularly through their centres s_j' and s_j'' by the segments $\overline{v_{j-1} v_j}$ and $\overline{v_j v_{j+1}}$, respectively.

The set $W_j = \operatorname{conv}(P_{j-1}'' \cup P_j')$ is a 3-dimensional cell satisfying the equations $W_j \cap V_{j-1} = P_{j-1}''$ and $W_j \cap V_j = P_j'$, whence it follows that the set $W = \bigcup_{j=0}^{n-1} (V_j \cup W_j)$ is homeomorphic to the product $\bar{B}^2 \times C$. Since $\overline{v_{j-1} v_j} \subset \operatorname{int}(V_{j-1} \cup W_j \cup V_j)$ for $j = 0, 1, \ldots, n-1$, we have $C \subset \operatorname{int} W$.

It is easy to see that for each j there is a deformation retraction of the difference $V_j \setminus (\overline{v_{j-1}v_j} \cup \overline{v_j v_{j+1}})$ to the set $\operatorname{bd} V_j \setminus (\operatorname{int} P'_j \cup \operatorname{int} P''_j)$ which, when restricted to $P'_j \setminus \{s'_j\}$ is a projection from the point s'_j onto $\operatorname{bd} P'_j$, and, when restricted to $P''_j \setminus \{s''_j\}$ is a projection from the point s''_j onto $\operatorname{bd} P''_j$. There also exists a deformation retraction of the difference $W_j \setminus \overline{v_{j-1}v_j}$ onto $\operatorname{bd} W_j \setminus (\operatorname{int} P''_{j-1} \cup \operatorname{int} P'_j)$, which, when restricted to any intersection with a plane P perpendicular to $\overline{v_{j-1}v_j}$ is a projection from the point $P \cap \overline{v_{j-1}v_j}$ onto $P \cap (\operatorname{bd} W_j \setminus (\operatorname{int} P''_{j-1} \cup \operatorname{int} P'_j))$. These retractions together determine a deformation retraction of the difference $W \setminus C$ to $\operatorname{bd} W$.

In the case when C is an arc the construction requires only minimal modification and so we omit the details. ■

Using the lemma above we examine the following.

4.3.2. EXAMPLE. Let S be the boundary of any two dimensional simplex Δ in the space $\mathbf{R}^3$. We show that for any point $x_0 \in \mathbf{R}^3 \setminus S$ the group $\pi_1(\mathbf{R}^3 \setminus S, x_0)$ is free and has one generator.

By Lemma 4.3.1 there is a polyhedron $W \subset \mathbf{R}^3$ homeomorphic to $\bar{B}^2 \times S$ such that $S \subset \operatorname{int} W$ and $\operatorname{bd} W$ is a deformation retract of the difference $W \setminus S$. We may moreover assume that the set $D = \Delta \setminus \operatorname{int} W$ is homeomorphic to the disc $\bar{B}^2$ and $K = \Delta \cap \operatorname{bd} W$ is a simple closed curve. Since the union $W \cup D$ is a deformation retract of the space $\mathbf{R}^3$, the union $(\operatorname{bd} W) \cup D$ is a deformation retract of the complement $\mathbf{R}^3 \setminus S$. The boundary $\operatorname{bd} W$ is homeomorphic to the torus and so by Example 3.4.14 its fundamental group is free and has two generators, the curve K being a possible representative for one of them. Using the simple connectedness of the set D we deduce from Van Kampen's Theorem (3.4.28) that the fundamental group of the union $(\operatorname{bd} W) \cup D$ is free and has one generator. Thus for every point $x_0 \in \mathbf{R}^3 \setminus S$ the group $\pi_1(\mathbf{R}^3 \setminus S, x_0)$ is free and has one generator. ■

We will now construct a polyhedron $C \subset \mathbf{R}^3$ which is a simple closed curve that is not equivalently imbedded to the curve S examined in the last example.

4.3.3. EXAMPLE. *A knotted simple closed curve.* We shall regard the space $\mathbf{R}^3$ as the metric product $\mathbf{R}^2 \times \mathbf{R}^1$. Let the points $s_1, s_2, s_3 \in S^1$ form the vertices of an equilateral triangle with barycentre at the origin. For $j = 1, 2, 3$ consider the points of $\mathbf{R}^3$: $p_j = (s_j, 1)$, $q_j = (s_j, -1)$, $c_j = (-2s_{j+2}, 0)$, $d_j = (-s_{j+2}, 0)$ and the broken lines $L_j = \overline{p_j d_j} \cup \overline{d_j q_{j+1}}$, $M_j = \overline{q_j c_j} \cup \overline{c_j p_{j+1}}$, where the indices are reduced modulo 3 as necessary. It is easy to check that the union $C = \bigcup_{j=1}^{3} (L_j \cup M_j)$ is a simple closed curve. We now determine the fundamental group $\pi_1(\mathbf{R}^3 \setminus C, x_0)$ where $x_0 = (0, 0, 0)$.

By Lemma 4.3.1 there is a polyhedron $W \subset \mathbf{R}^3$ which is homeomorphic to $\bar{B}^2 \times C$ such that $C \subset \operatorname{int} W$ and $\operatorname{bd} W$ is a deformation retract of the difference $W \setminus C$. We may moreover suppose that for every half-plane P in $\mathbf{R}^3$ whose edge is the x^3-axis the intersection $P \cap W$ has two components each of which is homeomorphic to $\bar{B}^2$. Let Q be any cube in the space $\mathbf{R}^3$ which contains the polyhedron W in its interior. Since Q is a deformation retract of $\mathbf{R}^3$, the set $Q' = Q \setminus \operatorname{int} W$ is a deformation retract of the set $\mathbf{R}^3 \setminus C$. By Theorem 3.4.10 and Example 3.2.18 it is therefore enough to determine the group $\pi_1(Q', x_0)$.

For $j = 1,2,3$ let $P_j \subset \mathbf{R}^3$ be a half-plane whose boundary is the x^3-axis and which passes through the point p_j. Let Q_j be the closure of the component of the set $Q\backslash(P_j \cup P_{j+1})$ which contains the point c_j. Taking $P'_j = P_j \cap Q'$ and $Q'_j = Q_j \cap Q'$ we have $Q' = \bigcup_{j=1}^{3} Q'_j$ and $Q'_{j-1} \cap Q'_j = P'_j$. As is easily seen, the set P'_j is homeomorphic to a punctured disc from the interior of which the interiors of two disjoint discs have been removed. A fixed orientation of the x^3-axis naturally determines orientations of each of the sets P'_j. By Example 3.4.25 the group $\pi_1(P'_j, x_0)$ is free and has two generators α_j, β_j; we make use of the representatives of these generators, which are described in Example 3.4.25: let $\alpha_j = [a_j]$ and $\beta_j = [b_j]$ where the loop a_j corresponds to the component of the set $P_j \cap W$ which contains p_j, and the loop b_j corresponds to the component containing q_j. The set P'_j is of course a deformation retract of the set Q'_j; we may therefore suppose that the loops a_j, b_j also represent the generators of the group $\pi_1(Q'_j, x_0)$.

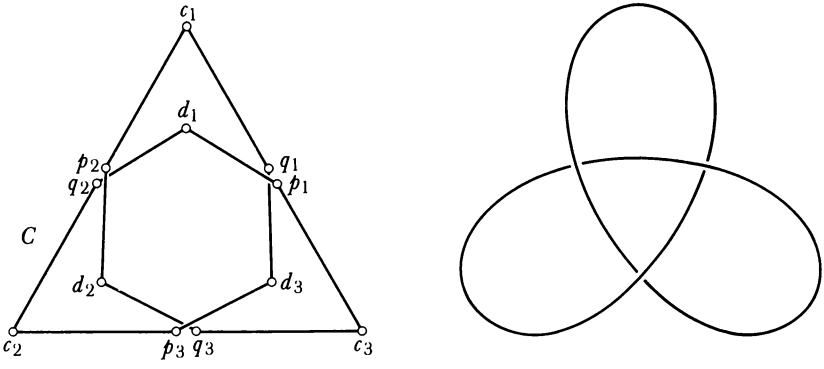


Fig.95. The knotted simple closed curve C (Example 4.3.3).
It is a trefoil knot reminiscent of a three leaf clover.

The inclusion map of P'_j into Q'_j induces an identity isomorphism of the fundamental groups; however, as is easily checked, the inclusion of P'_j into Q'_{j-1} takes the generator α_j to $\alpha_{j-1}\beta_{j-1}\alpha_{j-1}^{-1}$ and the generator β_j to α_{j-1}. Applying Van Kampen's Theorem (3.4.28) twice we deduce that the group $\pi_1(Q', x_0)$ has generators α_j, β_j and relations $\alpha_j\alpha_{j-1} = \alpha_{j-1}\beta_{j-1}$, $\beta_j = \alpha_{j-1}$ where $j = 1,2,3$. We may therefore suppose that the generators of the group are $\alpha_1, \alpha_2, \alpha_3$, the relations being $\alpha_1\alpha_3 = \alpha_3\alpha_2$, $\alpha_2\alpha_1 = \alpha_1\alpha_3$ and $\alpha_3\alpha_2 = \alpha_2\alpha_1$. From the last two relations we deduce that $\alpha_3 = \alpha_1^{-1}\alpha_2\alpha_1 = \alpha_2\alpha_1\alpha_2^{-1}$. The group $\pi_1(Q\backslash \operatorname{int} W, x_0)$, and thereby $\pi_1(\mathbf{R}^3\backslash C, x_0)$, thus has two generators α_1, α_2 and one relation $\alpha_1\alpha_2\alpha_1 = \alpha_2\alpha_1\alpha_2$.

Consider the group S_3 of permutations of the numbers $1,2,3$ and let $\beta_1 = (1,3,2)$, $\beta_2 = (3,2,1) \in S_3$. It is easy to check that these are generators of the group S_3 and that $\beta_1\beta_2\beta_1 = (2,1,3) = \beta_2\beta_1\beta_2$. Thus the formulas $h(\alpha_i) = \beta_i$ for $i = 1,2$ define an epimorphism $h: \pi_1(\mathbf{R}^3\backslash C, x_0) \to S_3$. Since the group S_3 is non-commutative, the group $\pi_1(\mathbf{R}^3\backslash C, x_0)$ is also non-commutative and in particular it is not the free group on one generator. Let S be the closed curve constructed in Example 4.3.2. Since the groups $\pi_1(\mathbf{R}^3\backslash C)$ and $\pi_1(\mathbf{R}^3\backslash S)$ are non-isomorphic, the sets $\mathbf{R}^3\backslash C$ and $\mathbf{R}^3\backslash S$ are not homeomorphic. It immediately follows that the simple closed curves C and S are not

equivalently imbedded in the space $\mathbf{R}^3$. An intuitive explanation of this fact boils down to this: the curve C may be obtained from the curve S by tying a knot which cannot be untied in $\mathbf{R}^3$. That is why C is called *knotted simple closed curve.* ■

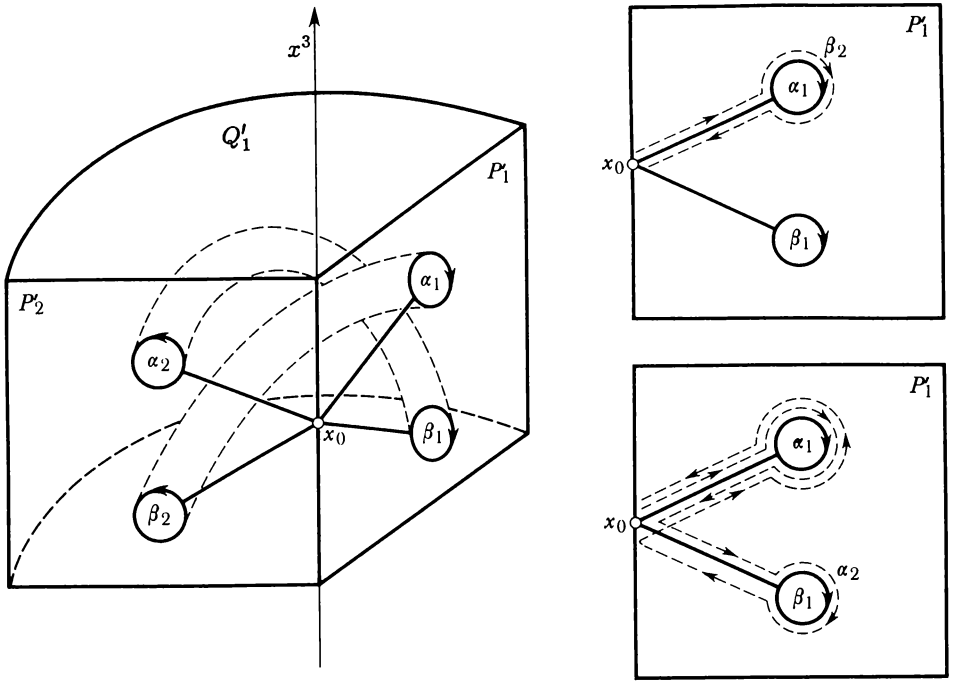


Fig.96. The inclusion of P_2' into Q_1' takes the generator β_2 to α_1 and the generator α_2 to $\alpha_1\beta_1\alpha_1^{-1}$ (cf. Example 4.3.3).

The knotted simple closed curve C described in Example 4.3.3 is in some sense the easiest simple closed curve in the space $\mathbf{R}^3$ which is not equivalently imbedded with the curve S. Actually, much more complicated knots can be tied on simple closed curves; their classification, from the point of view of equivalent imbeddings, is the concern of *knot theory* (see e.g. [14], Section 2.2). We might add that in the space $\mathbf{R}^m$ for $m \geq 4$ any two polyhedral simple closed curves are equivalently imbedded; the theory of polyhedral knots in such spaces is therefore trivial. If, however, we return to the case $m = 3$, but drop the polyhedral assumption, then peculiarities of a completely different sort appear which we will now discuss.

Let $X \subset \mathbf{R}^m$ be homeomorphic to a polyhedron. We say that the set X is *wildly* imbedded in the space $\mathbf{R}^m$ if it is not equivalently imbedded with any polyhedron in the space $\mathbf{R}^m$ (cf. Supplement 4.S.2). Since, as we have remarked earlier, any two simple closed curves and any two arcs in the plane $\mathbf{R}^2$ are equivalently imbedded, the plane $\mathbf{R}^2$ contains no wildly imbedded simple closed curves or arcs. We shall now show that there exist wildly imbedded arcs in Euclidean space $\mathbf{R}^3$.

We begin by calculating the fundamental group of the complement of a polyhedral arc in the space $\mathbf{R}^3$.

4.3.4. EXAMPLE. Let L be any polyhedral arc in the space $\mathbf{R}^3$. We show that the complement $\mathbf{R}^3 \setminus L$ is simply connected.

To this end we invoke Lemma 4.3.1 according to which there exists a polyhedron $W \subset \mathbf{R}^3$ homeomorphic to the product $\bar{B}^2 \times L$, or equivalently to $\bar{B}^3$, such that $L \subset \operatorname{int} W$ and $\operatorname{bd} W$ is a deformation retract of the difference $W \setminus L$. Let Q be any 3-dimensional cube in the space $\mathbf{R}^3$ which contains W in its interior. Since Q is a deformation retract of the space $\mathbf{R}^3$ it is enough therefore to prove that the difference $Q \setminus \operatorname{int} W$ is simply connected.

Consider any point $w \in \operatorname{int} W$. Since the boundary $\operatorname{bd} W$ is a deformation retract of the difference $W \setminus \{w\}$, the set $Q \setminus \operatorname{int} W$ is a deformation retract of $Q \setminus \{w\}$. Also the boundary $\operatorname{bd} Q$ is a deformation retract of $Q \setminus \{w\}$. It now follows that the sets $Q \setminus \operatorname{int} W$ and $\operatorname{bd} Q$ have the same homotopy type and by Example 3.4.26 the boundary $\operatorname{bd} Q$ is simply connected. ■

We now construct an arc in the space $\mathbf{R}^3$ whose complement is not simply connected.

4.3.5. EXAMPLE. *A wildly imbedded arc.* Let $Q_0 = [-1,1]^3 \subset \mathbf{R}^3$ and let $P_j = \{2j - 1\} \times [-1,1]^2$ for $j = 0,1$. Consider the points $a_j = (2j - 1, \frac{2}{3}, 0)$, $b_j = (2j - 1, \frac{1}{3}, 0)$, $c_j = (2j - 1, 0, 0)$, $d_j = (2j - 1, -\frac{1}{3}, 0)$, $e_j = (2j - 1, -\frac{2}{3}, 0)$ for $j = 0,1$ and the points $p = (0, \frac{1}{3}, \frac{1}{3})$, $q = (0,0,0)$, $s = (0, -\frac{1}{3}, -\frac{1}{3})$. The set

$$M_0 = (\overline{a_0 p} \cup \overline{p c_1}) \cup \overline{b_0 a_1} \cup (\overline{b_1 q} \cup \overline{q d_1}) \cup (\overline{c_0 s} \cup \overline{s e_0}) \cup \overline{d_0 e_1}$$

is the union of five polyhedral arcs in Q_0 with endpoints in $P_0 \cup P_1$.

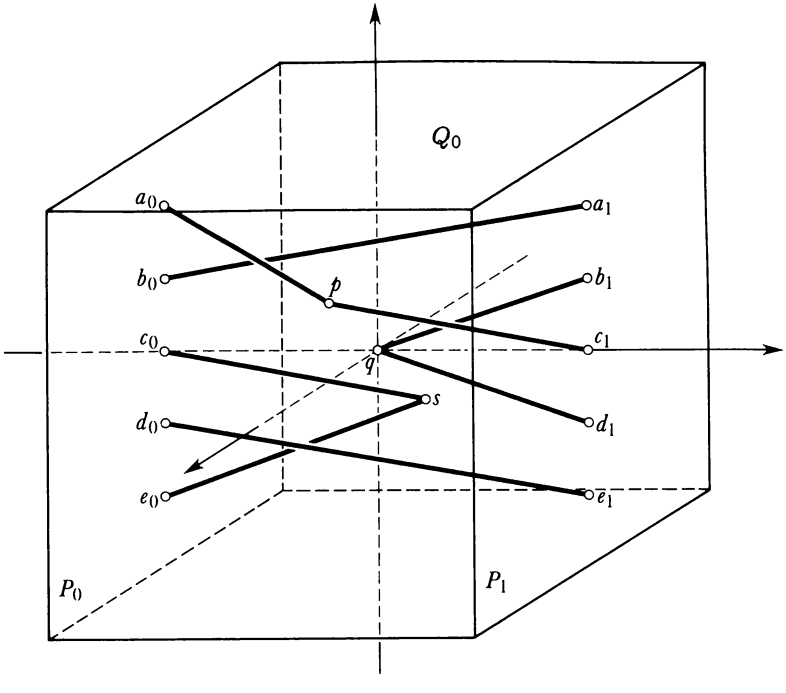


Fig.97. The first stage of the construction of a wildly imbedded arc (Example 4.3.5).

For $j = 0, \pm 1, \pm 2, \ldots$ consider the map $h_j: \mathbf{R}^3 \to \mathbf{R}^3$, which is the identity for $j = 0$, defined by the formula:

$$h_j(x^1, x^2, x^3) = (2^{-j}x^1 + 3(1 - 2^{-j}),\ 2^{-1-j}(3 - x^1)x^2,\ 2^{-1-j}(3 - x^1)x^3)$$

for $j > 0$ and by the formula:

$$h_j(x^1, x^2, x^3) = (2^j x^1 + 3(2^j - 1),\ 2^{-1+j}(3 + x^1)x^2,\ 2^{-1+j}(3 + x^1)x^3)$$

for $j < 0$. Observe that $h_{j-1}|P_i = h_j|P_0$ for $j = 0, \pm 1, \pm 2, \ldots$ Define

$$\begin{aligned} Q_j &= h_j(Q_0), \quad P_j = h_j(P_0), \quad a_j = h_j(a_0), \quad b_j = h_j(b_0), \\ c_j &= h_j(c_0), \quad d_j = h_j(d_0), \quad e_j = h_j(e_0), \quad M_j = h_j(M_0) \end{aligned}$$

for $j = 0, \pm 1, \pm 2, \ldots$ Let $M = \bigcup_{j=-\infty}^{\infty} M_j \cup \{u, v\}$ where $u = (-3, 0, 0)$ and $v = (3, 0, 0)$. It is easy to check that M is an arc with endpoints u, v. We show that the complement $\mathbf{R}^3 \backslash M$ is not simply connected.

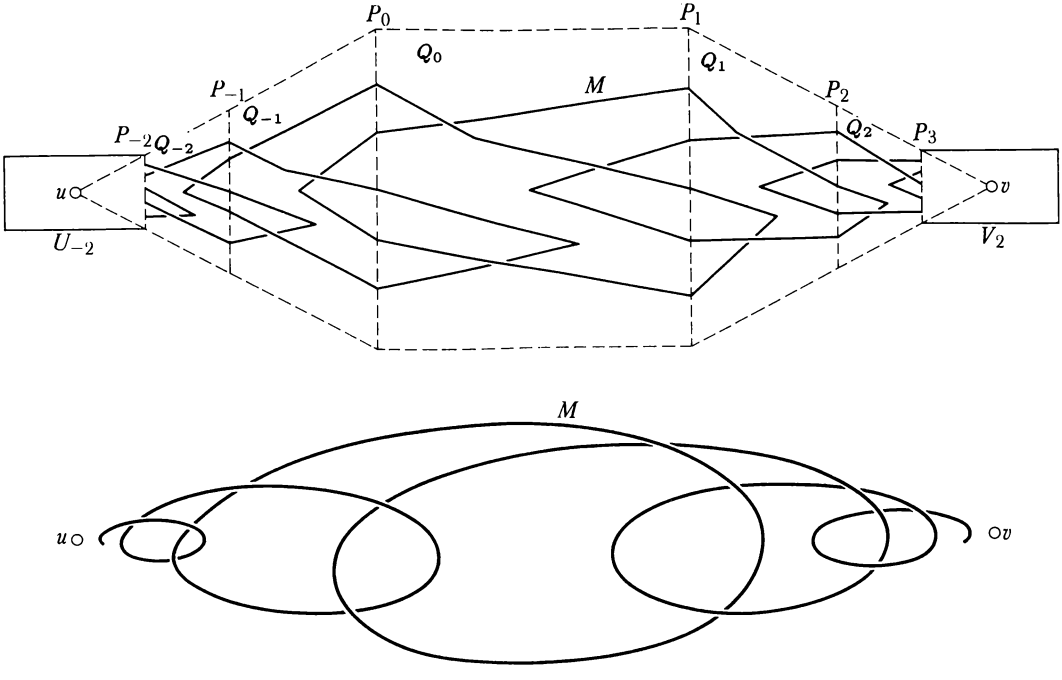


Fig.98. The wildly imbedded arc M in $\mathbf{R}^3$: above, the version used in Example 4.3.5, below, a more intuitive model.

For $m = 0, 1, \ldots$ let U_{-m} be a rectangular parallelepiped whose centre is the point u and one of whose side faces is the square P_{-m}, and let V_m be the parallelepiped whose centre is the point v and one of whose side faces is the square P_{m+1}. Let $X_m = \operatorname{bd} U_{-m} \cup \bigcup_{j=-m}^{m} Q_j \cup \operatorname{bd} V_m$ and $L_m = \bigcup_{j=-m}^{m} M_j$. We now calculate the fundamental group of the difference $X_m \backslash L_m$. By Lemma 4.3.1 there is a polyhedron $W_m \subset \mathbf{R}^3$ which is homeomorphic to the product $\bar{B}^2 \times L_m$ such that $L_m \subset \operatorname{int} W_m$ and $\operatorname{bd} W_m$

is a deformation retract of the difference $W_m \backslash L_m$. We may moreover suppose that $W_m \subset U_{-m} \cup \bigcup_{j=-m}^{m} Q_j \cup V_m$ and that for every index j, where $-m \leq j \leq m+1$, the intersection $P_j \cap W_m$ has five components each of which is homeomorphic to the disc $\bar{B}^2$. To calculate the fundamental group of the difference $X_m \backslash L_m$ it obviously suffices to examine the fundamental group of the set $X'_m = X_m \backslash \operatorname{int} W_m$. To do this we will use Van Kampen's Theorem (3.4.28).

Taking $P'_j = P_j \backslash \operatorname{int} W_m$ where $-m \leq j \leq m+1$ we deduce from Example 3.4.25 that the fundamental group of the set P'_j is free and has five generators $\alpha_j, \beta_j, \gamma_j, \delta_j, \eta_j$ corresponding respectively to the points a_j, b_j, c_j, d_j, e_j. Also the fundamental group of the set $Q'_j = Q_j \backslash \operatorname{int} W_m$ where $-m \leq j \leq m$ is free and has five generators; we may assume that they are $\alpha_j, \beta_j, \gamma_j, \delta_j$ and β_{j+1}. Finally we may assume that $\alpha_{-m}, \beta_{-m}, \gamma_{-m}, \eta_{-m}$ are the generators of the fundamental group of the set $V'_m = (\operatorname{bd} U_{-m}) \backslash \operatorname{int} W_m$; they are restricted by the one relation $\alpha_{-m}\beta_{-m}\gamma_{-m}\delta_{-m}\eta_{-m} = 1$. Similarly we may assume that $\alpha_{m+1}, \beta_{m+1}, \gamma_{m+1}, \delta_{m+1}$ and η_{m+1} are the generators of the fundamental group of the set $V'_m = (\operatorname{bd} V_m) \backslash \operatorname{int} W_m$; they are restricted by the one relation $\alpha_{m+1}\beta_{m+1}\gamma_{m+1}\delta_{m+1}\eta_{m+1} = 1$. Expanding appropriately the sets Q'_j and P'_j without altering their homotopy type we may suppose that the intersection $\bigcap_{j=-m}^{m} P'_j$ is non-empty and contains a fixed base point.

Since

$$X'_m = X_m \backslash \operatorname{int} W_m = (\operatorname{bd} U_{-m} \cup \bigcup_{j=-m}^{m} Q_j \cup \operatorname{bd} V_m) \backslash \operatorname{int} W_m = U'_{-m} \cup \bigcup_{j=-m}^{m} Q'_j \cup V'_m$$

we may assume by applying Van Kampen's Theorem that the fundamental group of the set X'_m has as generators $\alpha_j, \beta_j, \gamma_j, \delta_j$ for $-m \leq j \leq m+1$ and η_{-m}, η_{m+1}. As is easy to check, the inclusion of P'_j into Q'_{j-1} and into Q'_j for $-m+1 \leq j \leq m$, leads to the relations $\alpha_j = \alpha_{j-1}\beta_{j-1}\alpha_{j-1}^{-1}$, $\gamma_j = \alpha_{j-1}$, $\delta_j = \alpha_{j-1}^{-1}\beta_j^{-1}\alpha_{j-1}$, $\delta_{j-1} = \delta_j^{-1}\gamma_j^{-1}\delta_j$. The inclusion of P'_{-m} into U'_{-m} and into Q'_{-m} leads to the relation $\eta_{-m} = \delta_{-m}^{-1}\gamma_{-m}^{-1}\delta_{-m}$. The inclusion of P'_{m+1} into Q'_m and into V'_m leads to the relations $\alpha_{m+1} = \alpha_m\beta_m\alpha_m^{-1}$, $\gamma_{m+1} = \alpha_m$, $\delta_{m+1} = \alpha_m^{-1}\beta_{m+1}^{-1}\alpha_m$ and $\eta_{m+1} = \delta_m$. We thus have the relations:

$(1)_j$ $\alpha_j = \alpha_{j-1}\beta_{j-1}\alpha_{j-1}^{-1}$, where $-m+1 \leq j \leq m+1$,

$(2)_j$ $\gamma_j = \alpha_{j-1}$, where $-m+1 \leq j \leq m+1$,

$(3)_j$ $\delta_j = \alpha_{j-1}^{-1}\beta_j^{-1}\alpha_{j-1}$, where $-m+1 \leq j \leq m+1$,

$(4)_j$ $\delta_{j-1} = \delta_j^{-1}\gamma_j^{-1}\delta_j$, where $-m+1 \leq j \leq m$,

(5) $\eta_{-m} = \delta_{-m}^{-1}\gamma_{-m}^{-1}\delta_{-m}$,

(6) $\eta_{m+1} = \delta_m$,

(7) $\alpha_{-m}\beta_{-m}\gamma_{-m}\delta_{-m}\eta_{-m} = 1$,

(8) $\alpha_{m+1}\beta_{m+1}\gamma_{m+1}\delta_{m+1}\eta_{m+1} = 1$,

Now note that it is enough to limit the generators to α_j for $-m \leq j \leq m+1$ and β_{m+1}, γ_{-m}. Certainly from $(1)_{j+1}$ we may determine β_j as $\alpha_j^{-1}\alpha_{j+1}\alpha_j$ for $-m \leq j \leq m$. Relations $(2)_j$ say that $\gamma_j = \alpha_{j-1}$ for $-m+1 \leq j \leq m+1$. From $(3)_j$ we determine

δ_j as $\alpha_{j-1}^{-1}\alpha_j^{-1}\alpha_{j+1}^{-1}\alpha_j\alpha_{j-1}$ where $-m+1 \leq j \leq m$ and from $(4)_{-m+1}$ we determine δ_{-m} as $\alpha_{-m}^{-1}\alpha_{-m+1}^{-1}\alpha_{m+2}\alpha_{-m+1}\alpha_{-m}^{-1}\alpha_{-m+1}^{-1}\alpha_{-m+2}^{-1}\alpha_{-m+1}\alpha_{-m}$. From (6) we determine η_{m+1} as $\alpha_{m-1}^{-1}\alpha_m^{-1}\alpha_{m+1}^{-1}\alpha_m\alpha_{m-1}$. This leaves the generators α_j for $-m \leq j \leq m+1$, β_{m+1}, γ_{-m}, δ_{m+1}, η_{-m} and relations $(3)_{m+1}$, (5), (7) and (8). The relations $(4)_j$, where $-m+2 \leq j \leq m$ take the form

$$\alpha_{j-2}^{-1}\alpha_{j-1}^{-1}\alpha_j^{-1}\alpha_{j-1}\alpha_{j-2} = \alpha_{j-1}^{-1}\alpha_j^{-1}\alpha_{j+1}\alpha_j\alpha_{j-1}^{-1}\alpha_j^{-1}\alpha_{j+1}^{-1}\alpha_j\alpha_{j-1},$$

that is

$$(9)_j \qquad \alpha_j\alpha_{j-1}\alpha_{j-2}^{-1}\alpha_{j-1}^{-1}\alpha_j^{-1}\alpha_{j-1}\alpha_{j-2} = \alpha_{j+1}\alpha_j\alpha_{j-1}^{-1}\alpha_j^{-1}\alpha_{j+1}^{-1}\alpha_j\alpha_{j-1},$$

where $-m+2 \leq j \leq m$.

In view of relation (5) we have from relation (7)

$$(10) \qquad \alpha_{-m+2}\alpha_{-m+1}\alpha_{-m}^{-1}\alpha_{-m+1}^{-1}\alpha_{-m+2}^{-1}\alpha_{-m+1}\alpha_{-m} = 1.$$

By $(3)_{m+1}$ we obtain from relation (8)

$$(11) \qquad \alpha_{m+1}\alpha_m\alpha_{m-1}^{-1}\alpha_m^{-1}\alpha_{m+1}^{-1}\alpha_m\alpha_{m-1} = 1.$$

Moreover, relations (8) and (7) allow us to do away with the generators δ_{m+1} and η_{-m}. From relations $(9)_j$, (10) an (11) we obtain the equivalent set of relations

$$(12)_j \qquad \alpha_{j+1}\alpha_j\alpha_{j-1}^{-1}\alpha_j^{-1}\alpha_{j+1}^{-1}\alpha_j\alpha_{j-1} = 1 \quad \text{where} \quad -m+1 \leq j \leq m.$$

We now show that in the group with generators α_j where $-m \leq j \leq m+1$, β_{m+1}, γ_{-m} restricted by the relations $(12)_j$ where $-m+1 \leq j \leq m$, the elements α_j are different from unity. For this purpose consider the group S_5 of all permutations of the numbers $1,2,3,4,5$ and let $\lambda = (2,3,4,5,1)$, $\mu = (4,3,5,2,1) \in S_5$. It is easy to check that $\lambda\mu\lambda^{-1}\mu^{-1}\lambda^{-1}\mu\lambda = (1,2,3,4,5) = \mu\lambda\mu^{-1}\lambda^{-1}\mu^{-1}\lambda\mu$. Hence the formulas $h(\alpha_j) = \lambda$ for j even, $h(\alpha_j) = \mu$ for j odd, $h(\beta_{m+1}) = \lambda$ and $h(\gamma_{-m}) = \mu$ define a homomorphism $h: \pi_1(X_m \backslash L_m) \to S_5$. It follows that α_j is not the unity of the group $\pi_1(X_m \backslash L_m)$ for any j with $-m \leq j \leq m+1$.

Consider now the loop which represents the generator α_0 of the group $\pi_1(X_0 \backslash L_0)$. If it were homotopic to the trivial loop in the complement $\mathbf{R}^3 \backslash M$, then for some m it would be homotopic with the trivial loop in the set $\mathbf{R}^3 \backslash (U_{-m} \cup L_m \cup V_m)$. Since the set $U_{-m} \cup \bigcup_{j=-m}^{m} Q_j \cup V_m$ is a deformation retract of the space $\mathbf{R}^3$ the loop would also be homotopic to the trivial loop in the set $X_m \backslash L_m$ despite the fact that α_0 is not the unity of the group $\pi_1(X_m \backslash L_m)$. Thus the group $\pi_1(\mathbf{R}^3 \backslash M)$ is non-trivial.

Invoking Example 4.3.4, we deduce that the complement $\mathbf{R}^3 \backslash M$ is not homeomorphic to the complement of any polyhedral arc; the arc M is therefore wildly imbedded. ■

The construction presented in Example 4.3.5 can easily be exploited to obtain other examples.

4.3.6. EXAMPLE. We will use the notation of Example 4.3.5 in its entirety. As we have noted, the set M_0 is the union of five polyhedral arcs in Q_0 with endpoints in the set $P_0 \cup P_1$; denote these arcs by M_0^i for $i = 1,2,\ldots,5$. For $i = 1,2,\ldots,5$ it is

easy to construct a set $T_0^i \subset Q_0 \setminus M_0^i$ which is homeomorphic to the product $S^1 \times I$ such that the intersection $T_0^i \cap (P_0 \cup P_1)$ is the union of two circles with centres at the points of $M_0^i \cap (P_0 \cup P_1)$ and radii $1/8$, and so that $T_0^i \cap T_0^k = \emptyset$ for $i \neq k$. Let $T_0 = \bigcup_{i=1}^5 T_0^i$ and let $T_j = h_j(T_0)$ for $j = 0, \pm 1, \pm 2, \ldots$ It is easily checked that the set $T = \bigcup_{j=-\infty}^{\infty} T_j \cup \{u, v\}$ is homeomorphic with the sphere S^2. The argument in Example 4.3.5 yields the conclusion that the unbounded component of $\mathbf{R}^3 \setminus T$ is not simply connected; this component is therefore not homeomorphic to the unbounded component of the complement $\mathbf{R}^3 \setminus S^2$.

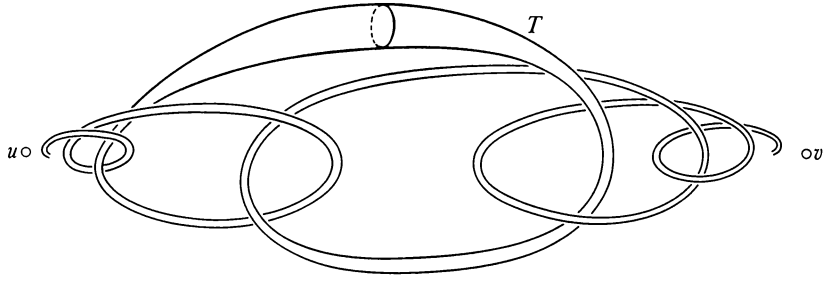


Fig.99. The set T is homeomorphic with the sphere S^2 but the unbounded component of $\mathbf{R}^3 \setminus T$ is not simply connected (Example 4.3.6).

In particular it follows that the sets T and S^2 are not equivalently imbedded. More in fact can be shown: as J. W. Alexander proves (*On the subdivision of a 3-space by a polyhedron*, Proc. Nat. Acad. Sci. USA 10(1924), 6–8), every polyhedron in the space $\mathbf{R}^3$ which is homeomorphic to the sphere S^2 separates the space $\mathbf{R}^3$ into two components whose closures are homeomorphic to the ball $\bar{B}^3$ and to the complement $\mathbf{R}^3 \setminus B^3$. It follows from this that the set T is wildly imbedded in the space $\mathbf{R}^3$. ■

4.3.7. EXAMPLE. Let T denote the space constructed in Example 4.3.6 and let B be any closed ball in $\mathbf{R}^3$ with boundary S and which contains T in its interior. Consider two sets $D' \subset T$, $D'' \subset S$ which are homeomorphic to the disc $\bar{B}^2$. There is a homeomorphism h of the product $S^1 \times I$ onto a subset W of the ball B such that $h(S^1 \times \{0\}) = \operatorname{bd} D'$ and $h(S^1 \times \{1\}) = \operatorname{bd} D''$. The set $T' = (S \cup T \cup W) \setminus (\operatorname{int} D' \cup \operatorname{int} D'')$ is homeomorpic to the sphere S^2. From the properties of the set T described in Example 4.3.6 it follows that the bounded component of the complement $\mathbf{R}^3 \setminus T'$ is not simply connected and so is not homeomorphic to the open ball B^3. As a consequence of Alexander's Theorem quoted in the last example the set T' is wildly imbedded in the space $\mathbf{R}^3$. ■

We conclude this section with one more example: a set $A \subset \mathbf{R}^3$ whose components are singletons but whose complement $\mathbf{R}^3 \setminus A$ is not simply connected.

4.3.8. EXAMPLE. *Antoine's necklace.* Let the set $T' \subset \mathbf{R}^3$ be the solid of revolution obtained by rotating around the x^3-axis the disc centred at $(3,0,0)$ of radius 1 lying in the x^1x^3-plane. Let the set $T'' \subset \mathbf{R}^3$ be the solid of revolution obtained from the disc in the x^1x^2-plane centred at $(0,0,0)$ and of radius 1 by rotating it around the line which is parallel to the x^2-axis and passes through the point $(3,0,0)$. If each of the

sets $X', X'' \subset \mathbf{R}^3$ is homeomorphic to the product $\bar{B}^2 \times S^1$ and the union $X' \cup X''$ is equivalently imbedded to the union $T' \cup T''$, then we shall say that the sets X' and X'' are *linked.*

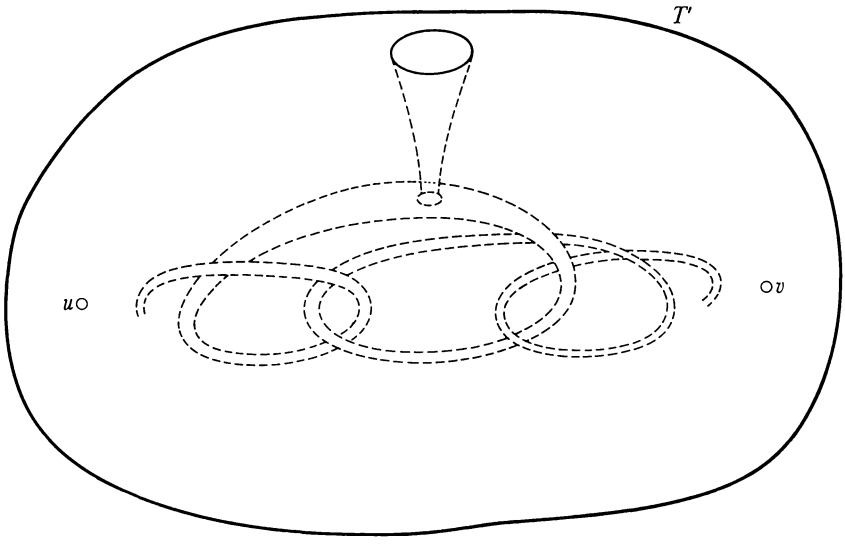


Fig.100. The set T' is homeomorphic to the sphere S^2 but the bounded component of $\mathbf{R}^3 \setminus T'$ is not simply connected (Example 4.3.7).

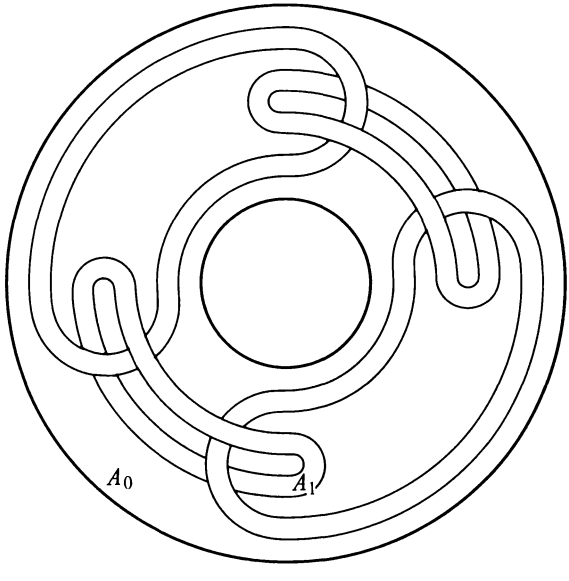


Fig.101. The first two steps in the construction of the Antoine necklace (Example 4.3.8).

We shall define a decreasing sequence of sets $A_n \subset \mathbf{R}^3$ for $n = 0, 1, \ldots$ We take $A_0 = T'$ and $A_1 = \bigcup_{j=1}^{4} T_j$ where the set $T_j \subset T'$ is homeomorphic to the product $\bar{B}^2 \times S^1$, $\operatorname{diam} T_j < \frac{3}{4} \operatorname{diam} T'$, the sets T_j, T_{j+2} are symmetric relative to the point $(0,0,0)$ and the sets T_j, T_{j+1} are linked for $j = 1,2,3,4$ (where the indices are to be reduced modulo 4). For $j = 1,2,3,4$ a homeomorphism $h_j: T' \to T_j$ may readily be defined so that $\operatorname{diam} h_j(T_k) < \frac{3}{4} \operatorname{diam} T_j$ for $k = 1,2,3,4$. Take $A_2 = \bigcup_{j=1}^{4} (h_j(A_1))$; this set is the union of 16 components each of which is homeomorphic to the product $\bar{B}^2 \times S^1$. This construction may be continued inductively in an obvious way so as to yield at the nth stage a set A_n which is the union of 4^n components each homeomorphic to the product $\bar{B}^2 \times S^1$ and of diameter less than $8(\frac{3}{4})^n$; moreover each component of the set A_{n-1} contains four components of the set A_n placed relative to each other just as the sets T_1, T_2, T_3, T_4 are in T'.

Since the sets A_n for $n = 0, 1, \ldots$ are compact and form a decreasing sequence, by Cantor's Theorem (1.8.9) it follows that the intersection $A = \bigcap_{n=0}^{\infty} A_n$ is non-empty. This set is known as *Antoine's necklace*. Since the diameters of the components of the sets A_n tend to zero as n tends to infinity, the set A does not contain any connected subsets other than singletons.

It turns out that the complement $\mathbf{R}^3 \backslash A$ is not a simply connected space. It may be proved for example that the loop a defined by the formula $a(r) = (3 + 3\cos 2\pi r, 0, 3\sin 2\pi r)$ for $r \in I$ is not homotopic to the trivial loop in the set $\mathbf{R}^3 \backslash A$. For this purpose it is enough to show that it is not homotopic to the trivial loop in the set $\mathbf{R}^3 \backslash A_n$ for any n. The proof of this intuitively obvious fact follows the lines of the previous examples and relies on an application of Van Kampen's Theorem; we leave this as an exercise for the reader (see Problem 4.P.15).

Since, as is easily seen, the complement $\mathbf{R}^3 \backslash X$ of any compact set X lying on a straight line in $\mathbf{R}^3$ is simply connected, the Antoine necklace is not equivalently imbedded to any set which lies on a straight line in $\mathbf{R}^3$. On the other hand it may be shown (see Problem 4.P.16) that the Antoine necklace is homeomorphic to the Cantor set (Example 4.4.1) which does lie on the real line $\mathbf{R}$. ■

Exercises

a) Show that any two sets, each consisting of k points on the line $\mathbf{R}^1$, are equivalently imbedded.

b) Show that any two countable dense subsets of $\mathbf{R}^1$ are equivalently imbedded. (Hint: Let $A = \{a_1, a_1, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$; take $h(a_1) = b_1$ and define the map $h: A \to B$ inductively by taking $h(a_n)$ to be the element of the set B with least index such that the inequalities $a_i < a_j$ and $h(a_i) < h(a_j)$ are equivalent for $i, j \leq n$. Using the map h construct a homeomorphism $h^*: \mathbf{R}^1 \to \mathbf{R}^1$ such that $h^*(a) = h(a)$ for $a \in A$.)

c) Describe in detail a homeomorphism of the interval I onto the arc M of Example 4.3.5.

d) Describe in detail a homeomorphism of the sphere S^2 onto the set T of Example 4.3.6.

e) Show that the Antoine necklace has no isolated points.

4.4. Various examples

In the current section we give some examples of sets in Euclidean spaces and of maps defined on them. We shall make use of these constructions in later parts of the book. We add the remark that the classic examples discussed here lend themselves to various modifications leading to new constructions that are of use in solving all kinds of topological problems.

4.4.1. EXAMPLE. *The Cantor set.* We construct by induction a decreasing sequence of closed subsets F_n of the unit interval I for $n = 0, 1, \ldots$ Let $F_0 = I$. We divide the interval I into three contiguous closed intervals $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$, $[\frac{2}{3}, 1]$ each of length $\frac{1}{3}$. We call the middle interval, that is, the interval $[\frac{1}{3}, \frac{2}{3}]$, the *interval excluded at the first step of the construction* and we denote the union of the other two by F_1. Each of these two intervals of length $\frac{1}{3}$ we divide into three contiguous subintervals of length $\frac{1}{9}$ each, the middle one of which we shall call *an interval excluded at the second step of the construction.* Denote by F_2 the union of all four intervals not excluded at the second step of the construction.

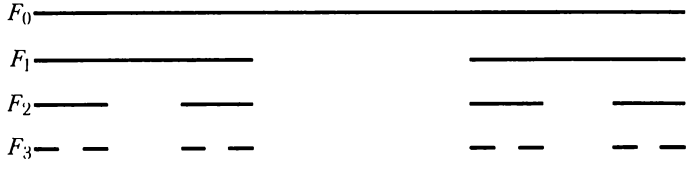


Fig.102. First steps in the construction of the Cantor set (Example 4.4.1).

Suppose that the set F_n is the union of 2^n pairwise disjoint closed intervals each of length 3^{-n}. Divide each of them into three contiguous closed subintervals each of length $\frac{1}{3}3^{-n} = 3^{-n-1}$, the middle one of which we shall call *an interval excluded at the* $(n+1)$*-st step of the construction.* We shall denote by F_{n+1} the union of all the $2 \cdot 2^n = 2^{n+1}$ subintervals not so excluded.

Obviously the sets F_n are closed and $F_{n+1} \subset F_n$ for $n = 0, 1, \ldots$ By Cantor's Theorem (1.8.9) the intersection $C = \bigcap_{n=0}^{\infty} F_0$ is non-empty; we call it the *Cantor set.*

From the construction of the Cantor set it follows immediately that it is a compact subspace of $\mathbf{R}^1$. Since the components of the set F_n have diameter 3^{-n} for $n = 0, 1, \ldots$, the Cantor set contains no connected subsets other than singletons. Since for each interval I'_n not excluded at the *n-th* step of the construction, for $n \geq 1$, there exists an interval I''_n different from I'_n which is also not excluded at the n-th step of the construction such that $\operatorname{diam}(I'_n \cup I''_n) = 3^{1-n}$, it follows that the Cantor set has no isolated points.

It is easy to see that the set F_n used in the construction of the set C consists of those real numbers $r \in I$ which have a base 3 expansion $r = 0.r_1 r_2 \ldots$, where $r_i \neq 1$ for $i = 1, 2, \ldots, n$. It follows that the set C consists of all real numbers $r \in I$ which have a base 3 expansion of the form $r = 0.r_1 r_2 \ldots$, where $r_i \neq 1$ for $i = 1, 2, \ldots$ The expansion is then uniquely determined by the number r. In other words, the set C consists of all real numbers $r = \sum_{i=1}^{\infty} r_i 3^{-i}$, where $r_i = 0$ or $r_i = 2$ for $i = 1, 2, \ldots$ Hence it follows in particular that Cantor's set contains uncountably many points. ■

4.4.2. EXAMPLE. *The Sierpiński curve.* The construction of this set is similar to that of the Cantor set except that the role of the interval I is taken by the square I^2. Let $F_0 = I^2$. Divide the square I^2 into nine contiguous squares each of side length $\frac{1}{3}$, and call the one which is interior to I^2 the square *excluded at the first step of the construction.* Denote by F_1 the union of the eight remaining squares not so excluded. Divide each of the squares making up F_1 into nine contiguous squares each of side length $\frac{1}{9}$ each and call the one interior to the square being subdivided a square *excluded at the second step of the construction.* Denote by F_2 the union of the 64 squares not excluded at the second step. Continuing the process we define inductively the set F_n which is a union of 8^n squares each of side length 3^{-n}. The intersection $S = \bigcap_{n=0}^{\infty} F_n$ is called the *Sierpiński curve* (or the *Sierpiński carpet*). Since each of the sets F_n for $n = 0, 1, \ldots$ is a continuum, it follows by Theorem 1.8.20 that the Sierpiński curve is a non-empty continuum. It is easy to see that it separates the plane $\mathbf{R}^2$ into $\aleph_0$ regions. ■

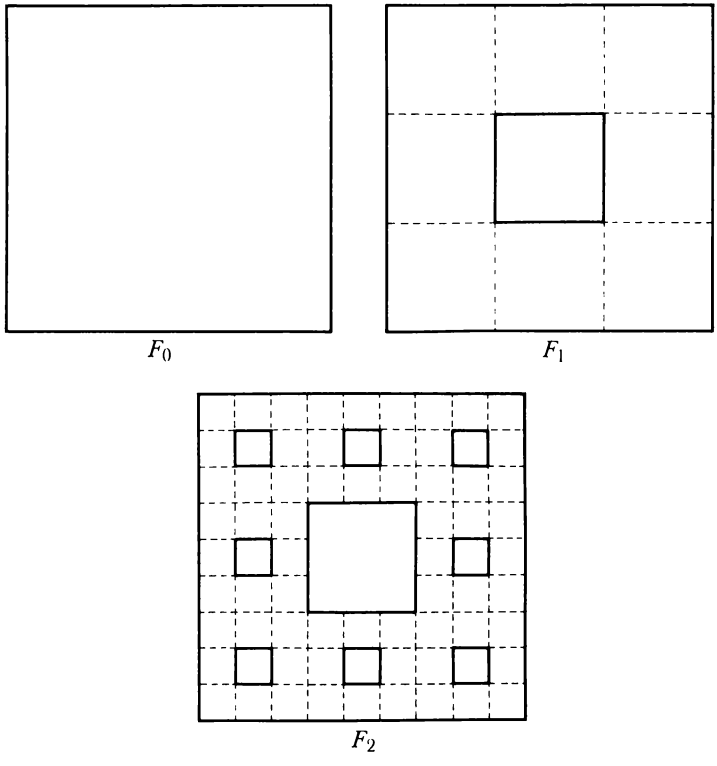


Fig.103. First steps in the construction of the Sierpiński curve (Example 4.4.2).

4.4.3. EXAMPLE. *The Menger Curve.* The construction resembles that of the Sierpiński curve except that the role of the square is this time taken over by the cube I^3. Let $F_0 = I^3$. Divide the cube I^3 into 27 contiguous cubes each of edge length $\frac{1}{3}$ and call those seven which do not meet any of the edges of the cube I^3 the cubes *excluded at the first step of the construction.*

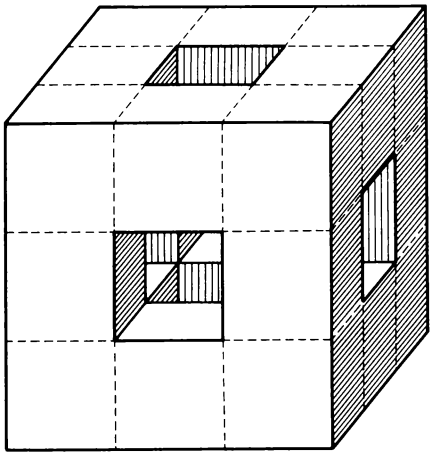


Fig.104. The set F_1 used in the construction of the Menger curve (Example 4.4.3).

Denote by F_1 the union of the remaining 20 cubes not excluded. We proceed analogously with each of the cubes making up F_1, obtaining a set F_2 etc. The intersection $M = \bigcap_{n=1}^{\infty} F_n$ is known as the *Menger curve*. Just as in the case of the Sierpiński curve it is easy to see that the Menger curve is a continuum. It may also be checked that it does not separate the space $\mathbf{R}^3$. ■

4.4.4. EXAMPLE. *The staircase function.* We construct a continuous map f of the Cantor set onto the unit interval. If a real number $r \in C$ has the base 3 expansion $r = \sum_{i=1}^{\infty} r_i 3^{-i}$ with $r_i \neq 1$ for $i = 1, 2, \ldots$, then we take $f(r) = \frac{1}{2} \sum_{i=1}^{\infty} r_i 2^{-i}$. We have of course $0 \leq f(r) \leq 1$ for each $r \in C$.

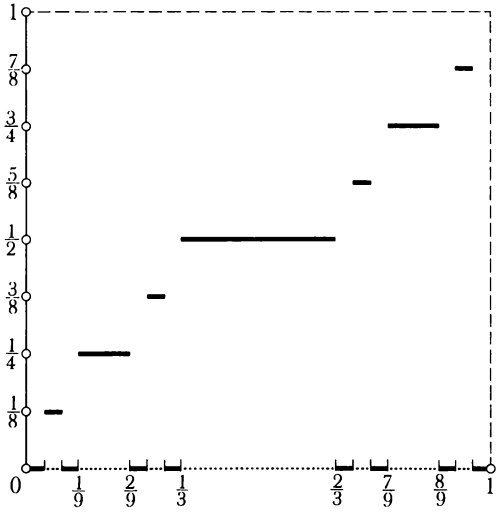


Fig.105. Graph of the staircase function (Example 4.4.4).

We show that the function f is uniformly continuous. Let $r = \sum_{i=1}^{\infty} r_i 3^{-i}$, $r' = \sum_{i=1}^{\infty} r_i' 3^{-i}$ and let $\epsilon > 0$. Choose a natural number k so that $2^{-k} < \epsilon$. If $|r - r'| < 3^{-k}$, then $r_i = r_i'$ for $i = 1, 2, \ldots, k$, hence $|f(r) - f(r')| \leq \frac{1}{2} \sum_{i=k+1}^{\infty} |r_i - r_i'| 2^{-i} \leq 2^{-k} < \epsilon$.

For every $s \in I$ consider any expansion of s to base 2, say $s = \sum_{i=1}^{\infty} s_i 2^{-i}$; thus $s_i = 0, 1$ for $i = 1, 2, \ldots$ Taking $r_i = 2s_i$ we have $r_i = 0, 2$ for $i = 1, 2, \ldots$, so

$$s = \sum_{i=1}^{\infty} s_i 2^{-i} = \frac{1}{2} \sum_{i=1}^{\infty} r_i 2^{-i} = f(r),$$

where $r = \sum_{i=1}^{\infty} r_i 3^{-i} \in C$. Thus $f(C) = I$.

The function $f: C \to I$ has a natural continuous extension $f^*: I \to I$ which we now describe. Notice that if r' is a left endpoint and r'' a right endpoint of an interval excluded at the n^{th} step of the construction of the Cantor set, then

$$r' = \sum_{i=1}^{n-1} r_i 3^{-i} + 2 \sum_{i=n+1}^{\infty} 3^{-i} \quad \text{and} \quad r'' = \sum_{i=1}^{n-1} r_i 3^{-i} + 2 \cdot 3^{-n},$$

where $r_i \neq 1$ for $i = 1, 2, \ldots, n-1$. Then

$$f(r') = \frac{1}{2} \sum_{i=1}^{n-1} r_i 2^{-i} + \sum_{i=n+1}^{\infty} 2^{-i} = \frac{1}{2} \sum_{i=1}^{n-1} r_i 2^{-i} + 2^{-n} = f(r''),$$

that is $f(r') = f(r'')$. Taking $f^*(r) = f(r') = f(r'')$ for $r' \leq r \leq r''$, where r', r'' are the endpoints of an excluded interval, we obtain a continuous extension $f^*: I \to I$ of the map $f: C \to I$. In view of the characteristic shape of its graph the function f^* is called the *staircase function*. The same name is also given to the map $f = f^* | C$. ■

It is worth drawing attention to the more general fact, in anticipation of a proof, that any non-empty compact metric space is the continuous image of the Cantor set (see Corollary 6.3.12 and Problem 6.P.39).

4.4.5. EXAMPLE. *The Peano map.* We construct a continuous map f of the unit interval onto I^2. The map will be the limit of a sequence of maps $f_n: I \to I^2$ which we now describe. Let $\mathcal{P}_1$ denote a subdivision of the interval I into 9 contiguous subintervals of length $\frac{1}{9}$ each and let $\mathcal{Q}_1$ denote a subdivision of the square I into 9 contiguous squares each of side length $\frac{1}{3}$. We define a map $g: I \to I^2$ by the conditions

$$g(0) = (0,0), \quad g(\tfrac{1}{9}) = (\tfrac{1}{3}, \tfrac{1}{3}), \quad g(\tfrac{2}{9}) = (0, \tfrac{2}{3}), \quad g(\tfrac{1}{3}) = (\tfrac{1}{3}, 1), \quad g(\tfrac{4}{9}) = (\tfrac{2}{3}, \tfrac{2}{3}),$$

$$g(\tfrac{5}{9}) = (1, \tfrac{1}{3}), \quad g(\tfrac{2}{3}) = (\tfrac{2}{3}, 0), \quad g(\tfrac{7}{9}) = (\tfrac{1}{3}, \tfrac{1}{3}), \quad g(\tfrac{8}{9}) = (\tfrac{2}{3}, \tfrac{2}{3}), \quad g(1) = (1,1),$$

and by linear extrapolation on the subintervals. The map g has the following properties:

(1) every interval of the subdivision $\mathcal{P}_1$ is taken to a diagonal of a square in the subdivision $\mathcal{Q}_1$, and

(2) every square of the subdivision $\mathcal{Q}_1$ has a diagonal which is the image of a subinterval of the subdivision $\mathcal{P}_1$.

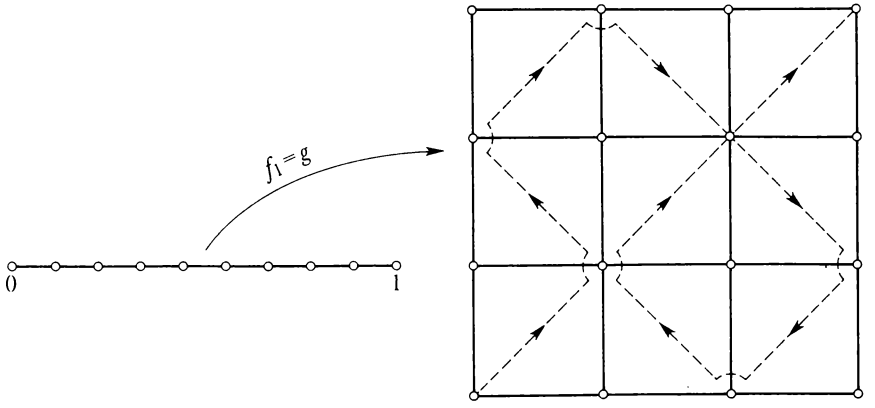


Fig.106. The first step in the construction of the Peano map (Example 4.4.5).

Define $f_1 = g$ and suppose we are given a subdivision $\mathcal{P}_n$ of the interval I into 9^n contiguous subintervals of length 9^{-n} each, a subdivision $\mathcal{Q}_n$ of the square I^2 into 9^n contiguous squares each of side 3^{-n}, and a continuous map $f_n: I \to I^2$ which have the following properties:

$(1)_n$ every interval of the subdivision $\mathcal{P}_n$ is taken to a diagonal of a square of the subdivision $\mathcal{Q}_n$, and

$(2)_n$ every square of the subdivision $\mathcal{Q}_n$ has a diagonal which is the image of a subinterval of the subdivision $\mathcal{P}_n$.

Consider an arbitrary subinterval $[a,b]$ of the subdivision $\mathcal{P}_n$ and let $f_n(a) = (x^1,x^2)$ and $f_n(b) = (y^1,y^2)$. Denote by g_j the composition of the map g with the rotation of the square I^2 about its centre through an angle of $\frac{1}{2}\pi j$ where $j = 0,1,2,3$. For each real number r with $a \leq r \leq b$ define

$$f_{n+1}(r) = \begin{cases} (x^1,x^2) + 3^{-n}g_0(r), & \text{if } x^1 < y^1,\ x^2 < y^2, \\ (y^1,x^2) + 3^{-n}g_1(r), & \text{if } y^1 < x^1,\ x^2 < y^2, \\ (y^1,y^2) + 3^{-n}g_2(r), & \text{if } y^1 < x^1,\ y^2 < x^2, \\ (x^1,y^2) + 3^{-n}g_3(r), & \text{if } x^1 < y^1,\ y^2 < x^2. \end{cases}$$

It may easily be checked that this definition gives a continuous map $f_{n+1}: I \to I^2$ satisfying conditions $(1)_{n+1}$ and $(2)_{n+1}$. From the construction of the sequence of maps f_n it follows immediately that $\rho(f_n(r), f_k(r)) \leq 3^{-n}$ for $k \geq n$ and $r \in I$. It follows from this that for each $r \in I$ the sequence $\{f_n(r)\}$ satisfies the Cauchy condition and hence is convergent; put $f(r) = \lim_n f_n(r)$. Since $\rho(f_n(r), f(r)) \leq 3^{-n}$, for $n = 1,2,\ldots$ and $r \in I$, the sequence of maps $\{f_n\}$ is unformly convergent to f and from Theorem 1.5.15 we deduce that the map $f: I \to I^2$ is continuous; we call it the *Peano map*. It follows from condition $(2)_n$ that the set $f(I)$ is dense in the square I^2 and, in view of the compactness of the interval I, we have $f(I) = I^2$. ■

Appropriate modification of the construction above easily yields a continuous map of the interval I' onto the cube I^m where m is any natural number. In Section 6.5 we shall give a topological characterization of metric spaces which are continuous images of the interval I (see Theorem 6.5.24 and Supplement 6.S.11).

4.4.6. EXAMPLE. *Common boundary of three plane regions.* The circle (or any simple closed curve) separates the plane into two components and is their common boundary. A bouquet of two circles separates the plane into three components but only one point of the bouquet lies in the closure of all three components of the complement at once. We will now construct a continuum $B \subset \mathbf{R}^2$ which separates the plane $\mathbf{R}^2$ into three components and is the boundary of each (see also Supplement 4.S.6).

Let $W \subset \mathbf{R}^2$ be a polyhedron which is connected, has connected interior and separates the plane $\mathbf{R}^2$ into three components S_1, S_2, S_3 with pairwise disjoint boundaries. We show that for any finite set $P \subset \operatorname{int} W$ there is a polyhedron $W' \subset W$ which is connected, has connected interior and separates the plane $\mathbf{R}^2$ into the three components S_1', S_2, S_3 with pairwise disjoint boundaries, with $P \subset S_1'$.

For the proof it is obviously enough to consider the case where P consists of one point p. By Theorem 1.10.7 there is a broken line $L \subset W$ which joins some point of $\operatorname{bd} S_1$ to the point p and, apart from its beginning, is contained in $\operatorname{int} W$. It is easy to construct a finite covering of the broken line L by open squares $K_1, K_2, \ldots, K_m$ such that $K_2, K_3, \ldots, K_m \subset \operatorname{int} W$, the intersection $K_1 \cap \operatorname{bd} W$ is connected and $K_i \cap K_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Then $W' = W \setminus \bigcup_{i=1}^m K_i$ is a polyhedron which is connected, has connected interior and separates the plane into three components $S_1' = S_1 \cup \bigcup_{i=1}^m K_i$ and S_2 and S_3.

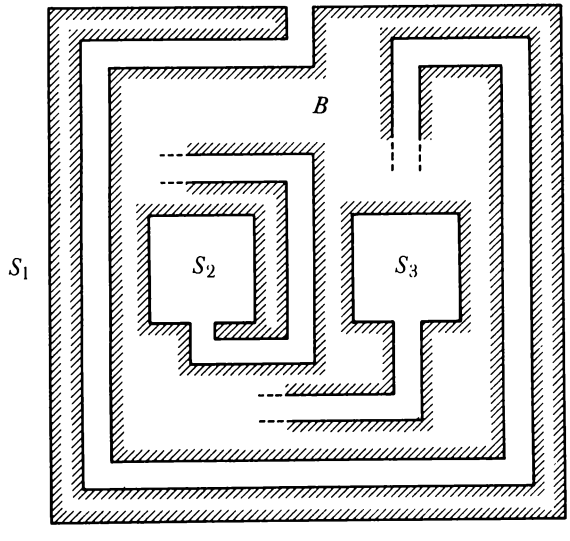


Fig.107. Construction of a common boundary B of three regions S_1, S_2, S_3 in the plane (Example 4.4.6).

Observe now that if a polyhedron $W \subset \mathbf{R}^2$ is connected, has connected interior and separates the plane $\mathbf{R}^2$ into three components S_1, S_2, S_3 with pairwise disjoint boundaries then for every positive real number ϵ there is a polyhedron $W' \subset W$ which is connected, has connected interior and separates the plane $\mathbf{R}^2$ into three components S_1', S_2, S_3 with pairwise disjoint boundaries and $\rho(x, \operatorname{bd} S_1') < \epsilon$ for every point $x \in W'$. To see this, just take a finite set $P \subset \operatorname{int} W$ such that $\rho(x, P) < \epsilon$ for every point $x \in W$ and then choose W' constructed as above.

Applying the above construction three times over we conclude that for every positive real number ϵ there is a polyhedron $W^* \subset W$ which is connected, has connected interior, separates the plane into three components S_1', S_2', S_3' which have pairwise disjoint boundaries and further $\rho(x, \operatorname{bd} S_j') < \epsilon$ for each $x \in W^*$ and $j = 1, 2, 3$.

Using this property we construct inductively a decreasing sequence of polyhedra $W_n \subset \mathbf{R}^2$ where for $n = 1, 2, \ldots$ the polyhedron W_n is connected, has connected interior and separates the plane $\mathbf{R}^2$ into three components $S_{1,n}, S_{2,n}, S_{3,n}$ which have pairwise disjoint boundaries and is such that $\rho(x, \operatorname{bd} S_{j,n}) < \frac{1}{n}$ for every $x \in W_n$ and $j = 1, 2, 3$. From Theorem 1.8.20 we deduce that the intersection $B = \bigcap_{k=1}^{\infty} W_n$ is a non-empty continuum. It separates the plane $\mathbf{R}^2$ into three components $S_j = \bigcup_{n=1}^{\infty} S_{j,n}$, where $j = 1, 2, 3$. If $x \in B$, then $x \in W_n$ for $n = 1, 2, \ldots$ and so $\rho(x, \operatorname{bd} S_j) = 0$, that is $x \in \operatorname{cl} S_j$ for $j = 1, 2, 3$. Every point of the set B thus belongs to the boundary of each of the components S_1, S_2, S_3.

Appropriate modification of the method above leads, for each natural number k, where $k \geq 2$, to the construction in the Euclidean space $\mathbf{R}^m$ for $m \geq 2$ of a continuum B which separates the space $\mathbf{R}^m$ into k components whose common boundary is B. ■

4.4.7. EXAMPLE. *An indecomposable space.* A metric space consisting of more than one point is called *indecomposable* if it is connected, yet cannot be expressed as a union $X = A \cup B$ where the sets A and B are closed, connected and different from X. We construct an example of an indecomposable continuum in the Euclidean plane $\mathbf{R}^2$ (see Supplement 4.S.3).

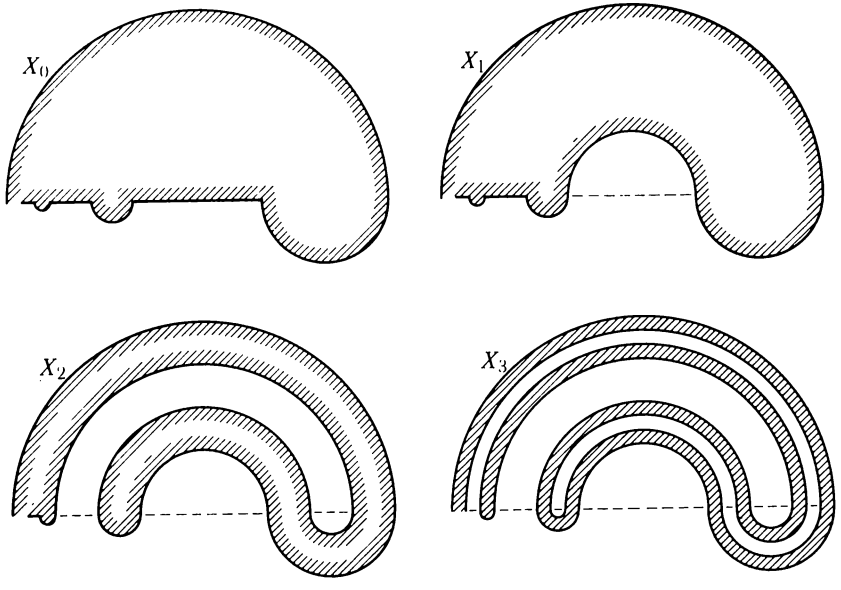


Fig.108. First steps in the construction of an indecomposable space (Example 4.4.7).

We use the notation of Example 4.4.1 and recall that the Cantor set C was defined as an intersection $C = \bigcap_{n=0}^{\infty} F_n$. For our purposes it is convenient to expand the definition of the sets F_n agreeing on the convention that $F_n = [-\frac{1}{2}, 1]$ for $n < 0$. Let $X_{n,0}$ for $n = 0, 1, \ldots$ be the intersection of the upper half-plane $\mathbf{R}^2_+ = \{(x^1, x^2) \in \mathbf{R}^2 : x^2 \geq 0\}$ with the union of the circles centred at $(\frac{1}{2}, 0)$ and radii $\frac{1}{6} + \frac{1}{4}c$ for $c \in F_{n-1}$. For $k = 1, 2, \ldots$ let $X_{n,k}$ be the intersection of the lower half-plane $\mathbf{R}^2_- = \{(x^1, x^2) \in \mathbf{R}^2 : x^2 \leq 0\}$ with the union of the circles centred at $(\frac{5}{2} \cdot 3^{-k}, 0)$ and radii $3^{-k}(\frac{1}{6} + \frac{1}{3}c)$ for $c \in F_{n-k-1}$. It is easy to check that the set $X_n = \bigcup_{k=0}^{\infty} X_{n,k}$ is homeomorphic with the disc $\bar{B}^2$ and $X_{n+1} \subset X_n$ for $n = 0, 1, \ldots$ From Theorem 1.8.20 it therefore follows that the intersection $X = \bigcap_{n=0}^{\infty} X_n$ is a non-empty continuum.

We shall show that the space X is indecomposable. Denote by $C' \subset C$ the set of endpoints of the intervals excluded during the construction of the Cantor set C together with the numbers 0 and 1. Let Y_0 be the intersection of the upper half-plane $\mathbf{R}^2_+$ with the union of the circles centred at $(\frac{1}{2}, 0)$ and of radii $\frac{1}{6} + \frac{1}{3}c$ for $c \in C'$. For $k = 1, 2, \ldots$ let Y_k be the intersection of the lower half-plane $\mathbf{R}^2_-$ with the union of all the circles centred at $(\frac{5}{2} \cdot 3^{-k}, 0)$ and of radii $3^{-k}(\frac{1}{6} + \frac{1}{3}c)$ for $c \in C'$. It is easy to see that the set $Y = \bigcup_{k=0}^{\infty} Y_k$ is connected, even pathwise connected, and has empty interior in X.

If $A \subset X$ is a continuum which contains the point $(0, 0)$, then either $A \subset Y$ or $A = X$. For, suppose that $x \in X \setminus A$; without loss of generality we may assume that $x = (c, 0)$ where $c \in C$. From the compactness of the set A it follows that there is an index n such that the component P_n of the set F_n which contains the point x is disjoint from A. The set $\{(x^1, x^2) \in \mathbf{R}^2 : x^1 \in P_n,\ x^2 = 0\}$ separates X_n into two components. Let X'_n be the component which contains $(0, 0)$; then $A \subset X \cap X'_n$ and since the set $X \cap X'_n$ is homeomorphic with the product $C \times [0, 1)$, the set A is contained in the component of $X \cap X'_n$ which contains the point $(0, 0)$. But it is easy to see that this component is contained in Y, hence $A \subset Y$.

Now suppose that $X = A \cup B$ where A and B are continua different from X. We may of course assume that $(0, 0) \in A$; from the argument above it therefore follows that $A \subset Y$. Hence in X the set A has no interior points; from this it follows that the set B is dense in X. This however is impossible as the set B is closed in X and different from X. ■

Exercises

a) Show that the Cantor set has empty interior in the line $\mathbf{R}^1$, that the Sierpiński curve has empty interior in the plane $\mathbf{R}^2$, and that the Menger curve has empty interior in the space $\mathbf{R}^3$.

b) Examine whether the staircase function $f^*: I \to I$ is differentiable.

c) Show that the preimage of any point under the Peano map consists of at most four points.

d) Examine whether the indecomposable space described in Example 4.4.7 is pathwise connected.

4.S. Supplements

4.S.1. Theorem 4.2.2 is a particular case of a more general theorem of Borsuk on the invariance of the number of components of a complement (see [10], p. 495–498). To be precise, if the compact sets $A, B \subset S^m$ are homeomorphic, then the complements $S^m \setminus A$ and $S^m \setminus B$ have the same number of components. From this follows a generalization of Corollary 4.2.3 which states that if the compact sets $A, B \subset \mathbf{R}^m$ are homeomorphic, then the complements $\mathbf{R}^m \setminus A$ and $\mathbf{R}^m \setminus B$ have the same number of components. In particular if a set $A \subset \mathbf{R}^m$ is homeomorphic to the sphere S^{m-1}, then A separates the space $\mathbf{R}^m$ into exactly two components. It is easily shown that (see Problem 4.P.13) A is then the boundary of both the components. The components are not in general homeomorphic to the corresponding components of $\mathbf{R}^m \setminus S^m$ (see Examples 4.3.6 and 4.3.7) except in the case $m = 2$ when Schönflies' Theorem, quoted in Section 4.3, may be applied. In the case $m = 2$ we thus obtain the following strengthening of Jordan's Theorem (4.2.5): *Every simple closed curve K lying in the Euclidean plane $\mathbf{R}^2$ separates the plane into two components, one bounded, say C', and one unbounded, say C''; moreover the curve K is the common boundary of both C' and C'' and the unions $K \cup C'$ and $K \cup C''$ are homeomorphic respectively to $\bar{B}^2$ and $\mathbf{R}^2 \setminus B^2$.*

4.S.2. The development of the theory of imbeddings originates with the work of A. Schönflies and L. Antoine in the first two decades of the century. Antoine's example (4.3.8) was described in 1921. In 1924 J. W. Alexander constructed the first wildly imbedded set in $\mathbf{R}^3$ homeomorphic to S^2; it is now known as Alexander's *horned sphere* (see e.g. [14], p. 69, 70). Systematic study of imbedding problems in $\mathbf{R}^3$ was begun in the late forties and early fifties with the work of E. Artin, R. Fox, R. H. Bing and E. E. Moise. The example of the wildly imbedded arc (4.3.5) is due to E. Artin and R. Fox (1948).

The opposite of a wild imbedding is a *tame* imbedding. The notion is of use in the study of the imbeddings of homeomorphic images of the sphere S^2 in the space $\mathbf{R}^3$, thanks largely to the theorem of J. W. Alexander mentioned in Example 4.3.6. It is still not known whether its analogue holds for higher dimensions. For this reason when studying imbeddings in the space $\mathbf{R}^m$ of sets X which are homeomorphic to the sphere S^{m-1}, tame imbedding is replaced by the stronger notion of *trivial imbedding*; a set X is *trivially imbedded* in $\mathbf{R}^m$ if it is equivalently imbedded to the sphere S^{m-1}.

4.S.3. The first example of an indecomposable continuum was given by L. E. J. Brouwer (1910). Example 4.4.7 is due to B. Knaster. Knaster also constructed (*Un continu dont tout sous-continu est indécomposable*, Fund. Math., 3(1922), 247–286) an example of a *hereditarily indecomposable* continuum; that is, a continuum every subcontinuum of which is indecomposable (see also Supplement 6.S.5).

A propos the notion of an indecomposable space we also mention that there exist connected spaces which cannot be expressed as a union of two connected disjoint subsets each containing more than one point. An example of a space with this property is the

Knaster-Kuratowski broom. Let C be the Cantor set constructed on the x^1-axis of the Euclidean plane $\mathbf{R}^2$ and denote by C' the set of endpoints of intervals excluded during the construction of the set and by C'' the complement $C\backslash C'$. Let $d = (\frac{1}{2}, 1)$. For each point $c \in C'$ define $L(c) = \{x = (x^1, x^2) \in \mathbf{R}^2 : x = (1-r)c + rd,\ r \in I,\ x^2 \text{ is rational}\}$; for each point $c \in C''$ define $L(c) = \{x = (x^1, x^2) \in \mathbf{R}^2 : x = (1-r)c + rd,\ r \in I,\ x^2 \text{ is irrational}\}$. The *Knaster-Kuratowski broom* is the union $M = \bigcup_{c \in C} L(c)$. It may also be shown that the complement $M\backslash\{d\}$ does not contain any connected subsets other than singletons; for this reason the point d is sometimes called an *explosion point* of the broom (cf. B. Knaster, K. Kuratowski, *Sur les ensembles connexes*, Fund. Math. 2(1921), 206–255; see also Problem 4.P.20).

During the discussion of Example 4.4.7 we tacitly made use of the concept of a composant. Let $p \in X$. The set of points $x \in X$ for which there exists a proper connected closed subset $C \subset X$ such that $p, x \in C$ is called a *composant* of the point p. Evidently the composants of a disconnected space coincide with the components of the space. In Example 4.4.7 the set Y is the composant of the point $(0,0)$ in the space X. It may be proved that every composant of a continuum X is dense and does not separate X (see Problem 4.P.18). If moreover a continuum X is indecomposable, then it has $\mathfrak{c}$ pairwise disjoint composants (see Problem 4.P.19).

4.S.4. We say that a continuum X is *snake-like* if for every positive real number ϵ there exists a finite open covering $U_1, U_2, \ldots, U_n$ of the space X such that $\operatorname{diam} U_j < \epsilon$ for $j = 1, 2, \ldots, n$ and $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Every snake-like, hereditarily indecomposable continuum of more than one point is called a *pseudoarc*. E. E. Moise proved (*An indecomposable plane continuum which is homeomorphic to each of its non-degenerate subcontinua*, Trans. Amer. Math. Soc., 63(1948), 581–594) that any two pseudoarcs are homeomorphic. Knaster's hereditarily indecomposable continuum mentioned already is snake-like and hence is a pseudoarc. Moise also proved in the paper cited above that a pseudoarc is homeomorphic to each of its non-singleton subcontinua; this property justifies the term pseudoarc, since the arc also has the property.

4.S.5. We say that a metric space is *topologically homogeneous* if for every two points $x, y \in X$ there exists a homeomorphism $h: X \to X$ such that $h(x) = y$. For example, the space S^m is topologically homogeneous for every m; from the theorem on the invariance of interior points (4.2.11) it follows easily that the ball $\bar{B}^m$ is not topologically homogeneous for $m > 0$. It may be proved (see Problem 4.P.22) that the Menger curve is topologically homogeneous, but the Sierpiński curve does not have this property. R. H. Bing has shown (*A homogeneous indecomposable plane continuum*, Duke Math. J., 15(1948), 729–742) that every pseudoarc is topologically homogeneous. Since the pseudoarc can be imbedded in the plane, there hence exist topologically homogeneous continua in the plane which are not simple closed curves.

4.S.6. The construction in Example 4.4.6 of a common boundary of three regions in the plane becomes more memorable once its more anecdotal rendering, due to the Japanese mathematician M. Wada, is noted. He describes the history of an island which

has two lakes – one filled with sweet water, the other with mineral water. Each of the islanders wanted, as near his home as possible, all three kinds of water: sweet, mineral and seawater. To accomplish this a multistage plan was drawn up to build canals filled with the various kinds of water in such a way that after the n^{th}-stage, which was to take 2^{-n} of a year to complete, the distance from every point of the island to each kind of water should not exceed 2^{-n} km. What was left of the island after a year was the common boundary of three regions.

It is worth noting that, as was proved by K. Kuratowski (see [10], p. 560), every continuum which is the common boundary of three regions in the plane either is itself indecomposable or is the union of two indecomposable continua.

4.P. Problems

4.P.1. Prove that every continuous map $f: S^m \to S^1$ for $m > 1$ is inessential.

4.P.2. Show that for every continuous inessential map $f: X \to S^1$ there exists a continuous map $g: X \to \mathbf{R}^1$ such that $f = pg$, where $p: \mathbf{R}^1 \to S^1$ denotes the covering map of Example 3.3.5. Is it true that for every inessential map $f: X \to S^2$ there exist continuous maps $g: X \to \mathbf{R}^2$ and $h: \mathbf{R}^2 \to S^2$ such that $f = hg$? (Cf. K. Borsuk, *On certain mapping of the 2-sphere into itself*, Ann. Soc. Polon. Math. 25 (1952), 268–272).

4.P.3. Using Theorem 4.1.3 prove the fundamental theorem of algebra. (Hint: Show that if $w(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ and $w_n(z) = z^n$, where z ranges through the complex numbers, and if S_r denotes the circle centred at 0 of radius r, then $w|S_r \simeq w_n|S_r$ for large enough r. Next observe that the map $w_n|S_r$ is essential for $n > 0$.)

4.P.4. Let $\mathcal{L}$ be a simplicial subcomplex of the complex K and suppose the complement $K \setminus \mathcal{L}$ does not contain simplices of dimension exceeding $m-1$. Prove that every continuous map $f: |\mathcal{L}| \to S^{m-1}$ has a continuous extension $f^*: |K| \to S^{m-1}$.

4.P.5. For every point $y \in \mathbf{R}^m$ define the map $p_y: \mathbf{R}^m \setminus \{y\} \to S^{m-1}$ by putting $p_y(x) = (x-y)/\|x-y\|$ for $x \in \mathbf{R}^m \setminus \{y\}$. Let X be a compact subset of the space $\mathbf{R}^m$ and suppose $y_0, y_1 \in \mathbf{R}^m \setminus X$. Show that the points y_0, y_1 lie in the same component of the complement $\mathbf{R}^m \setminus X$ if and only if $p_{y_0}|X \simeq p_{y_1}|X$.

4.P.6. Suppose $A, B \subset X$ are homeomorphic sets, where X denotes the Sierpiński curve. If A is open in X, must B be open in X?

4.P.7. Show that two polyhedral arcs in the space $\mathbf{R}^m$ are equivalently imbedded. (Hint: Show first that for any two closed balls Q_1, Q_2 in $\mathbf{R}^m$ which are contained in some region $U \subset \mathbf{R}^m$, there exists a homeomorphism $h: \mathbf{R}^m \to \mathbf{R}^m$ such that $h(Q_1) = Q_2$ and $h|\mathbf{R}^m \setminus U = \mathrm{id}$.)

4.P.8. Show that any two countable dense subsets of the space $\mathbf{R}^m$ are equivalently imbedded. (Hint: Show that for every countable $A \subset \mathbf{R}^m$ there is a homeomorphism $h: \mathbf{R}^m \to \mathbf{R}^m$ such that for any two distinct points $x, y \in h(A)$ where $x = (x^1, x^2, \ldots, x^m)$ and $y = (y^1, y^2, \ldots, y^m)$ we have $x^i \neq y^i$ for $i = 1, 2, \ldots, m$. Next observe that if the countable dense sets $A, B \subset \mathbf{R}^m$ have the property that for any two distinct points $x, y \in A$ or $x, y \in B$, where $x = (x^1, x^2, \ldots, x^m)$ and $y = (y^1, y^2, \ldots, y^m)$, it is the case that $x^i \neq y^i$ for $i = 1, 2, \ldots, m$, then there is an enumeration $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ such that $(a_j^i - a_k^i)(b_j^i - b_k^i) \geq 0$ for $i = 1, 2, \ldots, m$ and $j, k = 1, 2, \ldots$ where $a_j = (a_j^1, a_j^2, \ldots, a_j^m)$ and $b_j = (b_j^1, b_j^2, \ldots, b_j^m)$ for $j = 1, 2, \ldots$)

4.P.9. Show that if M denotes the arc of Example 4.3.5 and $q = (0, 0, 0)$, then the closure M' of the component of $M \backslash \{q\}$ which contains the point $v = (3, 0, 0)$ is a wildly imbedded arc in $\mathbf{R}^3$ whose complement $\mathbf{R}^3 \backslash M'$ is homeomorphic with the complement $\mathbf{R}^3 \backslash \{q\}$, and so is simply connected. Show that if M'' denotes the set symmetric to M' relative to the plane $x^1 = 0$, then the union $M' \cup M''$ is a wildly imbedded arc in $\mathbf{R}^3$ whose complement is simply connected but not homeomorphic to $\mathbf{R}^3 \backslash \{q\}$. (See [14], p. 65, 68.)

4.P.10. Construct an example of a set $X \subset \mathbf{R}^3$ which is homeomorphic to the sphere S^2 but neither the unbounded component of $\mathbf{R}^3 \backslash X$ is homeomorphic to the unbounded component of $\mathbf{R}^3 \backslash S^2$, nor is the bounded component of $\mathbf{R}^2 \backslash X$ homeomorphic to the bounded component of $\mathbf{R}^3 \backslash S^2$. (Hint: Combine Examples 4.3.6 and 4.3.7.)

4.P.11. Construct a set $D \subset \mathbf{R}^3$ homeomorphic to the disc $\bar{B}^2$ for which there does not exist a set $X \subset \mathbf{R}^3$ containing D that is homeomorphic to the sphere S^2. (Hint: Use Example 4.3.6; cut it into two sets homeomorphic to $\bar{B}^2$ and paste these components into two holes of a suitably imbedded disc.)

4.P.12. Construct a set $D \subset \mathbf{R}^3$ homeomorphic to the disc $\bar{B}^2$ for which there does not exist a set $D' \subset \mathbf{R}^3$ and a homeomorphism $h: D' \to \bar{B}^2$ such that $D \subset D'$ and $h(D) \subset B^2$. (Hint: Suitably densify the singular set constructed in Problem 4.P.11.)

4.P.13. Show that if a compact set $A \subset \mathbf{R}^m$ is homeomorphic to the sphere S^{m-1}, then it is the boundary of each component of the complement $\mathbf{R}^m \backslash A$ (see Supplement 4.S.1).

4.P.14. Show that if $A \subset X \subset \mathbf{R}^3$, where A is the Antoine necklace and the set X is homeomorphic to one of the sets, $I, \bar{B}^2, S^1, S^2$, then the set X is wildly imbedded in $\mathbf{R}^3$ (cf. [14], p. 72).

4.P.15. Determine the fundamental group of the complement $\mathbf{R}^3 \backslash A_n$ where A_n denotes the set used in Example 4.3.8 to construct the Antoine necklace. Show that the complement $\mathbf{R}^3 \backslash A$ is not simply connected.

4.P.16. Show that the Antoine necklace is homeomorphic to the Cantor set (cf. Problem 6.P.37).

4.P.17. Show that every set homeomorphic to the Cantor set and contained in the plane $\mathbf{R}^2$ is equivalently imbedded with a Cantor set lying on one of the coordinate axes. (Hint: If the set X in $\mathbf{R}^2$ is homeomorphic to the Cantor set, then $X = \bigcap_{n=1}^{\infty} X_n$, where X_n is a finite union of disjoint closed discs of radii less than $\frac{1}{n}$ with $X_{n+1} \subset \operatorname{int} X_n$ for $n = 1, 2, \ldots$)

4.P.18. Show that every composant (see Supplement 4.S.3) of a continuum X is dense in X and does not separate X (cf. [10], p. 209, 210).

4.P.19. Show that every indecomposable continuum has $\mathfrak{c}$ pairwise disjoint composants (see Supplement 4.S.3; cf. [10], p. 212, 213).

4.P.20. Show that the Knaster-Kuratowski broom M (see Supplement 4.S.3) is connected, but is not the union of two disjoint connected sets each containing more than one point. Show that the complement $M \setminus \{d\}$ does not contain any connected subset with more than one point.

4.P.21. Show that every snake-like continuum (see Supplement 4.S.4) has the fixed point property (cf. O. H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. 2(1951), 173–174).

4.P.22. Show that the Menger curve is topologically homogeneous, whereas the Sierpiński curve is not (see Supplement 4.S.5 and R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. 67(1958), 313–324).

Chapter 5
Manifolds

Manifolds are metric spaces which locally resemble the Euclidean spaces or half-spaces; from this point of view they represent the simplest objects in the class of metric spaces. It turns out however that the regularity of the local structure has little bearing on global properties, especially in higher dimensions. The class of manifolds as a result contains many interesting and important examples, and the associated body of problems is rich and rife with unusually difficult problems that are, by contrast, easy to state.

In this short chapter we touch on only the simplest questions in manifold theory. Section 5.1 is devoted to the concept of a topological manifold, some of the simplest examples, and the problem of topological homogeneity of topological manifolds. In Section 5.2 we examine triangulated manifolds and introduce the notions of orientation and orientability. Section 5.3 is technical in character: we define certain operations on complexes known as cutting and pasting; with their help it is easy to describe further examples of manifolds. In Section 5.4 we are concerned with the problem of topologically classifying 1- and 2-dimensional manifolds. In particular for each topological type of 2-dimensional manifold we construct what is called its normal form, obtained by pastings on the boundary of the disc. This allows us to associate with each 2-dimensional manifold numerical invariants and they in turn enable a topological classification of 2-dimensional manifolds.

5.1. The concept of a topological manifold

Let X be a compact metric space and m a non-negative integer. We say that X is an *m-dimensional topological manifold* or more briefly an *m-dimensional manifold*, if for every point $x \in X$ there is a set $U \subset X$, homeomorphic to the m-dimensional closed unit ball $\bar{B}^m$, such that $x \in \operatorname{int} U$. From Theorem 4.2.13 we obtain the following.

5.1.1. COROLLARY. *No n-dimensional and m-dimensional topological manifolds are ever homeomorphic for $n \neq m$.* ■

Every m-dimensional manifold thus uniquely determines a number m which we call the *dimension* of the manifold. The set of points x in an m-dimensional manifold X for which there is a set $U \subset X$ homeomorphic to the m-dimensional open ball B^m such that $x \in \operatorname{int} U$, is called the *interior* of the manifold X and is denoted $\operatorname{int} X$. The complement $X \setminus \operatorname{int} X$ is called the *boundary* of the manifold X and is denoted by $\operatorname{bd} X$. A manifold whose boundary is empty is called a *manifold without boundary* (cf. Supplement 5.S.1).

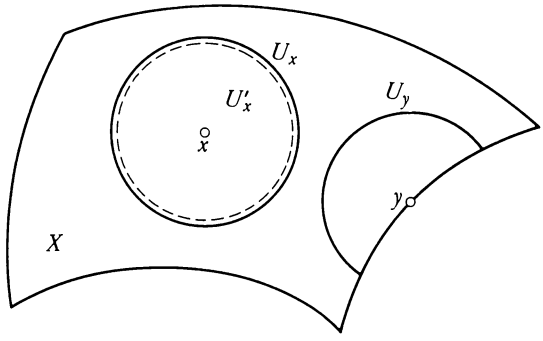


Fig.109. The points x and y lie in a 2-dimensional manifold X; thus there are sets U_x and U_y homeomorphic to B^2 with $x \in \operatorname{int} U_x$ and $y \in \operatorname{int} U_y$. Furthermore, the point x belongs to the interior $\operatorname{int} X$ since there is a set U'_x homeomorphic to B^2 with $x \in \operatorname{int} U'_x$; the point y does not have this property and so $y \in \operatorname{bd} X$.

5.1.2. EXAMPLE. *The $(m-1)$-dimensional sphere S^{m-1} is an $(m-1)$-dimensional manifold without boundary.* ■

5.1.3. EXAMPLE. *The m-dimensional projective space P^m is an m-dimensional manifold without boundary.* ■

5.1.4. THEOREM. *Let X be an m-dimensional manifold. For a point x to lie on the boundary* $\operatorname{bd} X$ *it is necessary and sufficient that there exists a set $U \subset X$ and a homeomorphism $h: U \to \bar{B}^m$ such that $x \in \operatorname{int} U$ and $h(x) \in S^{m-1}$.*

PROOF. Necessity of the condition is obvious. The proof of its sufficiency will be based on the theorem on the invariance of interior points (4.2.11).

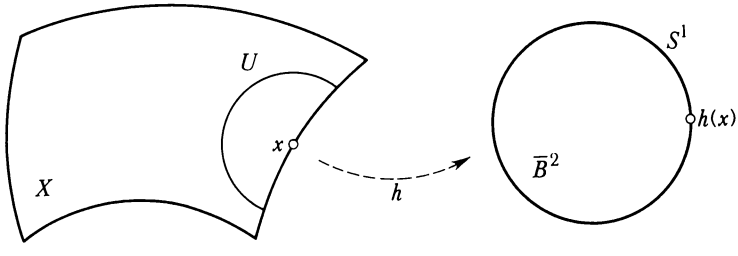


Fig.110. The point x lies on the boundary of the 2-dimensional manifold X and so according to Theorem 5.1.4 there is a set $U \subset X$ and a homeomorphism $h: U \to \bar{B}^2$ such that $x \in \operatorname{int} U$ and $h(x) \in S^1$.

Assume therefore that for some point x there is a set $U \subset X$ and a homeomorphism $h: U \to \bar{B}^m$ such that $x \in \operatorname{int} U$ and $h(x) \in S^{m-1}$. Assume further that $x \in \operatorname{int} X$ so that there exists a set $V \subset X$ and a homeomorphism $g: V \to \mathbf{R}^m$ such that $x \in \operatorname{int} V$. We may obviously assume without loss of generality that $U \subset V$. Consider the homeomorphism $f = hg^{-1}|g(U): g(U) \to \bar{B}^m$. Since $g(x) \in \operatorname{int} g(U) \subset \mathbf{R}^m$, it follows by Theorem

4.2.11 that $h(x) = fg(x) \in \operatorname{int} fg(U) \in \operatorname{int} \bar{B}^m = B^m$, contrary to the assumption that $h(x) \in S^{m-1}$. The contradiction so obtained completes the proof. ■

Using the theorem above we obtain:

5.1.5. EXAMPLE. *The m-dimensional closed unit ball $\bar{B}^m$ is an m-dimensional manifold whose interior is the open ball B^m and whose boundary is the sphere S^{m-1}.* ■

We prove now the following theorem on the metric product of manifolds.

5.1.6. THEOREM. *If X is an m-dimensional manifold and Y is an n-dimensional manifold, then the metric product $X \times Y$ is an $(m+n)$-dimensional manifold; moreover* $\operatorname{int}(X \times Y) = \operatorname{int} X \times \operatorname{int} Y$ *and* $\operatorname{bd}(X \times Y) = ((\operatorname{bd} X) \times Y) \cup (X \times \operatorname{bd} Y)$.

PROOF. For every non-negative integer k the k-dimensional closed unit ball $\bar{B}^k$ is homeomorphic to the k-dimensional unit cube I^k (see Example 1.10.11); hence the metric product $\bar{B}^m \times \bar{B}^n$ is homeomorphic to the ball $\bar{B}^{m+n}$. It thus follows that if X is an m-dimensional manifold and Y is an n-dimensional manifold, then the metric product $X \times Y$ is an $(m+n)$-dimensional manifold. Observe that similarly the metric product $B^m \times B^n$ is homeomorphic to the ball B^{m+n} and so $\operatorname{int} X \times \operatorname{int} Y \subset \operatorname{int}(X \times Y)$. To complete the proof it is enough to show that $((\operatorname{bd} X) \times Y) \cup (X \times (\operatorname{bd} Y)) \subset \operatorname{bd}(X \times Y)$.

Suppose for instance that $p = (x, y) \in (\operatorname{bd} X) \times Y$. By Theorem 5.1.4 there is a set $U_x \subset X$ and a homeomorphism $h_x \colon U_x \to \bar{B}^m$ such that $x \in \operatorname{int} U_x$ and $h_x(x) \in S^{m-1}$. Moreover there is a set $U_y \subset Y$ and a homeomorphism $h_y \colon U_y \to \bar{B}^n$ such that $y \in \operatorname{int} U_y$. Taking $U = U_x \times U_y$ and $h = h_x \times h_y$, we have $p \in \operatorname{int} U$, $h \colon U \to \bar{B}^m \times \bar{B}^n$ and $h(p) \in S^{m-1} \times \bar{B}^n$. From Theorem 1.10.9 it follows that there exists a homeomorphism $g \colon \bar{B}^m \times \bar{B}^n \to \bar{B}^{m+n}$ such that $g(S^{m-1} \times \bar{B}^n) \subset S^{m+n-1}$. Taking the composition $gh \colon U \to \bar{B}^{m+n}$, we have $gh(p) \in S^{m+n-1}$ and so again applying Theorem 5.1.4 we deduce that $p \in \operatorname{bd}(X \times Y)$. ■

We thus obtain the following examples.

5.1.7. EXAMPLE. *The torus $S^1 \times S^1$ is a 2-dimensional manifold without boundary.* ■

5.1.8. EXAMPLE. *The cylinder $S^1 \times I$ is a 2-dimensional manifold whose boundary is the union of the two circles $S^1 \times \{0\}$ and $S^1 \times \{1\}$.* ■

Now we prove the following.

5.1.9. THEOREM. *If the boundary of an m-dimensional manifold is non-empty, then the boundary is an $(m-1)$-dimensional manifold without boundary.*

PROOF. The interior, $\operatorname{int} X$, of any m-dimensional manifold X is an open set of X. It follows that the boundary $\operatorname{bd} X$ is a closed subset of the compact space X and so is a compact set.

Let $x \in \operatorname{bd} X$. By Theorem 5.1.4 there is a set $U \subset X$ and a homeomorphism $h \colon U \to \bar{B}^m$ such that $x \in \operatorname{int} U$ and $h(x) \in S^{m-1}$. Let $V_0 \subset S^{m-1}$ be a set homeomorphic to the ball B^{m-1} such that $h(x) \in V_0$ and $h^{-1}(V_0) \subset \operatorname{int} U$. Setting $U_0 = h^{-1}(V_0)$ we then have $x \in U_0$, but by Theorem 5.1.4 it follows that $U_0 \subset \operatorname{bd} X$.

Now $h(U \cap \operatorname{bd} X) \subset S^{m-1}$ and $h(U_0) = V_0$, so $h((U \cap \operatorname{bd} X) \setminus U_0) \subset S^{m-1} \setminus V_0$. It follows that x is not a boundary point of the set $(U \cap \operatorname{bd} X) \setminus U_0$, and since the point x is in $\operatorname{int} U$, it is also not a boundary point of the set $(\operatorname{bd} X) \setminus U_0$. The point x therefore belongs to the interior of the set U_0 relative to $\operatorname{bd} X$, which completes the proof. ■

A manifold need not in general be connected. The following however is the case.

5.1.10. THEOREM. *Every manifold has a finite number of components.*

PROOF. For each point x of an m-dimensional manifold X there is a connected set $U \subset X$ with $x \in U$, so every component of the manifold is an open set of X. It follows from the compactness of the space X that the components are finite in number. ■

The boundary of a connected manifold need not of course be connected (cf. Example 5.1.8); the following is however a consequence of Theorems 5.1.9 and 5.1.10.

5.1.11. COROLLARY. *The boundary of any manifold has a finite number of components.* ■

We now study the connectedness of the interior of a connected manifold. First we prove the following.

5.1.12. THEOREM. *If X is a connected m-dimensional manifold with $m > 0$, then for every pair of points $x, y \in X$ there is a continuous map $f: I \to X$ such that $f(0) = x$, $f(1) = y$ and $f(\operatorname{int} I) \subset \operatorname{int} X$.*

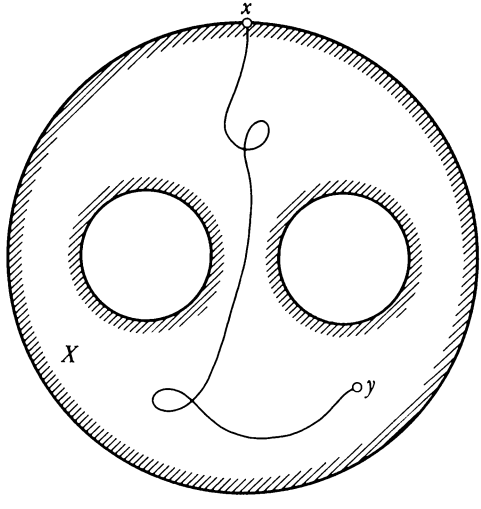


Fig.111. Since X is a connected manifold, for any two points $x, y \in X$ there exists a continuous map $f: I \to X$ such that $f(0) = x$, $f(1) = y$ and $f(\operatorname{int} I) \subset \operatorname{int} X$ (Theorem 5.1.12).

PROOF. Let $y \in X$ and let $A = \{x \in X$: there is a continuous map $f: I \to X$ such that $f(0) = x$, $f(1) = y$, $f(\operatorname{int} I) \subset \operatorname{int} X\}$. Observe that for every point $p \in \bar{B}^m$, for $m > 0$, there is a continuous map $\varphi: I \to \bar{B}^m$ such that $\varphi(0) = p = \varphi(1)$ with $\varphi(\operatorname{int} I) \subset B^m$. It follows that $y \in A$. To prove that $A = X$ it is enough to show that the set A is open-and-closed in X.

If $x \in A \subset X$, then there is a set $U \subset X$ such that $x \in \operatorname{int} U$ and there exists a homeomorphism $h: U \to \bar{B}^m$; moreover there is a continuous map $f: I \to X$ such that $f(0) = x$, $f(1) = y$ and $f(\operatorname{int} I) \subset \operatorname{int} X$. We show that $U \subset A$. There exists a real number $r_0 \in I$ such that $f(r_0) \in U \cap \operatorname{int} X$. If $x' \in U$, then the map $f': I \to X$ defined by the formula:

$$f'(r) = \begin{cases} h^{-1}(2rhf(r_0) + (1-2r)h(x')), & \text{for } 0 \leq r \leq \frac{1}{2}, \\ f((2-2r_0)r + 2r_0 - 1), & \text{for } \frac{1}{2} \leq r \leq 1, \end{cases}$$

is continuous and satisfies the conditions: $f'(0) = x'$, $f'(1) = y$, and $f'(\operatorname{int} I) \subset \operatorname{int} X$. Hence $x' \in A$.

We have thus shown that the set A is open.

In order to show that the set $X \setminus A$ is open suppose that $x \in X \setminus A$. There exists a set $U \subset X$ such that $x \in \operatorname{int} U$, and there is a homeomorphism $h: U \to \bar{B}^m$. It is evidently sufficient to show that $\operatorname{int} U \subset X \setminus A$. If however there was a point $x' \in (\operatorname{int} U) \cap A$, then by an argument analogous to the previous case we could show that $x \in A$, contrary to hypothesis. ■

The theorem above has a number of immediate corollaries.

5.1.13. COROLLARY. *The interior of a connected manifold is itself connected.* ■

5.1.14. COROLLARY. *For a manifold to be pathwise connected it is necessary and sufficient that it be connected.* ■

We now prove a theorem on the topological homogeneity of a manifold. We begin with the following lemma.

5.1.15. LEMMA. *For each point $y \in B^m$ there is a homeomorphism $h: \bar{B}^m \to \bar{B}^m$ such that $h|S^{m-1} = \mathrm{id}$ and $h(y) = 0$.*

PROOF. We may of course suppose that $y = (y^1, 0, \ldots, 0)$ with $-1 < y^1 < 1$ (see Example 1.3.16). Consider a homeomorphism $g: [-1,1] \to [-1,1]$ such that $g(-1) = -1$, $g(1) = 1$, $g(y^1) = 0$ and define a function φ on $\bar{B}^m$ by the formula:

$$\varphi(x^1, x^2, \ldots, x^m) = \begin{cases} \sqrt{1 - \sum_{i=2}^m (x^i)^2}\; g\left(x^1 / \sqrt{1 - \sum_{i=2}^m (x^i)^2}\right), & \text{if } \sum_{i=2}^m (x^i)^2 \neq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the map $h: \bar{B}^m \to \bar{B}^m$ defined by the formula:

$$h(x^1, x^2, \ldots, x^m) = (\varphi(x^1, x^2, \ldots, x^m), x^2, \ldots, x^m) \text{ for } (x^1, x^2, \ldots, x^m) \in \bar{B}^m$$

has the required properties. ■

Next we prove the following.

5.1.16. LEMMA. *If X is a manifold, then for every point $x \in \operatorname{int} X$ there exists a set $V \subset \operatorname{int} X$ such that $x \in \operatorname{int} V$ and for every point $x' \in \operatorname{int} V$ there exists a homeomorphism $g: X \to X$ such that $g(x) = x'$.*

PROOF. Let $x \in \operatorname{int} X$ where X is an m-dimensional manifold. There therefore exists a set $U \subset X$ such that $x \in \operatorname{int} U$ and a homeomorphism $k: U \to B^m$. Define a

function $f: U \to \mathbf{R}^m$ by taking $f(u) = a(k(u) - k(x))$ for $u \in U$, where the number $a > 0$ is chosen in such a way that $\bar{B}^m \subset f(U)$. Setting $V = f^{-1}(\bar{B}^m)$ we have $V \subset \operatorname{int} X$ and $x \in \operatorname{int} V$. If $x' \in \operatorname{int} V$, then $y = f(x') \in B^m$. By Lemma 5.1.15 there is a homeomorphism $h: \bar{B}^m \to \bar{B}^m$ such that $h|S^{m-1} = \mathrm{id}$ and $h(y) = 0$. The map $g: X \to X$ defined by the formula:

$$g(\xi) = \begin{cases} f^{-1}h^{-1}f(\xi), & \text{if } \xi \in V, \\ \xi, & \text{if } \xi \in X \backslash V, \end{cases}$$

has the required properties. ■

We may now prove the promised homogeneity theorem.

5.1.17. THEOREM. *If X is a connected manifold, then for any pair of points $x, y \in \operatorname{int} X$, there exists a homeomorphism $h: X \to X$ such that $h(x) = y$.*

PROOF. Let $y \in \operatorname{int} X$ be a fixed point and let $A = \{x \in \operatorname{int} X :$ there is a homeomorphism $h: X \to X$ such that $h(x) = y\}$. Now obviously $y \in A$, so in order to show that $A = \operatorname{int} X$ it is enough by Corollary 5.1.13 to prove that the set A is open and closed in $\operatorname{int} X$.

Let $x \in A$ and suppose the homeomorphism $h: X \to X$ satisfies the condition $h(x) = y$. By Lemma 5.1.16 there is a set $V \subset \operatorname{int} X$ such that $x \in \operatorname{int} V$ and for each point $x' \in \operatorname{int} V$ there is a homeomorphism $g: X \to X$ such that $g(x) = x'$. We show that $\operatorname{int} V \subset A$. Certainly, if $x' \in \operatorname{int} V$, the homeomorphism $hg^{-1}: X \to X$ satisfies the condition $hg^{-1}(x') = y$ and so $x' \in A$. Thus A is open in $\operatorname{int} X$.

To prove that $(\operatorname{int} X) \backslash A$ is open in $\operatorname{int} X$, suppose that $x \in (\operatorname{int} X) \backslash A$. By Lemma 5.1.16 there is a set $V \subset \operatorname{int} X$ such that $x \in \operatorname{int} V$ and a homeomorphism $g: X \to X$ such that $g(x) = x'$. We show that $\operatorname{int} V \subset (\operatorname{int} X) \backslash A$. For, if there was a point $x' \in (\operatorname{int} V) \cap A$ and the homeomorphism $h: X \to X$ satisfied the condition $h(x') = y$, then the homeomorphism $hg: X \to X$ would satisfy $hg(x) = y$, contrary to the assumption that $x \notin A$. ■

Since every point belonging to the interior of an m-dimensional topological manifold has a neighbourhood homeomorphic to the space $\mathbf{R}^m$, the theorem on the invariance of interior points (4.2.11) carries across immediately to cover the case of the interior of any topological manifold. In particular the following analogue of the theorem on invariance of open sets (4.2.12) emerges.

5.1.18. THEOREM. *Suppose that X is a topological manifold and the sets $U, V \subset \operatorname{int} X$ are homeomorphic. If the set U is open in X, then the set V is also open in X.* ■

5.1.19. COROLLARY. *A manifold without boundary is never homeomorphic to a proper subset of itself.*

PROOF. Let k be the number of components of the manifold (cf. Theorem 5.1.10) and suppose the homeomorphism h takes X onto a proper subset of X. Since the set $h(X)$ is open and closed in X, the set $h(X)$ has less than k components which is impossible. ■

In the subsequent discussion of this chapter we shall be concerned with manifolds which are at the same time polyhedra. The problem of the existence of a triangulation

of an arbitrary manifold will be considered more fully in Supplement 5.S.2. Here we limit ourselves to a proof of the following.

5.1.20. THEOREM. *If a simplicial complex K is a triangulation of a connected m-dimensional manifold then:*

(1) $\dim K = m$;

(2) *every simplex of the complex K is the face of an m-dimensional simplex in K;*

(3) *every $(m-1)$-dimensional simplex of K is the common face of at most two m-dimensional simplices;*

(4) *for any two m-dimensional simplices $\Delta', \Delta'' \in K$ there is a sequence of m-dimensional simplices $\Delta_1, \Delta_2, \ldots, \Delta_k \in K$ such that $\Delta' = \Delta_1$, $\Delta'' = \Delta_k$ and Δ_j and Δ_{j+1} have a common $(m-1)$-dimensional face for $j = 1, 2, \ldots, k-1$; and*

(5) *those $(m-1)$-dimensional simplices of K which are the faces of precisely one m-dimensional simplex of K, together with all of their faces form a triangulation of the boundary* bd X *of the manifold X.*

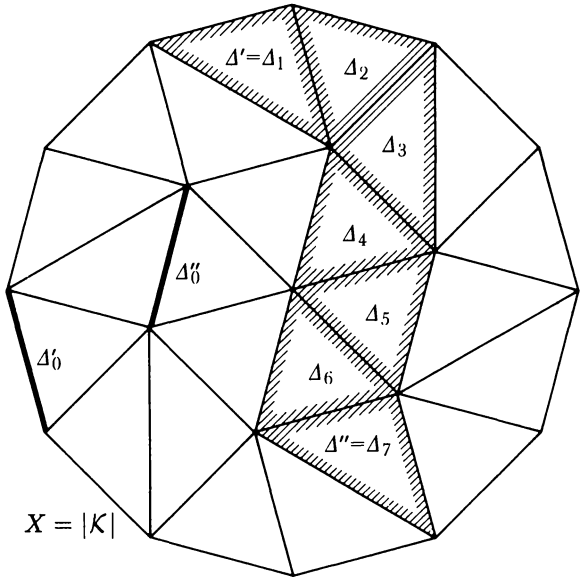


Fig.112. The complex K is a triangulation of the 2-dimensional manifold X. The 1-dimensional simplex Δ_0'' is the face of two 2-dimensional simplices; the 1-dimensional simplex Δ_0' is the face of one 2-dimensional simplex and so lies on the boundary bd X (cf. conditions (3) and (5) of Theorem 5.1.20). In the sequence of 2-dimensional simplices $\Delta' = \Delta_1, \Delta_2, \ldots, \Delta_7 = \Delta''$ every two consecutive simplices have a common edge (cf. condition (4) of Theorem 5.1.20).

PROOF. Let $\Delta \in K$ be a k-dimensional simplex which is not a proper face of any simplex of K. Let b denote the barycentre of the simplex Δ; then b lies in the interior of the simplex Δ relative to X. Moreover there exists a set $U \subset X$ homeomorphic to the ball $\bar{B}^m$ such that b lies in the interior of U relative to X. From Theorem 4.2.13 it follows immediately that $k = m$ and hence we obtain properties (1) and (2).

To prove property (3) suppose that the m-dimensional simplices Δ_1 and Δ_2 have a common $(m-1)$-dimensional face Δ. It is easy to see that the union $\operatorname{int}\Delta_1 \cup \operatorname{int}\Delta \cup \operatorname{int}\Delta_2$ is homeomorphic to the interior of an m-dimensional simplex, and so by Theorem 5.1.18 is an open subset of X. Hence it follows that Δ is not the face of any m-dimensional simplex different from Δ_1 or Δ_2.

We proceed now to a proof of property (4). Let $\Delta' \in K$ be a fixed m-dimensional simplex and denote by K_1 the subcomplex of the complex K comprising all those m-dimensional simplices Δ'' for which there is a sequence of m-dimensional simplices $\Delta_1, \Delta_2, \ldots, \Delta_k \in K$ such that $\Delta' = \Delta_1$, $\Delta'' = \Delta_k$ and Δ_j and Δ_{j+1} have a common $(m-1)$-dimensional face for $j = 1, 2, \ldots, k-1$, together with all the faces of such Δ''. Suppose there exists an m-dimensional simplex not belonging to K_1 and let K_2 denote the subcomplex of the complex K comprising all the m-dimensional simplices which do not belong to K_1 together with all their faces.

By property (2) we have $K = K_1 \cup K_2$ and, since the complex K is connected, the intersection $K_1 \cap K_2$ contains a non-empty simplex. Let Δ_0 be a simplex of maximal dimension n lying in the intersection $K_1 \cap K_2$; obviously $0 \leq n < m-1$. Let U_0 denote the union of the interiors of the simplices $\Delta \in K$ which have Δ_0 as a face; the difference $U_0 \setminus \Delta_0$ is a disconnected set.

Let b be the barycentre of the simplex Δ_0. There is a set $U \subset X$ such that $b \in \operatorname{int} U$ and a homeomorphism $h\colon U \to \bar{B}^m$. Replacing if necessary the ball $\bar{B}^m$ by another ball of smaller radius we may suppose without loss of generality that $U \subset U_0$ and that $h^{-1}(S^{m-1})$ meets $|K_1| \setminus \Delta_0$ and $|K_2| \setminus \Delta_0$. Hence it follows that the difference $h^{-1}(S^{m-1}) \setminus \Delta_0$ is a disconnected set and so the sphere S^{m-1} is separated by the compact set $h(\Delta_0) \cap S^{m-1}$, which is homeomorphic to some subspace of the space $\mathbf{R}^n$ with $n < m-1$. It is easy to see that this contradicts the separation invariance theorem for spheres (4.2.2).

To complete the proof it remains to check property (5). Note first that the boundary $\operatorname{bd} X$ is disjoint from the interior of every m-dimensional simplex of K and so is contained in the union of the $(m-1)$-dimensional simplices of the complex K. Let $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_p$ denote the $(m-1)$-dimensional simplices of the complex K which are the faces of precisely one m-dimensional simplex of K; let $\bar{\bar{\Delta}}_1, \bar{\bar{\Delta}}_2, \ldots, \bar{\bar{\Delta}}_q$ denote the $(m-1)$-dimensional simplices of the complex K which are the faces of precisely two m-dimensional simplices of K.

If $x \in \operatorname{int}\bar{\Delta}_j$ where $1 \leq j \leq p$, then by Theorem 5.1.4 we have $x \in \operatorname{bd} X$; thus $\bigcup_{j=1}^{p} \operatorname{int}\bar{\Delta}_j \subset \operatorname{bd} X$. Since the boundary $\operatorname{bd} X$ is a compact space, we obtain $\bigcup_{j=1}^{p} \bar{\Delta}_j \subset \operatorname{bd} X$.

If $x \in \operatorname{bd} X \setminus \bigcup_{j=1}^{p} \bar{\Delta}_j$, then there exists an index k with $1 \leq k \leq q$ such that $x \in \bar{\bar{\Delta}}_k$. Observe that if $x \in \operatorname{int}\bar{\bar{\Delta}}_k$, then, denoting by Δ' and Δ'' the two m-dimensional simplices whose common face is $\bar{\bar{\Delta}}_k$, we obtain the open set $\operatorname{int}\Delta' \cup \operatorname{int}\bar{\bar{\Delta}}_k \cup \operatorname{int}\Delta''$ which is homeomorphic to the ball B^m and contains the point x; in this case we have $x \in \operatorname{int} X$ contrary to hypothesis. Hence $(\operatorname{bd} X) \setminus \bigcup_{j=1}^{p} \bar{\Delta}_j \subset \bigcup_{k=1}^{q} \bar{\bar{\Delta}}_k$ and, since by Theorem 5.1.9 the boundary $\operatorname{bd} X$ is either empty or an $(m-1)$-dimensional manifold, whereas the polytope $\bigcup_{k=1}^{q} \operatorname{bd}\bar{\bar{\Delta}}_k$ is of dimension less than $m-1$, we have $\operatorname{bd} X = \bigcup_{j=1}^{p} \bar{\Delta}_j$. ■

Exercises

a) Prove that the interior, $\operatorname{int} X$, of any m-dimensional manifold X is an open set of X.

b) Carry through a detailed proof of Theorem 5.1.18.

c) For $n = 1, 2, \ldots$, give an example of a manifold whose boundary has exactly n components.

d) Show that if a 1-dimensional simplicial complex K satisfies conditions (1)–(5) of Theorem 5.1.20, then the polyhedron $|K|$ is a connected 1-dimensional manifold.

5.2. Orientability of a manifold

A simplex equipped with an ordering of its vertices is called an *ordered* simplex. We say that two ordered simplices corresponding to the same (unordered) simplex have *coherent* or *opposing orientations* depending on whether the ordering of the vertices agrees up to an even or an odd permutation. It is easy to see that the relation of coherent orientation is an equivalence on the set of all orderings of the vertices of a simplex. Each equivalence class of the relation is called an *orientation* of the simplex. It is obvious that every simplex of positive dimension has precisely two orientations, which are said to be *opposing*.

A simplex with a given orientation is called an *oriented* simplex. The simplex $\Delta(a_0, a_1, \ldots, a_n)$ with orientation determined by the ordering $a_0, a_1, \ldots, a_n$ of its vertices will be denoted by $\Delta[a_0, a_1, \ldots, a_n]$. The simplex with opposing orientation will be denoted by $-\Delta$. If $\dim \Delta = 0$ we agree that $\Delta = -\Delta$. The (-1)-dimensional simplex will not be oriented.

We now study the n-dimensional oriented simplex $\Delta[a_0, a_1, \ldots, a_n]$. We prove the following.

5.2.1. THEOREM. *For $j = 0, 1, \ldots, n$ the orientation of the simplex $(-1)^j \Delta[a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$ depends only on the orientation of the simplex $\Delta[a_0, a_1, \ldots, a_n]$ and not on the ordering of its vertices.*

PROOF. Since every permutation of the vertices of the simplex $\Delta(a_0, a_1, \ldots, a_n)$ is a composition of a number of transpositions of consecutive vertices, it is therefore enough to show that the transposition of two consecutive vertices, which obviously changes the orientation of the simplex $\Delta[a_0, a_1, \ldots, a_n]$ to its opposite, also changes the orientation of the simplex $(-1)^j \Delta[a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$ to its opposite.

This is obviously true when the two vertices to be transposed either have both indices less than j or both greater than j. It remains to check the two cases when transposing a_{j-1}, a_j and a_j, a_{j+1}. In both cases the simplex $\Delta[a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$ remains unchanged; however the sign of $(-1)^j$ is reversed and this completes the proof. ■

The simplices $(-1)^j \Delta[a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$ for $j = 0, 1, \ldots, n$ are called the *oriented facets* of the n-dimensional oriented simplex $\Delta[a_0, a_1, \ldots, a_n]$. Theorem 5.2.1 assures the validity of the definition. The following is obvious.

5.2.2. ASSERTION. *A change of orientation of a simplex yields a change of orientation of its oriented facets.* ■

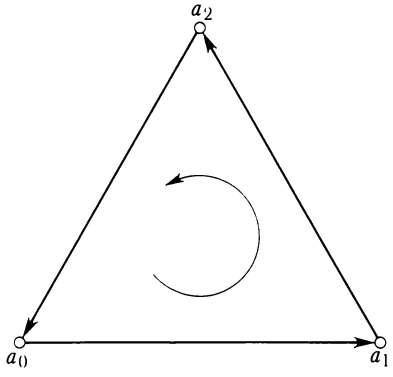


Fig.113. The oriented simplices $\Delta[a_1, a_2]$, $\Delta[a_2, a_0]$ and $\Delta[a_0, a_1]$ are oriented facets of the oriented simplex $\Delta[a_0, a_1, a_2]$ since $\Delta[a_1, a_2] = (-1)^0\Delta[a_1, a_2]$, $\Delta[a_2, a_0] = (-1)^1\Delta[a_0, a_2]$ and $\Delta[a_0, a_1] = (-1)^2\Delta[a_0, a_1]$.

Let K be a simplicial complex in which every simplex is oriented in some arbitrary, but fixed, way. If $\Delta, \Delta_0 \in K$, $\dim \Delta = n$ and $\dim \Delta_0 = n - 1$, then the number $[\Delta : \Delta_0]$ defined by

$$[\Delta : \Delta_0] = \begin{cases} 0, & \text{if } \Delta_0 \text{ is not a facet of } \Delta, \\ 1, & \text{if } \Delta_0 \text{ is an oriented facet of } \Delta, \\ -1, & \text{if } -\Delta_0 \text{ is an oriented facet of } \Delta, \end{cases}$$

is called the *incidence coefficient* of the oriented simplices Δ and Δ_0. The following is obvious.

5.2.3 ASSERTION. *If* $\Delta, \Delta_0 \in K$, $\dim \Delta = n$ *and* $\dim \Delta_0 = n - 1$, *then* $[-\Delta : \Delta_0] = [\Delta : -\Delta_0] = -[\Delta : \Delta_0]$. ■

In addition assume now that the n-dimensional simplices $\Delta', \Delta'' \in K$ have a common facet Δ_0; such simplices are called *contiguous*. If $[\Delta' : \Delta_0] = -[\Delta'' : \Delta_0]$ then the contiguous oriented simplices are said to have *coherent* orientations. If, however, $[\Delta' : \Delta_0] = [\Delta'' : \Delta_0]$ then the contiguous oriented simplices are said to have *opposing* orientations. By Assertion 5.2.3 it follows immediately that these definitions do not depend on the choice of orientation of the facet Δ_0 but only on the orientations of the simplices Δ' and Δ'' themselves.

Let X be an m-dimensional manifold with triangulation K. Any function which assigns to each m-dimensional simplex of the complex K one of its two possible orientations, in such a way that every two contiguous m-dimensional simplices have coherent orientations, is called an *orientation* of the manifold X with triangulation K. If a triangulated manifold has at least one orientation, then we call it a *orientable* manifold. By properties (3) and (4) of Theorem 5.1.20 we obtain the following.

5.2.4. COROLLARY. *Every connected orientable m-dimensional triangulated manifold for* $m > 0$ *has exactly two orientations.* ■

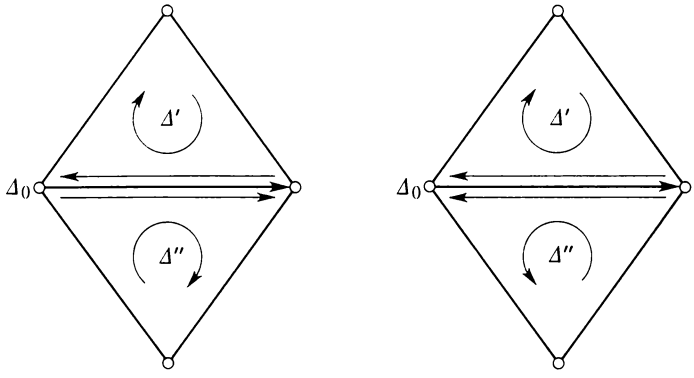


Fig.114. Left: the contiguous simplices Δ' and Δ'' have coherent orientations since $[\Delta' : \Delta_0] = -1$ and $[\Delta'' : \Delta_0] = 1$, so that $[\Delta' : \Delta_0] = -[\Delta'' : \Delta_0]$. Right: the simplices Δ' and Δ'' have opposing orientations since $[\Delta' : \Delta_0] = -1$ and $[\Delta'' : \Delta_0] = -1$ so that $[\Delta' : \Delta_0] = [\Delta'' : \Delta_0]$. In either case the choice of orientation for the face Δ_0 is of no significance.

Using Theorem 5.1.9 and property (5) of Theorem 5.1.20 we obtain the following.

5.2.5. COROLLARY. *The boundary of any orientable, triangulated manifold is an orientable manifold.*

PROOF. Let the simplicial complex K be a triangulation of the orientable m-dimensional manifold X and let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be the $(m-1)$-dimensional simplices of K with the property that for $j = 1, 2, \ldots, k$ each simplex Γ_j is the face of exactly one m-dimensional simplex Δ_j of K. Fix an orientation ω of the complex K and for $j = 1, 2, \ldots, k$ let the function ω' assign to the simplex Γ_j an orientation $\omega'(\Gamma_j)$ such that the oriented simplex $(\Gamma_j, \omega'(\Gamma_j))$ is an oriented face of the oriented simplex $(\Delta_j, \omega(\Delta_j))$. It is easy to check that ω' is an orientation of the boundary bd X. ■

Hence we obtain the following.

5.2.6. EXAMPLE. The boundary of the m-dimensional simplex Δ^m for $m > 0$ with the triangulation described in Example 2.3.2 is an $(m-1)$-dimensional orientable manifold. ■

A manifold with a triangulation that does not have any orientation is called a *non-orientable* manifold. In the next section we develop some techniques for building manifolds which enable us to give easily examples of non-orientable manifolds.

Exercises

a) Give a detailed proof of Corollary 5.2.4.

b) Prove that every triangulated manifold lying in the Euclidean plane $\mathbf{R}^2$ is orientable.

c) Give an example of a triangulation of a polyhedron homeomorphic to the torus $S^1 \times S^1$, and show that it is an oriented manifold.

5.3. Pastings and cuttings

Let K be a simplicial complex. We say that an equivalence relation R on the complex K is a *simplicial relation* if $\Delta' R \Delta''$ holds if and only if $\dim \Delta' = \dim \Delta''$ and there is an ordering $a_0, a_1, \ldots, a_n$ of the vertices of the simplex Δ' and an ordering $b_0, b_1, \ldots, b_n$ of the vertices of the simplex Δ'' such that $a_j R b_j$ holds for $j = 0, 1, \ldots, n$. A simplicial relation is therefore determined by its definition on the vertices. In order to simplify the description of R, we shall omit conditions of the type aRa, also of the two conditions aRb, bRa we shall write only one, and we shall omit conditions which may be inferred from the transitivity of the relation.

Recall that by Corollary 2.3.14 the complex K is a nerve of the covering of the polyhedron $|K|$ consisting of the stars of the vertices of K. Let R be a simplicial relation on the complex K; the equivalence class of the simplex Δ under the relation R will be denoted by $[\Delta]$. Form the covering $\mathcal{U}$ of the polyhedron $|K|$ by sets of the form $U_{[b]} = \bigcup\{\operatorname{st} a : a \in [b]\}$, where b is a vertex of the complex K. Any nerve of this covering will be denoted by K/R and will be called the complex K *pasted* according to the relation R. Since all nerves of the same covering are simplicially isomorphic, the terminology does not lead to any ambiguity. The relation R is often defined by means of some construction, and we shall avoid unnecessary formalism by saying that the complex K/R arises from the complex K by *pasting* according to the construction.

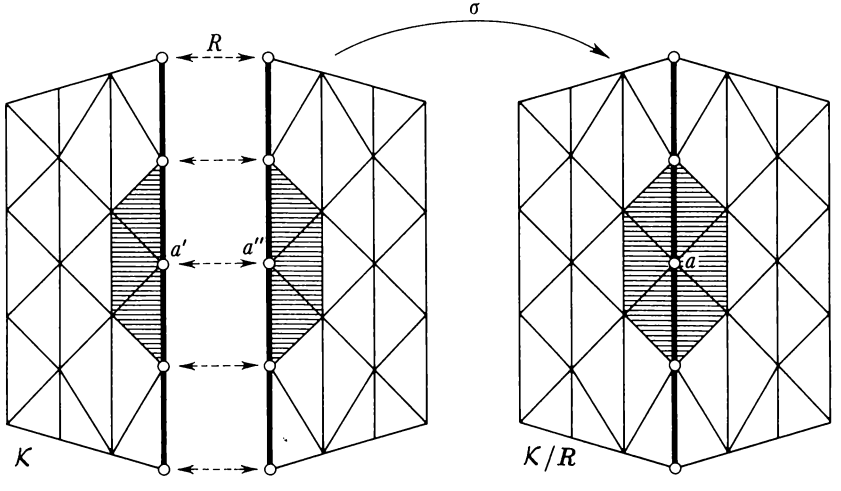


Fig.115. A pasting σ of the complex K according to the relation R. The two stars $\operatorname{st} a'$, $\operatorname{st} a''$ of the vertices a' and a'' give rise to a single star $\operatorname{st} a$.

Let a be any vertex of the complex K endowed with a simplicial relation R. Denote by $\sigma^0(a)$ the vertex of the complex K/R, that is of the nerve of the covering $\mathcal{U} = \{U_{[b]} : b \in K\}$, which corresponds to the element $U_{[a]}$ of this covering. The following is the case.

5.3.1. THEOREM. *The map σ^0 is a simplicial map of the vertices of the complex K into K/R.*

PROOF. If $\Delta(a_0, a_1, \ldots, a_n) \in K$, then by Lemma 2.3.13 we have $\operatorname{st} a_0 \cap \operatorname{st} a_1 \cap \ldots \cap \operatorname{st} a_n \neq \emptyset$, hence of course $U_{[a_0]} \cap U_{[a_1]} \cap \ldots \cap U_{[a_n]} \neq \emptyset$ and so $\Delta\{\sigma^0(a_0), \sigma^0(a_1), \ldots, \sigma^0(a_n)\} \in K/R$. ■

Thus the map σ^0 defines a natural simplicial map $\sigma: K \to K/R$; we call it the *pasting* of the complex K according to the relation R. Since, of course, every vertex of the complex K/R is of form $\sigma^0(a)$, where a is a vertex of the complex K, we have $\sigma(K) = K/R$.

We now use the method of pasting complexes to construct various examples. Throughout Examples 5.3.2–5.3.6 we shall be using one and the same simplicial complex K which we now describe. In the Euclidean plane $\mathbf{R}^2$ consider the points $a_0 = (0,0)$, $a_1 = (1,1)$, $a_2 = (1,-1)$ and form a triangulation of the square with vertices $a_1, a_2, -a_1, -a_2$ consisting of the simplices $\Delta(a_0, a_1, a_2)$, $\Delta(a_0, a_1, -a_2)$, $\Delta(a_0, -a_1, -a_2)$, $\Delta_0(a_0, -a_1, a_2)$ and all of their faces. Let K be the barycentric subdivision of order 2 of this triangulation. (For clarity, in Figure 116 only the barycentric subdivision of order 1 is sketched.) We distinguish from among the vertices of K the following: $b = (1,0)$, $b_1 = (1, \frac{1}{2})$, $b_2 = (1, -\frac{1}{2})$, $c = (0,1)$, $c_1 = (\frac{1}{2}, 1)$, $c_2 = (-\frac{1}{2}, 1)$. Let K_0 be the simplicial subcomplex of the complex K consisting of the simplices which lie on the segments $\overline{a_1(-a_2)}$ and $\overline{(-a_1)a_2}$. Let K_1 be the simplicial subcomplex of the complex K consisting of the simplices which lie on the segment $\overline{(-b)b}$.

5.3.2. EXAMPLE. *The tube.* Define a relation R on the complex K by taking: $a_1 R(-a_2)$, $b_1 R(-b_2)$, $bR(-b)$, $b_2 R(-b_1)$, $a_2 R(-a_1)$. The polyhedron $|K/R|$ is called the *tube*; it is easily observed that it is homeomorphic with the cylinder $S^1 \times I$ (Example 5.1.8) and its special name is on account of certain constructions considered in Section 5.4. The relation R restricted to the subcomplex K_0 induces a relation R_0 in K_0; it is easily observed that the polyhedron $|K_0/R_0|$, which is the boundary of $|K/R|$, has two components each of which is homeomorphic to a circle. ■

5.3.3. EXAMPLE. *The Möbius band.* Define a relation R on the complex K by taking: $a_1 R(-a_1)$, $b_1 R(-b_1)$, $bR(-b)$, $b_2 R(-b_2)$, $a_2 R(-a_2)$. It is easily observed that the polyhedron $|K/R|$ is homeomorphic with the Möbius band as defined in Example 3.3.2; we shall call it too the *Möbius band.* The relation R restricted to the subcomplex K_i for $i = 0, 1$ defines a relation R_i on K_i. It is easily observed that each of the polyhedra $|K_0/R_0|$ and $|K_1/R_1|$ is homeomorphic to a circle; the former is called the *edge* and the latter the *equator* of the Möbius band $|K/R|$ (see also Example 3.3.2).

Note that the Möbius band $|K/R|$ is a 2-dimensional polyhedron with boundary $|K_0/R_0|$. Certainly the polyhedron $|K|$ is a 2-dimensional manifold whose boundary is the union of sixteen 1-dimensional simplexes. After the process of pasting $\sigma: K \to K/R$, each of the eight 1-dimensional simplices of the subcomplex K_0 remains a face of exactly one 2-dimensional simplex in K/R, whereas the remaining 1-dimensional simplices join up in pairs pasted by σ. Each of these images is a 1-dimensional simplex in K/R which is a common face of exactly two 2-dimensional simplices.

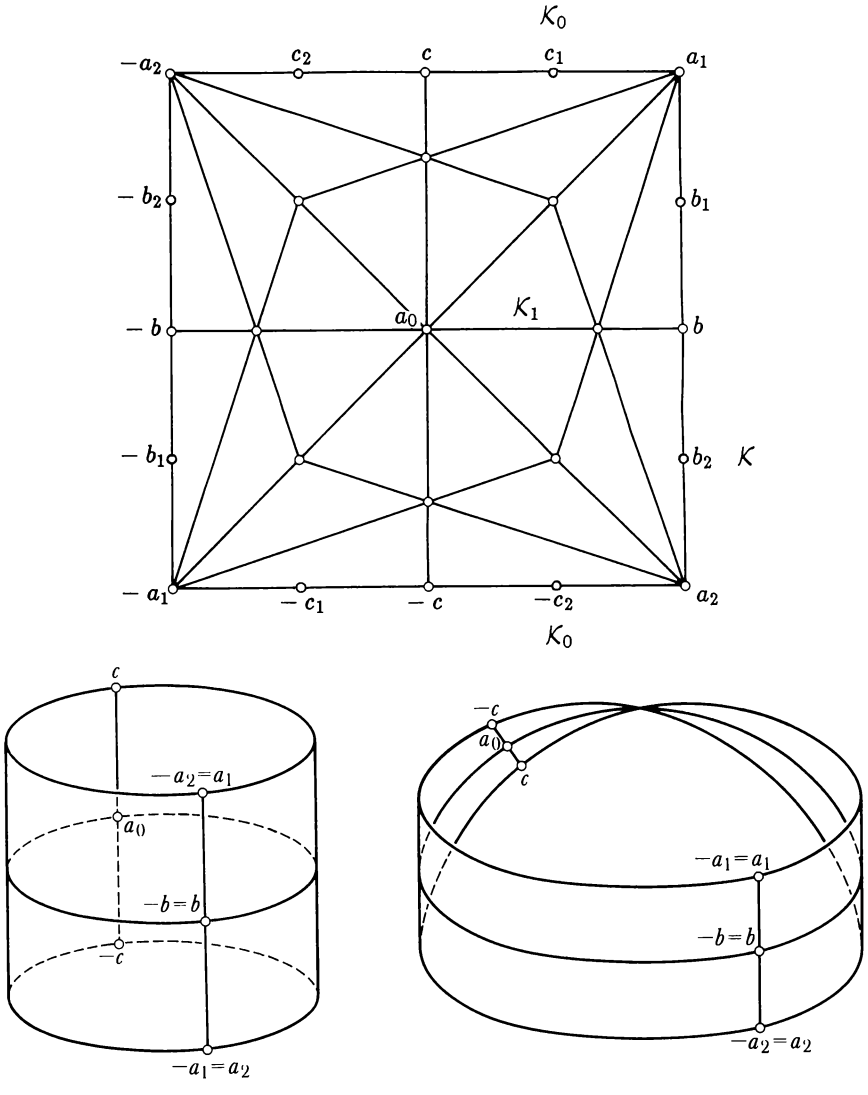


Fig.116. By pasting along the boundary of the square we obtain the tube (Example 5.3.2) and the Möbius band (Example 5.3.3).

The Möbius band with the triangulation described above is not orientable. To prove this, suppose that an orientation of the band exists; in an obvious way it determines an orientation of the manifold $|K|$ with the triangulation K. Consider the simplex $\Delta(a_1, b_1, d) \in K$, where the vertex d is determined by a_1, b_1 in an obvious way. Suppose that under the orientation the oriented simplex $\epsilon\Delta[a_1, b_1, d]$, where $\epsilon = \pm 1$, corresponds to the simplex $\Delta(a_1, b_1, d)$. Consider any sequence of 2-dimensional simplices of the complex K whose first term is the simplex $\Delta(a_1, b_1, d)$ and last term is $\Delta(-a_1, -b_1, -d)$ with consecutive simplices contiguous. It is then easily seen that the oriented simplex $\epsilon\Delta[-a_1, -b_1, -d]$ corresponds to the simplex $\Delta(-a_1, -b_1, -d)$. How-

ever, since $\sigma(a_1) = \sigma(-a_1)$ and $\sigma(b_1) = \sigma(-b_1)$, in the complex K/R the simplices $\Delta(\sigma(a_1), \sigma(b_1), d)$ and $\Delta(\sigma(-a_1), \sigma(-b_1), -d)$ are contiguous and have a common face with vertices $\sigma(a_1) = \sigma(-a_1)$ and $\sigma(b_1) = \sigma(-b_1)$. It follows immediately that the oriented simplices $\Delta[\sigma(a_1), \sigma(b_1), d]$ and $\Delta[\sigma(-a_1), \sigma(-b_1), -d]$ have opposite orientations contrary to hypothesis. ■

5.3.4. EXAMPLE. *The torus.* We add to the conditions defining R in Example 5.3.2 the following: $a_1 R a_2$, $c_1 R(-c_2)$, $cR(-c)$, $c_2 R(-c_1)$. It is easily seen that the polyhedron $|K/R|$ is homeomorphic to the torus $S^1 \times S^1$ (cf. Example 5.1.7). ■

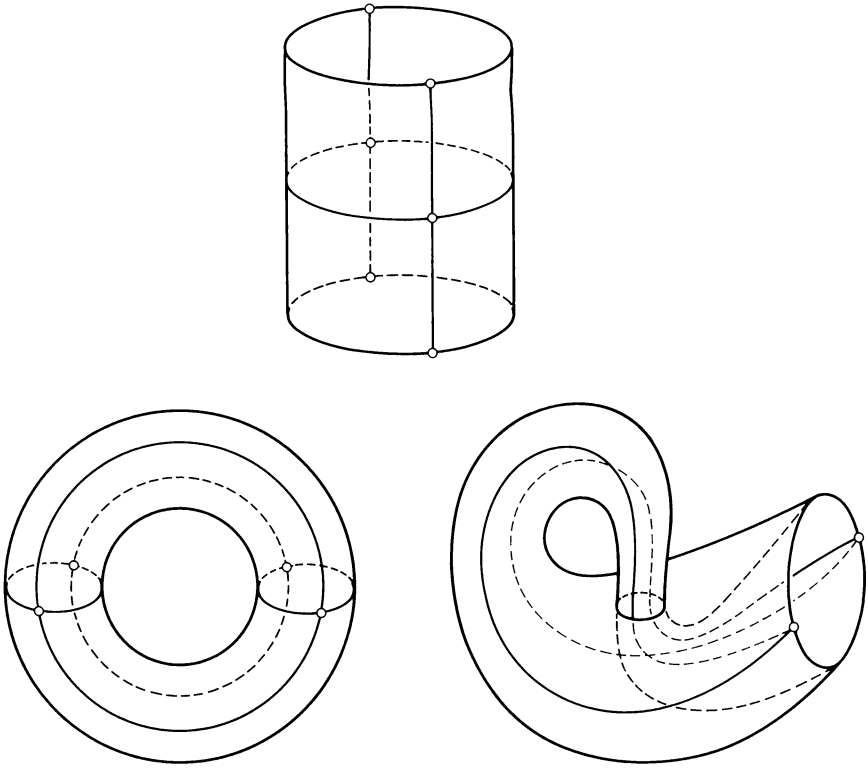


Fig.117. By further pasting of the tube (Example 5.3.2) we obtain the torus (Example 5.3.4) and the Klein bottle (Example 5.3.5). (In 'reality' the self intersection of the Klein bottle does not occur, and results here from the attempt at modeling the bottle in the space $\mathbf{R}^3$.)

5.3.5. EXAMPLE. *The Klein bottle.* We add to the conditions defining the relation R of Example 5.3.2 the following: $a_1 R - (a_1)$, $c_1 R(-c_1)$, $cR(-c)$, $c_2 R(-c_2)$. The polyhedron $|K/R|$ is called the *Klein bottle*. By an argument similar to that of Example 5.3.3 we discover that it is a non-orientable manifold without boundary. ■

5.3.6. EXAMPLE. *The projective plane.* We add to the conditions defining the relation R of Example 5.3.3 the following: $c_1 R(-c_1)$, $cR(-c)$, $c_2 R(-c_2)$. It is easy to see that the

polyhedron $|K/R|$ is homeomorphic with the projective plane P^2 (see Example 1.5.13). By an argument similar to that of Example 5.3.3 we learn that it is a non-orientable manifold without boundary. ■

We will now be concerned with some special kinds of simplicial relations. Let the simplicial complex K be a triangulation of an m-dimensional manifold X and let the subcomplex K_0 be a triangulation of the boundary $\operatorname{bd} X$. Suppose we are given a simplicial isomorphism $\psi\colon K_0 \to K_0$ which is an involution without a fixed simplex – that is, it satisfies $\psi^{-1} = \psi$ and $\psi(\Delta) \neq \Delta$ for all $\Delta \in K_0$. The isomorphism then determines a simplicial relation R on the complex K such that if v', v'' are distinct vertices of K_0, then $v'Rv''$ holds if and only if $v'' = \psi(v')$. An easily proved property of this relation is given in the following.

5.3.7. ASSERTION. *If a simplicial relation R is determined by an involution which does not fix any simplex of the subcomplex K_0 of a complex K, where $|K|$ is an m-dimensional manifold with boundary $|K_0|$, then the polyhedron $|K/R|$ is an m-dimensional manifold without boundary.* ■

We apply the assertion above in our discussion of the next examples.

5.3.8. EXAMPLE. *Pasting two manifolds along their boundaries.* Let the simplicial complex K be a triangulation of an m-dimensional manifold X and let the subcomplex K_0 be a triangulation of the boundary, $\operatorname{bd} X$. Similarly let the simplicial complex $\mathcal{L}$ be a triangulation of an m-dimensional manifold Y and let the subcomplex $\mathcal{L}_0$ be a triangulation of the boundary, $\operatorname{bd} Y$. Suppose that $|K| \cap |\mathcal{L}| = \emptyset$ and consider the simplicial complex $\mathcal{M} = K \cup \mathcal{L}$; evidently the polyhedron $Z = |\mathcal{M}|$ is a manifold with boundary $\operatorname{bd} Z = \operatorname{bd} X \cup \operatorname{bd} Y$, whose triangulation $\mathcal{M}_0 = K_0 \cup \mathcal{L}_0$ is a subcomplex of $\mathcal{M}$.

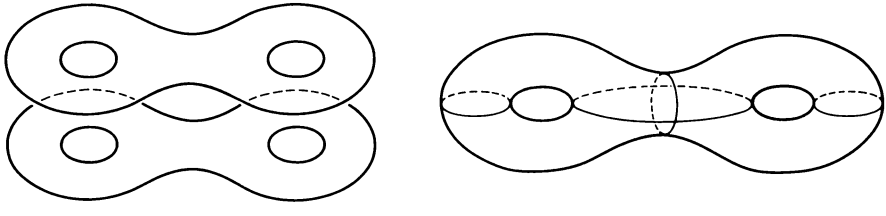


Fig.118. Pasting together two manifolds along their boundaries gives rise to a manifold without boundary (Example 5.3.8).

Suppose that the subcomplexes K_0 and $\mathcal{L}_0$ are simplicially isomorphic and say $\varphi\colon K_0 \to \mathcal{L}_0$ is the simplicial isomorphism. Then we may put $\psi(\Delta) = \varphi(\Delta)$ for $\Delta \in K_0$ and $\psi(\Delta) = \varphi^{-1}(\Delta)$ for $\Delta \in \mathcal{L}_0$ to obtain a simplicial isomorphism $\psi\colon \mathcal{M}_0 \to \mathcal{M}_0$ which is involutory without a fixed simplex. By Assertion 5.3.7 the polyhedron $|\mathcal{M}/R|$, where R is the simplicial relation determined by the involution ψ, is a manifold without boundary. We say that it was obtained by *pasting together the manifolds X and Y along their boundaries.*

The reader can easily apply these general considerations to the case when X is a disc and Y is the Möbius band (see Example 5.3.3). Both manifolds have boundaries

isomorphic to a circle and it is easy to select triangulations of the manifolds X and Y so that their boundaries will have simplicially isomorphic triangulations. Pasting together X and Y along their boundaries gives rise, as may easily be checked, to a manifold homeomorphic to the projective plane (see Example 5.3.6). ■

5.3.9. EXAMPLE. *The m-dimensional projective space.* Let $X = \{(x^1, x^2, \dots, x^m) \in \mathbf{R}^m : \sum_{i=1}^m |x^i| \leq 1\}$. Let $a_i = (\delta_i^1, \delta_i^2, \dots, \delta_i^m)$ for $i = 1, 2, \dots, m$ and let $a_0 = (0, 0, \dots, 0) \in \mathbf{R}^m$. Consider the natural triangulation K of the polyhedron X consisting of all the m-dimensional simplices $\Delta(a_0, \epsilon_1 a_1, \dots, \epsilon_m a_m)$, where $\epsilon_i = \pm 1$ for $i = 1, 2, \dots, m$, together with all their faces. Let K_0 be the simplicial subcomplex of the complex K comprising all the $(m-1)$-dimensional simplices of the form $\Delta(\epsilon_1 a_1, \epsilon_2 a_2, \dots, \epsilon_m a_m)$, where $\epsilon_i = \pm 1$ for $i = 1, 2, \dots, m$, together with all their faces. It may readily be checked via Theorem 1.10.9 that there is a homeomorphism of the set $|K|$ onto the ball $\bar{B}^m$ which takes $|K_0|$ onto the sphere S^{m-1}. The set $X = |K|$ is thus a manifold with boundary $\operatorname{bd} X = |K_0|$.

Consider the barycentric subdivision K' of the complex K. The central symmetry of the complex K' relative to the point a_0 determines, in an obvious way, a simplicial isomorphism $\psi: K_0' \to K_0'$ which is an involution without a fixed simplex. By Assertion 5.3.7 the polyhedron $|K'/R|$, where the simplicial relation R is determined by the involution ψ, is an m-dimensional manifold without boundary. By reference to Example 1.5.13 we easily infer that this polyhedron is homeomorphic to the m-dimensional projective space P^m.

Let $\Delta', \Delta'' \in K_0'$ be $(m-1)$-dimensional simplices which are symmetric relative to the point a_0. Consider an arbitrary orientation of the manifold X with the triangulation K. This evidently determines an orientation of Δ' and of Δ''. The pasting $\sigma: K' \to K'/R$ takes both simplices to the same $(m-1)$-dimensional simplex in K'/R. It now follows that in order for the manifold $|K'/R|$ to be orientable it is necessary that the antipodal map reverses the orientation of the space $\mathbf{R}^m$. Writing down the matrix of this map, we immediately notice that its determinant is $(-1)^m$. Thus the m-dimensional projective space P^m with the triangulation earlier described is an orientable manifold if and only if m is odd. ■

We conclude the section by defining an operation which, in a sense, is the inverse of the operation of pasting. Let K_0 be a simplicial subcomplex of a complex K. Consider a covering $\mathcal{U}$ of the complement $|K| \setminus |K_0|$ whose members are the components of the differences $\operatorname{st} a \setminus |K_0|$, where a runs through the vertices of the complex K. Let N be any nerve of the covering $\mathcal{U}$. We say that the complex N arises from the complex K by *cutting along the subcomplex* K_0.

Exercises

a) Justify why it was necessary in Example 5.3.2–5.3.6 to take the barycentric subdivision of order 2 of the initial triangulation of the square.

b) Give additional conditions which, together with the defining conditions for R in Example 5.3.3, will yield the Klein bottle by way of the Möbius band rather than of the tube as in Example 5.3.5.

c) Define a simplicial relation R on the complex K described in the paragraph preceding Examples 5.3.2–5.3.6 such that the polyhedron $|K/R|$ is homeomorphic with the sphere S^2.

d) Show that pasting together two suitably subdivided 2-dimensional simplices along their boundaries yields a manifold homeomorphic to the sphere S^2.

e) Show that by cutting the Möbius band along its equator (under the triangulation described in Example 5.3.3) we obtain a connected complex. Is its underlying polyhedron an orientable manifold?

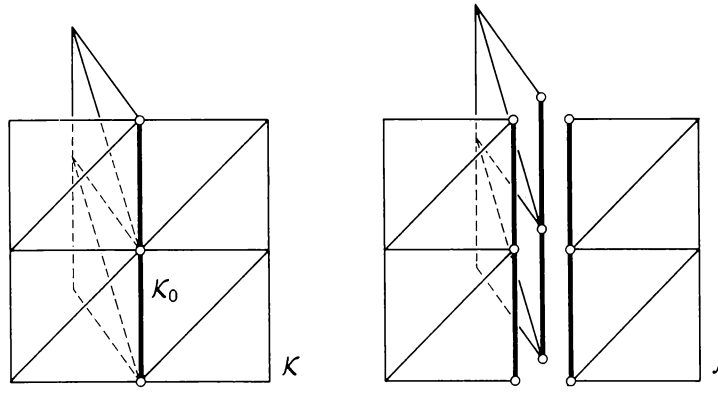


Fig.119. The complex N arises from the complex K by cutting along the subcomplex K_0.

5.4. Classification of 1- and 2-dimensional manifolds

The problem of classifying manifolds topologically consists of finding a set of topological invariants associated with the manifold so that any two manifolds with identical invariants are homeomorphic. For each topological type of manifold, a manifold is selected whose description is particularly simple and which is known as the *normal form.* Consequently from the point of view of the topology of manifolds it is enough to study their normal forms.

The problem of classifying manifolds topologically has not so far been solved for dimensions higher than 2. In this section we will be concerned with the classification of 1- and 2-dimensional manifolds. The classification of 1-dimensional manifolds is very simple. It is based on the following theorem.

5.4.1. THEOREM. *Every connected 1-dimensional manifold is homeomorphic either to the unit circle S^1 or to the unit interval I.*

PROOF. Let X be a 1-dimensional manifold. For each point $x \in X$ there is thus a set $U_x \subset X$ homeomorphic to the interval I such that $x \in \operatorname{int} U_x$. Using the compactness of the space X, apply Theorem 1.8.12 to the covering $\{\operatorname{int} U_x\}_{x \in X}$ and choose a finite system of sets $U_1, U_2, \ldots, U_k$ with the property that $X = \bigcup_{j=1}^{k} U_j$ and there exist homeomorphisms $h_j: I \to U_j$ for $j = 1, 2, \ldots, k$. Let $B_j = h_j(\{0, 1\})$ for $j = 1, 2, \ldots, k$.

Dropping, if necessary, some of the terms of the sequence $U_1, U_2, \ldots, U_k$ we may at once suppose that the covering $\{U_j\}_{j=1}^k$ of the space X is irreducible; that is, the set $X_i = \bigcup_{j \neq i} U_j$ is different from X for $i = 1, 2, \ldots, k$. Choosing, if necessary, an appropriate subset of the set U_i we may also suppose that $U_i \cap X_i \subset B_i$ for $i = 1, 2, \ldots, k$. From the definition of a 1-dimensional manifold we infer that $U_j \cap U_i = B_j \cap B_i$ for all $i \neq j$ and that, moreover, there do not exist three pairwise distinct indices h, i, j for which $U_h \cap U_i \cap U_j \neq \emptyset$.

If $k = 1$ or $k = 2$ the proposition in the theorem is obviously true. If $k > 2$, then by renumbering the sequence $U_1, U_2, \ldots, U_k$ if necessary we may suppose that the intersection $U_j \cap U_{j+1}$ consists of only one point for $j = 1, 2, \ldots, k-1$ and the intersection $U_k \cap U_1$ either consists of one point or is empty. By an easy induction we conclude that the union $X = \bigcup_{j=1}^k U_j$ is homeomorphic in the first case to the circle S^1 and in the second case to the interval I. ■

5.4.2. COROLLARY. *Every connected 1-dimensional manifold without boundary is homeomorphic to the circle S^1. Every connected 1-dimensional manifold with a non-empty boundary is homeomorphic to the interval I.* ■

5.4.3. COROLLARY. *Every 1-dimensional manifold with arbitrary triangulation is orientable.* ■

The 2-dimensional manifolds are also called *surfaces*. We now take up the problem of topologically classifying connected surfaces and, to simplify the discussion, we limit ourselves to the case of surfaces without boundary. We shall also assume that all surfaces under consideration have a triangulation; this assumption does not in any way limit the generality of our discussion (cf. Supplement 5.S.2).

We first prove the following.

5.4.4. THEOREM. *For every connected surface X without boundary, endowed with a triangulation K, there is a simplicial complex $\mathcal{L}$ such that the polyhedron $|\mathcal{L}|$ is homeomorphic with the disc $\bar{B}^2$ and there is a simplicial relation R on the triangulation of the boundary* $\operatorname{bd}|\mathcal{L}|$ *such that the complex $\mathcal{L}/R$ is simplicially isomorphic to K.*

PROOF. Let $\Delta_i = \Delta(a_{0i}, a_{1i}, a_{2i})$, where $i = 1, 2, \ldots, k$, be the 2-dimensional simplices of the complex K. Renumbering, if necessary, the terms of this sequence and the vertices of each simplex, we may straight off assume that for $j = 2, 3, \ldots, k$ there is an index $i_j < j$ such that the simplex Δ_j meets the simplex Δ_{i_j} along the edge $\Delta(a_{1j}, a_{2j})$; moreover, let $a_{1j} = a_{h_j^1 i_j}$ and $a_{2j} = a_{h_j^2 i_j}$.

Consider now a sequence of pairwise disjoint 2-dimensional simplices $\Lambda_i = \Delta(b_{0i}, b_{1i}, b_{2i})$ for $i = 1, 2, \ldots, k$. Let the simplicial complex M_i consist of the simplex Λ_i together with all its faces for $i = 1, 2, \ldots, k$. In the simplicial complex $\mathcal{L}_j = \bigcup_{i=1}^j M_i$ define a simplicial relation R_j by induction on $j = 1, 2, \ldots, k$. We define the relation R on the complex $\mathcal{L}_1$ assuming that no two distinct simplices are related. Next, if $\Lambda', \Lambda'' \in \mathcal{L}_{j-1}$ and $\Lambda' R_{j-1} \Lambda''$ holds, then let $\Lambda' R_j \Lambda''$ hold. Finally we require that the conditions $b_{1j} R_j b_{h_j^1 i_j}$ and $b_{2j} R_j b_{h_j^2 i_j}$ shall hold.

It can be checked that the underlying space of the complex $\mathcal{L}_j/R_j$ is homeomorphic to the disc $\bar{B}^2$ for $j = 1, 2, \ldots, k$. Let us define $\mathcal{L} = \mathcal{L}_k/R_k$ and let $\sigma\colon \mathcal{L}_k \to \mathcal{L}$ be the

corresponding pasting. If $c', c'' \in \mathrm{bd}\,|\mathcal{L}|$ are vertices of the complex $\mathcal{L}$ let us define $c'Rc''$ as holding if and only if $c' = \sigma(b_{h'i'})$ and $c'' = \sigma(b_{h''i''})$ where $a_{h'i'} = a_{h''i''}$. Let $\tau\colon \mathcal{L} \to \mathcal{L}/R$ be the pasting of the complex $\mathcal{L}$ according to the relation R. It can be verified that the map sending a vertex $a_{hi} \in K$ to the vertex $\tau\sigma(b_{hi}) \in \mathcal{L}/R$, where $h = 0, 1, 2$ and $i = 1, 2, \ldots, k$, defines a simplicial isomorphism of the complex K onto the complex $\mathcal{L}/R$. ■

Let X be any surface without boundary, endowed with a triangulation K. Every pair $(\mathcal{L}, R)$, where $\mathcal{L}$ is a simplicial complex such that $|\mathcal{L}|$ is homeomorphic to the disc $\bar{B}^2$, and R is a simplicial relation on the triangulation of the boundary $\mathrm{bd}\,|\mathcal{L}|$ such that the complex $\mathcal{L}/R$ is simplicially isomorphic with the complex K, will be called a *model* of the surface X. Evidently a surface may have several models.

Let $(\mathcal{L}, R)$ be a model of the surface X. We regard the polyhedron $|\mathcal{L}|$ as a manifold with boundary $|\mathcal{L}_0|$, where $\mathcal{L}_0$ is the appropriate simplicial subcomplex of the complex $\mathcal{L}$. Let $I_1, I_2, \ldots, I_p$ be the 1-dimensional simplices of the complex $\mathcal{L}_0$; suppose moreover that $I_i = \Delta(c_i, c_{i+1})$ for $i = 1, 2, \ldots, p-1$ and $I_p = \Delta(c_p, c_1)$. It is easy to see that the number p is even and that the simplices of the sequence $I_1, I_2, \ldots, I_p$ fall into disjoint pairs (I_i, I_j) where $I_i R I_j$. Let $p = 2r$; from each pair (I_i, I_j) where $I_i R I_j$ choose one simplex. Associate arbitrarily with each of the selected simplices the symbols $\alpha_1, \alpha_2, \ldots, \alpha_r$. If $I_i R I_j$, then: either $c_i R c_j$ and $c_{i+1} R c_{j+1}$, or $c_i R c_{j+1}$ and $c_{i+1} R c_j$. If the symbol α_ν corresponds to the simplex I_i then, in the former case, we associate the symbol α_ν with the simplex I_j and in the latter case, the symbol α_ν^{-1}. To the relation R there corresponds a sequence $a_1 a_2 \ldots a_p$ which is an arrangement without repetitions of the symbols $\alpha_1, \alpha_2, \ldots, \alpha_r$, $\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_r^{-1}$ written in the same order as the occurrence of the corresponding simplices in the list $I_1, I_2, \ldots, I_p$. This sequence of symbols is traditionally written without separating commas between the individual terms and is called a *description* of the model $(\mathcal{L}, R)$. Evidently a model may have several different descriptions.

Let us agree to the convention that $(\alpha_i^{-1})^{-1} = \alpha_i$ for $i = 1, 2, \ldots, r$; the symbols α_i and α_i^{-1} are said to be *inverse* to each other. The following operations on a description $a_1 a_2 \ldots a_p$ yield a description of the same model.

1) *Cyclic shift*: consists of replacing a description $a_1 a_2 \ldots a_p$ by the description $a_p a_1 \ldots a_{p-1}$;
2) *Change of orientation of the simplices I_i, I_j*: consists of replacing the description $a_1 a_2 \ldots a_i \ldots a_j \ldots a_p$ where $I_i R I_j$ by the description $a_1 a_2 \ldots a_i^{-1} \ldots a_j^{-1} \ldots a_p$;
3) *Change of orientation of the complex $\mathcal{L}$*: consists of replacing the description $a_1 a_2 \ldots a_p$ by the description $a_p^{-1} a_{p-1}^{-1} \ldots a_1^{-1}$;

To simplify the notation it is also convenient to agree the following *substitution convention*. Namely, suppose that, in the description $a_1 a_2 \ldots a_p$, two disjoint segments $a_i a_{i+1} \ldots a_{i+k}$ and $a_j a_{j+1} \ldots a_{j+k}$ occur so that $i + k < j$ and $a_j = a_{i+k}^{-1}$, $a_{j+1} = a_{i+k-1}^{-1}, \ldots$, and $a_{j+k} = a_i^{-1}$; then the description $a_1 a_2 \ldots a_p$ will be regarded as identical with $a_1 a_2 \ldots a_{i-1} b a_{i+k+1} \ldots a_{j-1} b^{-1} a_{j+k+1} \ldots a_p$. Thus the symbol b replaces the string of symbols $a_i a_{i+1} \ldots a_{i+k}$ and the symbol b^{-1} replaces the string $a_{i+k}^{-1} a_{i+k-1}^{-1} \ldots a_i^{-1}$. Since agreement on the substitution convention leads to the loss of the bijective corre-

spondence between the symbols of the description and the 1-dimensional simplices of the subcomplex $\mathcal{L}_0$, let us introduce the convenient notion of a vertex of a symbol: if a symbol b replaces the string $a_i a_{i+1} \dots a_{i+k}$ (including the case $k = 0$) then the point a_i will be called the *beginning of the symbol* b and the point a_{i+k} its *end*. The beginning and end of a symbol will be referred to as the *vertices* of the symbol. We also note that by accepting the substitution convention we allow descriptions consisting of only two symbols.

Using descriptions, we shall now perform various operations on models; these operations will not alter the topological type of the surface. We begin by describing an elementary operation from which we shall compose more complicated ones. Let $a_1 a_2 \dots a_p$ be a description of the model $(\mathcal{L}, R)$ of the surface X and suppose $p \geq 4$. Let the point c' be the end of the symbol a_k and c'' the end of the symbol a_ℓ, where $1 \leq k < \ell \leq p$. Take an arbitrary broken line J, formed of 1-dimensional simplices in $\mathcal{L}$, which is a 1-dimensional manifold with boundary $\{c', c''\}$ and whose interior lies in the interior of the manifold $|\mathcal{L}|$. In order not to complicate the terminology, we shall identify the broken line with its natural triangulation.

Cutting the complex $\mathcal{L}$ along the broken line J we obtain two simplicial complexes $\bar{\mathcal{L}}$ and $\bar{\bar{\mathcal{L}}}$ whose underlying spaces are homeomorphic to the disc $\bar{B}^2$; the broken line thus splits into two broken lines, with which we associate the symbols a_0 and a_0^{-1}. The boundaries of the polyhedra $|\bar{\mathcal{L}}|$ and $|\bar{\bar{\mathcal{L}}}|$ now have corresponding to them, in a natural way, the strings $a_0 a_{\ell+1} \dots a_p a_1 \dots a_k$ and $a_0^{-1} a_{k+1} \dots a_\ell$. These are not, of course, descriptions of any models. Suppose that possibly after applying to both these strings the substitution convention, they take the form $\bar{b}_1 \bar{b}_2 \dots \bar{b}_m$ and $\bar{\bar{b}}_1 \bar{\bar{b}}_2 \dots \bar{\bar{b}}_n$. Next suppose that for some indices i and j, where $1 \leq i \leq m$ and $1 \leq j \leq n$, one of the equations $\bar{b}_i = \bar{\bar{b}}_j$ or $\bar{b}_i = \bar{\bar{b}}_j^{-1}$ holds. Define a relation S on the simplicial complex $\bar{\mathcal{L}} \cup \bar{\bar{\mathcal{L}}}$ by requiring in the case $\bar{b}_i = \bar{\bar{b}}_j$, that the beginning of $\bar{b}_i$ is S-related to the beginning of $\bar{\bar{b}}_j$ and that the end of $\bar{b}_i$ is S-related to the end of $\bar{\bar{b}}_j$, and by requiring in the case $\bar{b}_i = \bar{\bar{b}}_j^{-1}$ that the beginning of $\bar{b}_i$ is S-related to the end of $\bar{\bar{b}}_j$ and the end of $\bar{b}_i$ is S-related to the beginning of $\bar{\bar{b}}_j$. This relation naturally extends to all the 0- and 1-dimensional simplices corresponding to the symbols $\bar{b}_i$ and $\bar{\bar{b}}_j$. The simplicial complex $(\bar{\mathcal{L}} \cup \bar{\bar{\mathcal{L}}})/S$ has underlying space homeomorphic to the disc $\bar{B}^2$. The boundary of this polyhedron has the corresponding string of symbols $\bar{b}_1 \dots \bar{b}_{i-1} \bar{\bar{b}}_{j-1}^{-1} \dots \bar{\bar{b}}_1^{-1} \bar{\bar{b}}_n^{-1} \dots \bar{\bar{b}}_{j+1}^{-1} \bar{b}_{i+1} \dots \bar{b}_m$ when $\bar{b}_i = \bar{\bar{b}}_j$ or the string of symbols $\bar{b}_1 \dots \bar{b}_{i-1} \bar{\bar{b}}_{j+1} \dots \bar{\bar{b}}_n \bar{\bar{b}}_1 \dots \bar{\bar{b}}_{j-1} \bar{b}_{i+1} \dots \bar{b}_m$ when $\bar{b}_i = \bar{\bar{b}}_j^{-1}$. It can be verified that this is the description of a model of the same surface X. The operation of replacing the old model by the new is called *cutting from* c' *to* c'' *and pasting* $\bar{b}_i$ *to* $\bar{\bar{b}}_j$.

We now proceed to a description of some further operations on models.

1) *Cancellation.* Suppose that a description consisting of at least four symbols has a segment of the form $\dots a a^{-1} \dots$ Let c' be the vertex which is the common end of a and the beginning of a^{-1}; let c'' be an arbitrary vertex which is neither the beginning of a nor the end of a^{-1}. Cutting from c' to c'' we obtain two strings of form

$\ldots ax \ldots$ and $\ldots x^{-1}a^{-1} \ldots$ Substituting $b = ax$ we obtain strings of the form $\ldots b \ldots$ and $\ldots b^{-1} \ldots$ After pasting b to b^{-1} we obtain a model whose description differs from the initial description of the model only in that the neighbouring symbols aa^{-1} have been suppressed. Repeated application of this procedure leads either to a description which has no consecutive symbols which are inverse to each other, or to one of the two descriptions aa^{-1} or $a^{-1}a$.

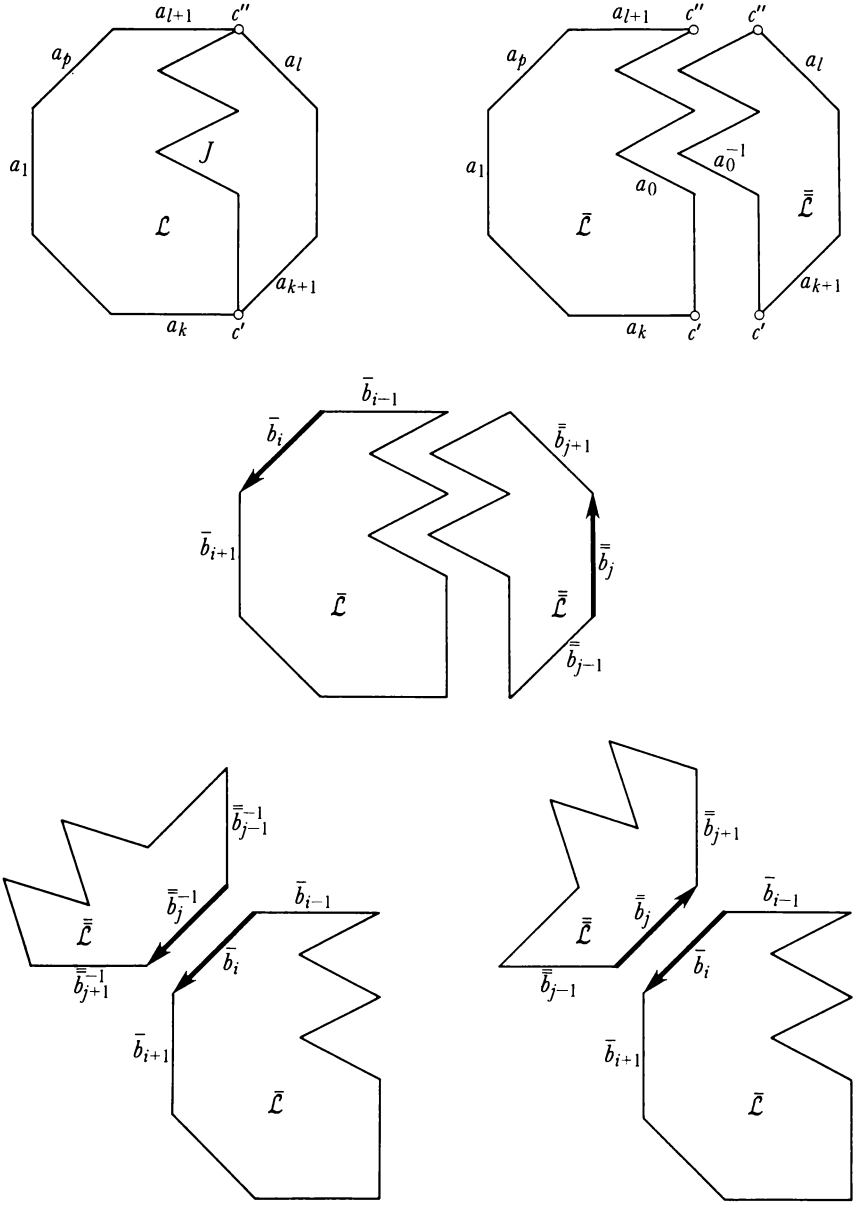


Fig.120. Cutting the model of the surface from c' to c'' and pasting $\bar{b}_i$ to $\bar{\bar{b}}_j$.

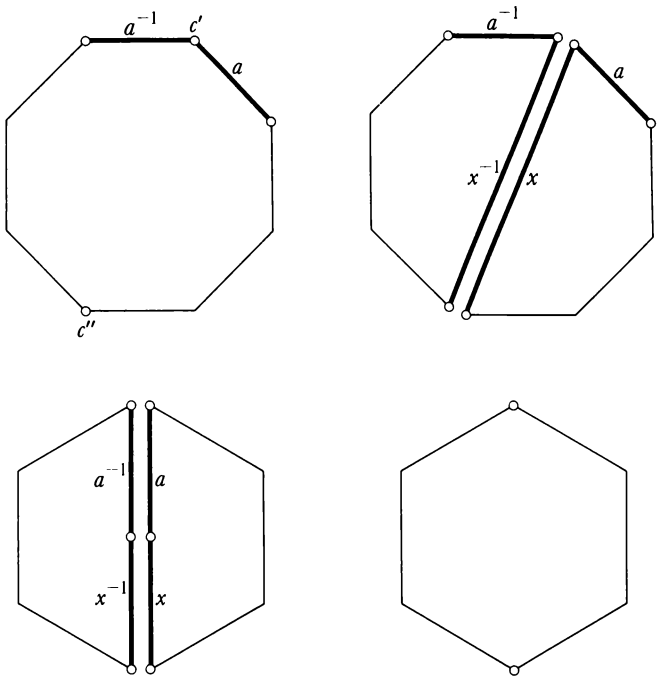


Fig.121. Successive stages in the process of cancellation within a description. A pair of consecutive symbols a, a^{-1} have been suppressed.

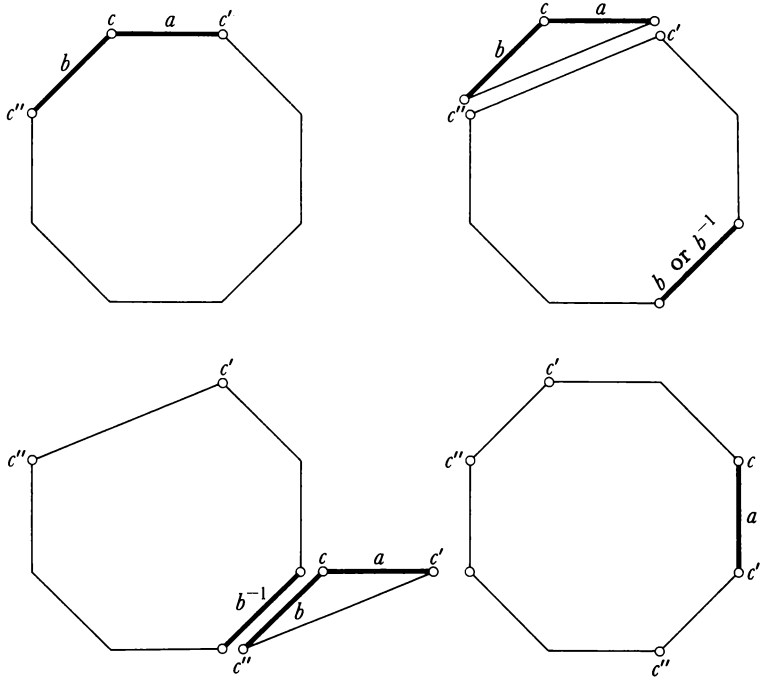


Fig.122. Successive stages in the operation of reduction to a single vertex. The number of vertices equivalent to c has dropped by 1 and the number of vertices equivalent to c' has risen by 1.

2) *Reduction to a single vertex.* Suppose that a description consisting of at least four symbols does not contain consecutive symbols that are inverses of each other. Suppose further that there are two vertices c and c' of a symbol of the description which are not equivalent; we may of course suppose without loss of generality that c' is the beginning and c the end of a symbol a. The symbol b with beginning c is thus by hypothesis different from a^{-1}, but we also have $b \neq a$ since the vertices c' and c are not equivalent. Let c'' be the end of b. Cut from c' to c''; then the symbol b and its corresponding symbol (b or b^{-1}) lie in different strings of symbols arising from the cut. Pasting the symbol b to its corresponding symbol (b or b^{-1}) we obtain a model of the same surface in which the vertex c is equivalent to a smaller number of vertices, while the vertex c' is equivalent to a larger number of vertices than was originally the case. Applying this operation repeatedly we obtain a model in which all the vertices are equivalent.

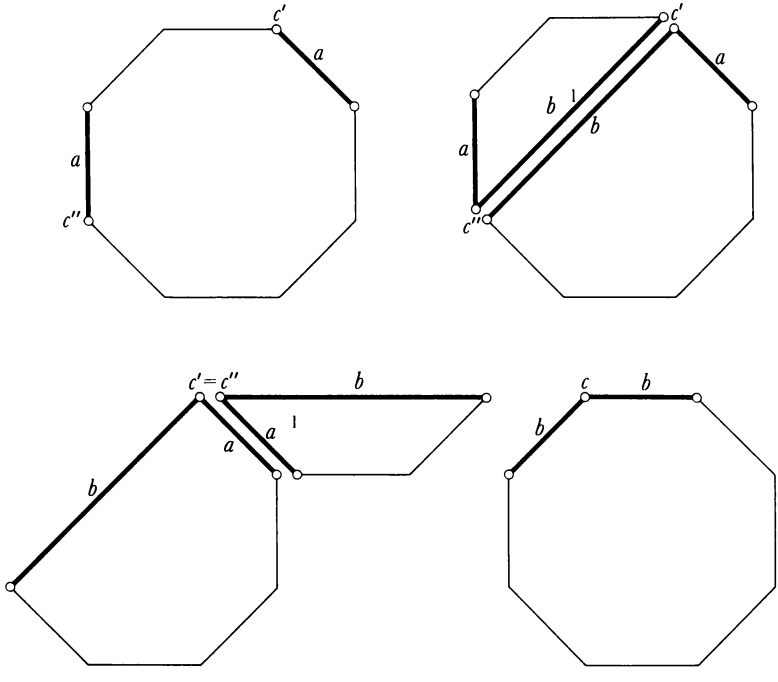


Fig.123. Successive stages in the extraction of a Möbius band. In place of the symbol a appearing twice, though not consecutively, we obtain the symbol b appearing twice consecutively.

3) *Extraction of the Möbius band.* Suppose that the description of the model consists of at least four symbols without consecutive symbols being inverses of each other, and that all vertices are equivalent. Suppose further that a symbol a occurs twice in the description. It is easy to see that this is a necessary and sufficient condition for the surface to be non-orientable. Let the vertex c' be the end of the first occurrence of a and let the vertex c'' be the end of the second occurrence of a. Cutting from c' to c'' we obtain two strings $\ldots ab \ldots$ and $\ldots ab^{-1} \ldots$ and pasting the two symbols a we obtain a model which has the form $\ldots bb \ldots$

We say that in the model the pair bb is an *extracted Möbius band.* Comparison with Example 5.3.3 easily explains this term. Note also that if the description contains other pairs of identical symbols, then they may be extracted in the form of new Möbius bands without upsetting bands already extracted. We can thus arrive at a situation where all pairs of identical symbols have been extracted in the form of Möbius bands.

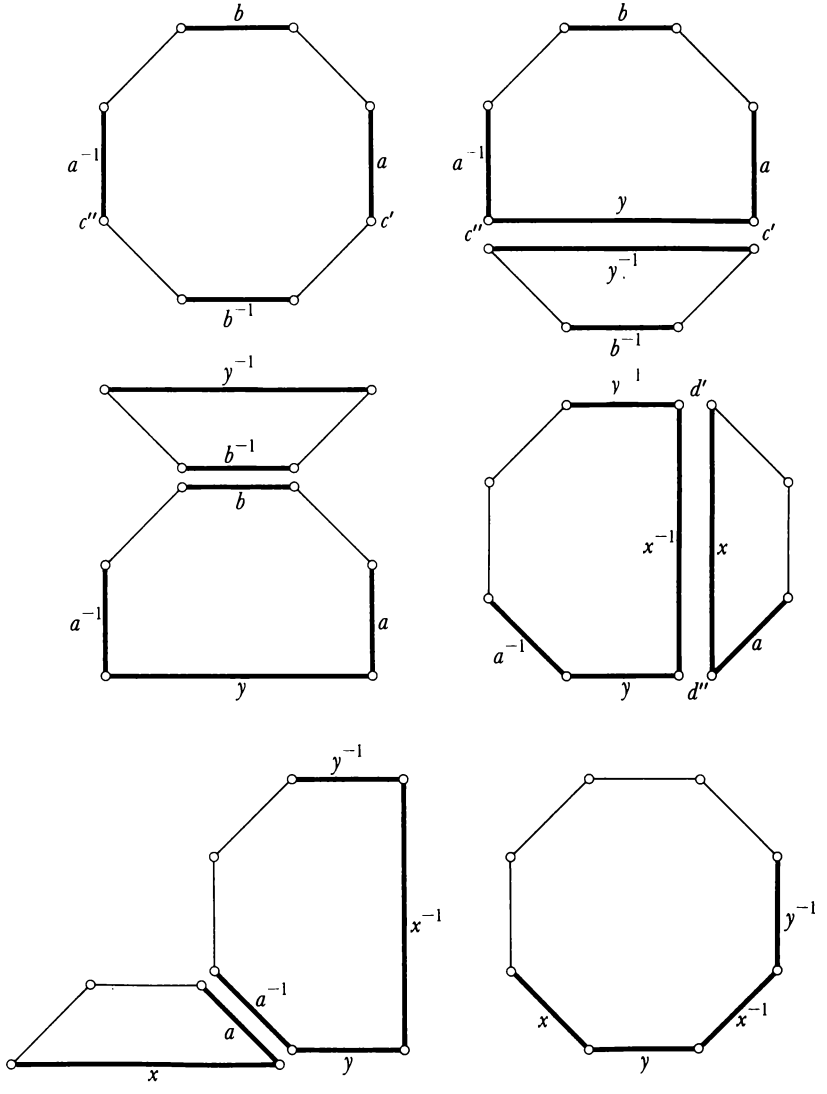


Fig. 124. Successive stages in the operation of extracting tubes. In place of the not necessarily consecutive pairs a, b, a^{-1}, b^{-1} we obtain a consecutive run of symbols $xyx^{-1}y^{-1}$.

4) *Extraction of tubes.* Suppose that after performing the operations 1) to 3) on the description of the model there is a pair of inverse symbols a and a^{-1}. Then the description contains at least one more pair of inverse symbols b and b^{-1} and the description has the form $\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots$ For otherwise, all the symbols of the

substring $a \dots a^{-1}$ would appear in pairs of the form: b and b^{-1}. The beginning of the symbol a and the end of the symbol a^{-1} would not then be equivalent to the remaining vertices of the string $a \dots a^{-1}$, contrary to assumption.

So suppose the description takes the form $\dots a \dots b \dots a^{-1} \dots b^{-1} \dots$ Let c' be the beginning of a and c'' the end of a^{-1}. Cutting from c' to c'' we obtain the two strings $\dots a^{-1} y a \dots b \dots$ and $\dots b^{-1} \dots y^{-1} \dots$ Pasting b to b^{-1}, we obtain a description of the form $\dots y^{-1} \dots a^{-1} y a \dots$ Let d' be the beginning of y^{-1} and d'' the end of y and the beginning of a. Cutting from d' to d'' we obtain the strings $\dots x a \dots$ and $a^{-1} y x^{-1} y^{-1} \dots$ Pasting a to a^{-1} we obtain a description of the form $\dots x y x^{-1} y^{-1} \dots$

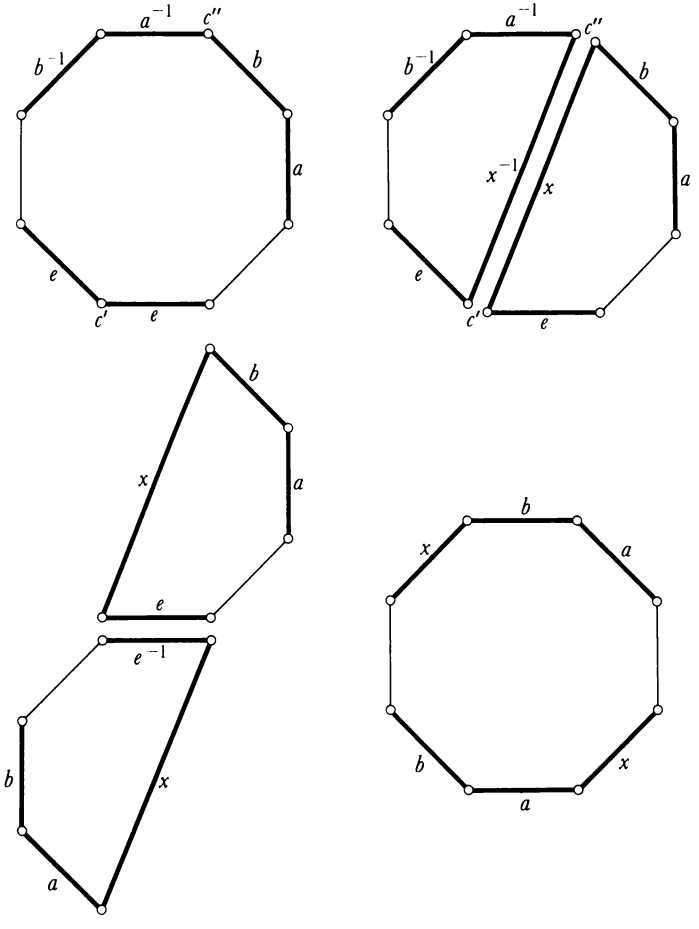


Fig.125. Successive stages in the operation of converting a tube into a Möbius band. Instead of the extracted tube and Möbius band $\dots aba^{-1}b^{-1} \dots ee \dots$, we obtain a description of the form $\dots abx \dots bax$ from which we can then extract three Möbius bands.

We say that the four symbols in succession $xyx^{-1}y^{-1}$ form an *extracted tube.* Comparison with Example 5.3.2 easily explains the term. Note moreover that if the description contains more pairs of inverse symbols, then they can be extracted in the form of new tubes without disturbing already extracted tubes and extracted Möbius

bands. So we can arrive at a situation where all pairs of identical symbols have been extracted in the form of Möbius bands and all pairs of inverse symbols have been extracted as tubes.

5) *Conversion of tubes into Möbius bands.* Suppose a description contains an extracted tube and an extracted Möbius band; that is, it has the form $\dots aba^{-1}b^{-1}\dots ee\dots$ Let c' be the end of the first symbol e and the beginning of the second symbol e. Let c'' be the end of b and the beginning of a^{-1}. Cutting from c' to c'' we obtain the strings $\dots abxe\dots$ and $ex^{-1}a^{-1}b^{-1}\dots$ Pasting the two symbols e we obtain a description of the form $\dots abx\dots bax\dots$ In this description we have three pairs of identical symbols. So, we can extract three Möbius bands instead of one extracted tube and one extracted Möbius band. Thus if the description contains at least one extracted Möbius band it can be processed to a form in which it consists of only extracted Möbius bands.

Using operations (1) to (5) we arrive at the following theorem.

5.4.5. THEOREM. *Every connected surface without a boundary has a model whose description takes one of the forms:*

$(1)_0$ $$aa^{-1},$$

$(1)_{2r}$ $$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_rb_ra_r^{-1}b_r^{-1}, \quad \textit{or}$$

$(2)_r$ $$a_1a_1a_2a_2\dots a_ra_r. \blacksquare$$

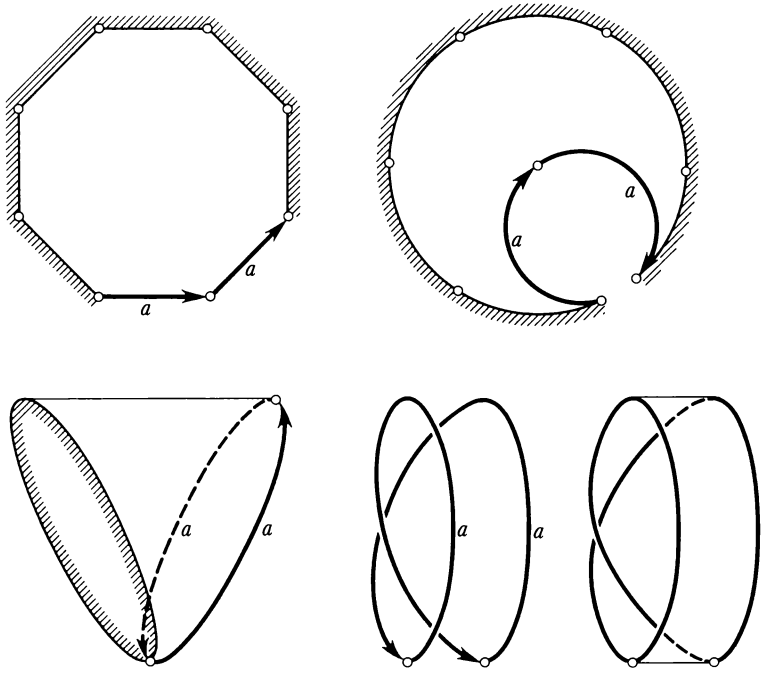


Fig.126. Geometric interpretation of an extracted Möbius band. The pair of symbols aa corresponds to a Möbius band pasted along its boundary onto a circular hole in the surface.

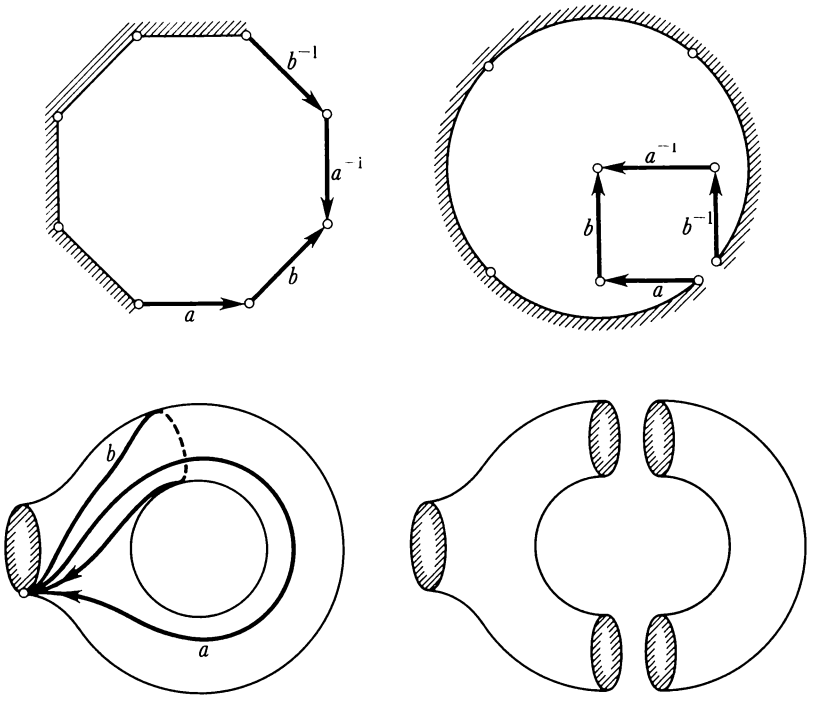


Fig.127. Geometric interpretation of an extracted tube. The string of symbols $aba^{-1}b^{-1}$ corresponds to a tube pasted along its boundary onto two circular holes in the surface.

The form $(1)_0$ is called the *normal form of the first kind of degree* 0; it is easy to see this corresponds to a surface homeomorphic to the sphere S^2. The form $(1)_{2r}$ is called the *normal form of the first kind of degree* $2r$. It corresponds to a surface which may intuitively be described as a sphere punctured by $2r$ disjoint holes which have then been pairwise connected by means of r disjoint tubes.

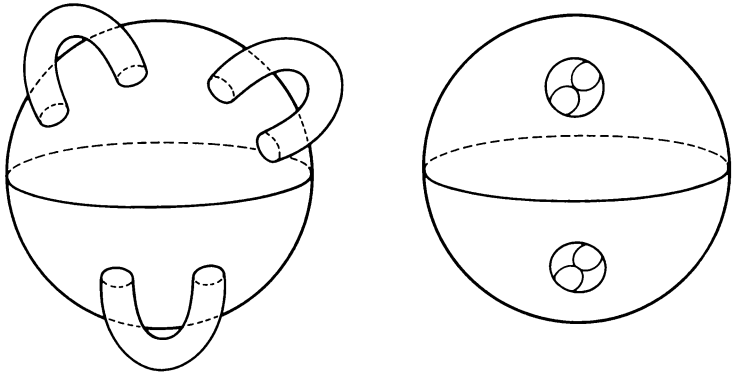


Fig.128. How to interpret surfaces corresponding to descriptions in normal form: a sphere with three tubular handles corresponds to a normal form of the first kind and degree 6; the sphere with two Möbius bands pasted in corresponds to a normal form of the second kind of degree 2.

The form $(2)_r$ is called the *normal form of the second kind of degree r*. It corresponds to a surface which may be intuitively described as a sphere punctured by r circular holes which have then each been filled by pasting edge to edge a copy of the Möbius band.

To apply Theorem 5.4.5 to the topological classification of connected surfaces without boundary, a proof is required that every surface of the stated type uniquely defines the kind and degree of its normal form. In other words proof is needed that homeomorphic surfaces cannot have normal forms differing in kind or degree. This is an immediate consequence of Example 5.4.6 below, where we compute the fundamental group of a surface of a given normal form. A more elementary proof is sketched in Supplement 5.S.5. We also give in Supplement 5.S.5 information on the normal forms and the classification of surfaces which have non-empty boundary.

5.4.6. EXAMPLE. Consider the description in normal form of a model $(\mathcal{L}, R)$ of a connected surface X endowed with a triangulation K. Let $\mathcal{L}_0$ be a subcomplex of a complex $\mathcal{L}$ such that $|\mathcal{L}_0| = \operatorname{bd}|\mathcal{L}|$ and let R_0 denote the relation R restricted to $\mathcal{L}_0$. Obviously $|\mathcal{L}_0/R_0|$ is a bouquet of r circles where r denotes the degree of the normal form of the description.

Take a 2-dimensional simplex $\Delta \in \mathcal{L}$, two vertices of which lie in the interior $\operatorname{int}\mathcal{L}$, and the third vertex v_0 is a vertex of any symbol of the description. Let $X_1 = |K\backslash\{\Delta\}|$ and $X_2 = \Delta$. Since the boundary $\operatorname{bd}|\mathcal{L}|$ is a deformation retract of the polyhedron $|\mathcal{L}\backslash\{\Delta\}|$, the group $\pi_1(X_1, v_0)$ is isomorphic to the group $\pi_1(|\mathcal{L}_0/R_0|, v_0)$ and so, according to Example 3.4.24, is free and has r generators; the generators may be identified with those symbols of the description which have exponents equal to $+1$. It is easy to see that the inclusion $i\colon(\operatorname{bd}\Delta, v_0) \to (X_1, v_0)$ takes the only generator of the group $\pi_1(\operatorname{bd}\Delta, v_0)$ onto that product in the group $\pi_1(X_1, v_0)$ of generators and their inverses which is determined by the description.

Applying Corollary 3.4.30, we discover that if the surface X has a description of the first kind of order $2r$, then the group $\pi_1(X)$ has $2r$ generators $a_1, b_1, a_2, b_2, \ldots, a_r, b_r$ and one relation $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_r b_r a_r^{-1} b_r^{-1} = 1$. The case $r = 0$ requires separate argument, but its conclusion is included in the general case, since the group $\pi_1(X)$ is then trivial. If, however, the surface X has a description of the second kind of degree r, then the group $\pi_1(X)$ has r generators $a_1, a_2, \ldots, a_r$ and one relation $(a_1)^2(a_2)^2 \ldots (a_r)^2 = 1$. ■

Exercises

a) Find the description in normal form of the model of each of the surfaces considered in Examples 5.3.4–5.3.6.

b) Show that the existence of at last one extracted Möbius band is a necessary condition for performing the operation of converting an extracted tube into extracted Möbius bands.

c) Describe the inverse operation to that of converting tubes into Möbius bands. Can its repeated application replace all extracted Möbius bands by extracted tubes?

5.S. Supplements

5.S.1. In Section 5.1 we made the assumption that every topological manifold is a compact space. For many applications this assumption is very restrictive; for example, the Euclidean spaces are not manifolds in this sense. The assumption of compactness is often replaced by the weaker hypothesis that the space has a countable dense set. In such a setting the compact manifolds are known under the name of *closed* manifolds. Some authors use the word manifold to mean a space which in our terminology is without boundary. In such a setting manifolds, in the sense in which we use the word, are known as manifolds *with boundary*.

5.S.2. It follows from Corollary 5.4.2 that every 1-dimensional topological manifold is homeomorphic with a polyhedron. Also every 2-dimensional topological manifold, or surface, has this property. This is a classical result obtained by J. Gawehna (*Über unberandete zweidimensionale Mannigfaltigkeiten*, Math. Ann. 98 (1927), 321–354). Using more refined techniques, E. E. Moise proved that every 3-dimensional topological manifold is homeomorphic to a polyhedron (*Affine structures in 3-manifolds*, V. *The triangulation problem and Hauptvermutung*, Ann. of Math. 56 (1952), 96–114); another proof was given by R. H. Bing (*An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. 69 (1959), 37–65). The question whether every m-dimensional topological manifold for $m > 3$ has the property is so far unresolved.

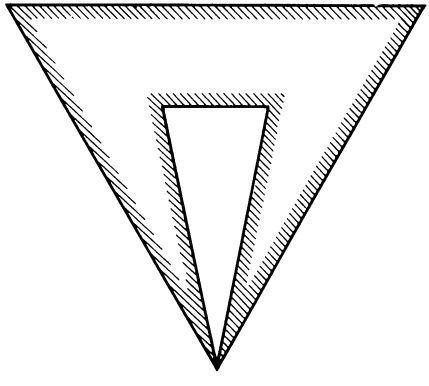


Fig.129. A polyhedron which is a 2-dimensional pseudomanifold (cf. Supplement 5.S.2) but is not a 2-dimensional manifold.

To combine the asssumptions that a space is a polyhedron and a topological manifold is sometimes pointless. On the one hand, it is enough for the purposes of introducing orientability to suppose that the polyhedron is an *m-dimensional pseudomanifold*; that is, it has a triangulation satisfying conditions (1)–(5) of Theorem 5.1.20. On the other hand, the assumption that a topological manifold has a triangulation is for many purposes not sufficient, since it does not in general give information about the topological structure of the star of an arbitrary vertex. For this reason the notion of a combinatorial manifold was introduced in the twenties (J. W. Alexander, M. H. A. Newman). We say

that an m-dimensional topological manifold X is a *combinatorial manifold* if it has a triangulation K such that for every vertex $v \in K$ the subcomplex comprising the simplices $\Delta \in K$ of which v is a vertex together with their faces, is simplicially isomorphic with a subdivision of the natural triangulation of the m-dimensional simplex described in Example 2.3.1.

As Moise shows in the paper cited above, every m-dimensional topological manifold for $m \leq 3$ is a combinatorial manifold. It is not true for $m \geq 4$. (For $m = 4$ see: M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982), 357–453; for $m > 4$ see: R. Kirby, L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. 75 (1965), 742–749.)

It is worth noting that J. W. Milnor's counter-example to the Hauptvermutung (see Supplement 2.S.1) is not a manifold; it is therefore still not known whether the Hauptvermutung holds for triangulated manifolds.

5.S.3. Any m-dimensional manifold without boundary which has the homotopy type of the m-dimensional sphere is called an *m-dimensional homotopic sphere*. The conjecture that every m-dimensional homotopic sphere is homeomorphic with the sphere S^m is known as the *generalized Poincaré conjecture in homotopic form* (the original conjecture due to Poincaré concerned the case $m = 3$ and was stated using different terminology). The conjecture was found to be true for all dimensions $m \geq 4$ (on the assumption of extra structure – combinatorial or smooth manifolds; cf. Supplements 5.S.2 and 5.S.4); for $m = 2$ its truth easily follows from Theorem 5.4.5 and Example 5.4.6; nonetheless it remains unresolved even now for the case $m = 3$. Attempts at proving the Poincaré conjecture contributed to the development of manifold theory and of related areas of topology.

5.S.4. Equipping a topological manifold with the somehow alien structure of a simplicial complex, or of a combinatorial manifold, permits a wider range of techniques to be brought into play, and yields deeper results. Another step in the same direction is the introduction of a smooth manifold. We say that an m-dimensional topological manifold X (where we assume, in order to simplify the definition, that X is a manifold without boundary) is a *smooth manifold* or a *differentiable manifold* if there is a family $\{U_t, \varphi_t\}_{t \in T}$, known as an *atlas*, where $\{U_t\}_{t \in T}$ is an open covering of the space X and φ_t, for each $t \in T$, is a homeomorphism of the set U_t onto the ball B^m such that, for each $s \in S$, the composition $\varphi_s \varphi_t^{-1} : \varphi_t(U_t \cap U_s) \to B^m$ is a differentiable map of class C^∞.

S. S. Cairns proved (*The triangulation problem and its role in analysis*, Bull. Amer. Math. Soc. 52 (1946), 545–571) that every m-dimensional triangulated topological manifold is a smooth manifold if $m \leq 4$. On the other hand there do exist triangulated topological manifolds which are not smooth; the first such example (of dimension 10) was given by M. Kervaire (*A manifold which does not admit any differentiable structure*, Comment. Math. Helv. 34(1960), 257–270). However, as Cairns proved (*Triangulation of the manifold of class one*, Bull. Amer. Math. Soc. 41(1935), 549–552) every smooth manifold is homeomorphic to a polyhedron; as we have noted in Supplement 5.S.2 it is still not known whether the smoothness assumption is essential.

5.S.5. The problem of topologically classifying m-dimensional manifolds consists in essence of two questions. First, there is the problem of constructing normal forms, that is, defining a sequence of m-dimensional manifolds $X_1, X_2, \ldots$ such that every m-dimensional manifold is homeomorphic to exactly one manifold in the sequence. Then, there is the homeomorphism problem; this amounts to specifying a set of topological invariants which will decide in a finite number of steps whether any two m-dimensional manifolds are homeomorphic or not.

As A. Markov showed (*O nierazreshimosti niekotoryh problem topologii*, Dokl. Akad. Nauk SSSR 123(1958), 978–980) the homeomorphism problem is undecidable for $m \geq 4$. Hence the cases $m \leq 3$ are particularly interesting. The case $m = 1$ has a trivial solution following from Corollary 5.4.2: for 1-dimensional manifolds without boundary there is just one normal form, viz. the circle S^1, similarly for 1-dimensional manifolds with non-empty boundary there is just one normal form, viz. the interval I; two 1-dimensional manifolds are homeomorphic if and only if they are both without boundary or both have non-empty boundary.

For the 2-dimensional manifolds, or surfaces, we limited ourselves in Section 5.4 to the case when the boundary is empty, and specified a sequence of normal forms. Using the concept of the fundamental group, the homeomorphism problem is easily solved for these manifolds (see Example 5.4.6; cf. also Corollaries 3.4.11 and 3.4.21). The problem can be solved another way using some facts from algebraic topology. Namely, it turns out that orientability of a triangulated manifold is a property of the manifold itself and not of the triangulation; similarly the degree of the normal form of the description of a surface depends only on the surface and not on the triangulation nor on any method of reducing the description to normal form. The degree is in fact related by the formula $r = 2 - \chi(K)$ with the Euler-Poincaré characteristic $\chi(K)$ (cf. Supplement 2.S.3) of the triangulation K of the corresponding normal form (see Problem 5.P.13). The desired set of invariants is thus orientability and the Euler-Poincaré characteristic.

The classification of surfaces with non-empty boundary is carried out using similar methods to those for surfaces without boundary. The corresponding normal forms differ from the ones discussed here in that there are some holes punched in the disc, on whose boundary identifications are made. The set of invariants in this case is provided by: orientability, the Euler-Poincaré characteristic and the number of components of the boundary.

5.S.6. The topological classification problem for 3-dimensional manifolds is still unsolved even for manifolds without boundary. There is a far greater abundance of types for these manifolds as compared to surfaces. We describe in some detail one family of 3-dimensional manifolds.

In the Euclidean plane $\mathbf{R}^2$, let $a_0, a_1, \ldots, a_{p-1}$ be the vertices of a regular p-gon centred at the origin. We regard these points as points of $\mathbf{R}^3$ under the natural inclusion $\mathbf{R}^2 \subset \mathbf{R}^3$. Let $c_+ = (0,0,1)$ and $c_- = (0,0,-1)$. Consider the polyhedron $X = \mathrm{conv}\{a_0, a_1, \ldots, a_{p-1}, c_+, c_-\}$; its boundary has a natural triangulation with vertices $a_0, a_1, \ldots, a_{p-1}, c_+, c_-$ which we may regard as a simplicial subcomplex of a triangulation of X with vertices $a_0, a_1, \ldots, a_{p-1}, c_+, c_-, 0$. Let K be the barycentric subdivision of this triangulation. Let q be an integer which is coprime to p. Consider the composition

map of the revolution around the x^3-axis through an angle of $2\pi q/p$ together with the reflection relative to the x^1x^2-plane. This isometry takes the triangulation K onto itself and determines an equivalence relation on that subcomplex K_0 of the complex K which is the triangulation of the boundary $\operatorname{bd} X$. The underlying space of the complex K/R is called the *lens space* corresponding to the numbers p and q and is denoted by $L(p,q)$; it is easy to check that it is a 3-dimensional manifold without boundary (see Problems 5.P.14–5.P.16).

Some progress at solving the classification problem for 3-dimensional manifolds is associated with the name of P. Heegaard. He observed (*Sur l'Analysis Situs*, Bull. Soc. Math. France 44(1916), 161–242) that every connected, orientable, 3-dimensional manifold X contains a manifold B_k which is homeomorphic to a ball $\bar{B}^3$ with k solid handles grafted onto it such that $X \setminus \operatorname{int} B_k$ is homeomorphic to B_k.

The manifold is thus the union of two copies of the manifold B_k pasted together along the boundary $\operatorname{bd} B_k$. The manifolds X for which $k = 1$ were topologically classified by K. Reidemeister (*Homotopie von Linsenräume*, Abh. Math. Sem. Univ. Hamburg 11(1935), 102–109). These are, up to homeomorphism, the sphere S^3, the metric product of spheres $S^1 \times S^2$, and the lens spaces. Unfortunately for $k > 1$ the situation is significantly more complicated.

5.S.7. Let $V'_{n,k}$ be the set of all affinely independent systems of $k+1$ points $a_0 = 0$, $a_1, \ldots, a_k$ in Euclidean space $\mathbf{R}^n$. Associate with every member of the set $V'_{n,k}$ a matrix of size $k \times n$ which we may identify with a point of $\mathbf{R}^{kn}$. This endows the set $V'_{n,k}$ with a metric. It turns out that with this metric the set $V'_{n,k}$ is a manifold; it is called a *Stiefel manifold* (see Problem 5.P.17). Restriction to orthonormal systems of points also yields a manifold $V_{n,k} \subset V'_{n,k}$. For example, the manifold $V_{n,1}$ is homeomorphic to the sphere S^{n-1}.

Let $M_{n,k}$ denote the set of k-dimensional linear subspaces of Euclidean space $\mathbf{R}^n$. It is easy to see that to each member of the set $M_{n,k}$ there corresponds a system of minors of order k of the matrices of size $k \times n$. This allows the introduction of a metric on the set $M_{n,k}$. It turns out that under this metric the set $M_{n,k}$ is a manifold; it is called a *Grassman manifold* (see Problem 5.P.18). For example the manifold $M_{n,1}$ is homeomorphic to the projective space P^{n-1}.

5.P. Problems

5.P.1. Prove that the boundary, $\operatorname{bd} X$, of any m-dimensional manifold has empty interior in X.

5.P.2. Prove that if X is a connected manifold, then for every pair of points $x, y \in \operatorname{int} X$ there is a homeomorphism $h\colon X \to X$ such that $h(x) = y$ and $h \mid \operatorname{bd} X = \mathrm{id}$ (see Theorem 5.1.17).

5.P.3. Let X be a connected manifold and suppose $x, y \in \operatorname{bd} X$. Does there always exist a homeomorphism $h\colon X \to X$ such that $h(x) = y$?

5.P.4. Investigate for which k and n is the k-dimensional skeleton of an n-dimensional simplex a k-dimensional manifold?

5.P.5. Prove that the metric product of two orientable manifolds is, with the triangulation defined in Supplement 2.S.8, an orientable manifold.

5.P.6. Suppose that all the simplices of a simplicial complex K are oriented in an arbitrary but fixed manner. Prove that if the simplex $\Delta' \in K$ is of dimension $n+1$, and if $\Delta'' \in K$ is of dimension $n-1$ and Δ runs through all the n-dimensional simplices of the complex K, then $\sum_{\Delta}[\Delta' : \Delta][\Delta : \Delta''] = 0$.

5.P.7. Show that by pasting together two copies of the m-dimensional simplex Δ^m along their boundary, $\operatorname{bd}\Delta^m$, we obtain an m-dimensional manifold which is homeomorphic to the sphere S^m.

5.P.8. Show that any connected surface X without a boundary is homeomorphic to a subset of the Euclidean space $\mathbf{R}^4$ and that the surface X is orientable, if and only if, it is homeomorphic to a subspace of the Euclidean space $\mathbf{R}^3$.

5.P.9. Show that any connected surface X without a boundary and with a triangulation K has a model whose description takes the form $a_1 a_2 \dots a_r a_1^{-1} a_2^{-1} \dots a_{r-1}^{-1} a_r^{\epsilon}$ where $\epsilon = \pm 1$. Explain the connection between the number r and the Euler-Poincaré characteristic (cf. Supplement 5.S.5) and also the connection between the number ϵ and the orientability of the manifold X with triangulation K.

5.P.10. Prove that if a surface X with triangulation K is orientable (non-orientable), then the surface X with triangulation K', where K' denotes the barycentric subdivision of the complex K, is also orientable (non-orientable).

5.P.11. Show that a connected surface X without a boundary and with triangulation K is non-orientable, if and only if, there is a subcomplex K_0 of the complex K such that the polyhedron $|K_0|$ is homeomorphic with the circle S^1 and that, after cutting the complex K along the subcomplex K_0, we obtain a surface with a boundary which is also homeomorphic with the circle.

5.P.12. Show that every triangulated surface is a combinatorial manifold (cf. Supplement 5.S.2).

5.P.13. Show that the degree of the normal form of the description of a surface without a boundary equals $2-\chi(K)$ where χ is the Euler-Poincaré characteristic (cf. Supplement 2.S.3) and K is the complex obtained by pasting according to the relation determined by the description.

5.P.14. Show that for every pair of relatively prime number p, q, where $p > 0$, the lens space $L(p, q)$ (see Supplement 5.S.6) is a 3-dimensional manifold without boundary.

5.P.15. Prove that in order that two lens spaces $L(p, q)$ and $L(p, q')$ (see Supplement 5.S.6) be homeomorphic, it is necessary and sufficient that the number q' equal one of

the numbers $\pm q^{\pm 1}$ (where q and q' are treated as elements of the field Z_p of integers mod p).

5.P.16. Prove that in order that two lens spaces $L(p,q)$ and $L(p,q')$ (see Supplement 5.S.6) have the same homotopy type, it is necessary and sufficient that for some integer $m \in Z_p$ the equation $q' = \pm m^2 q$ holds in the field Z_p. Deduce that the spaces $L(7,1)$ and $L(7,2)$ are 3-dimensional manifolds which have the same homotopy type, but are not homeomorphic (cf. [5], p. 223–225).

5.P.17. Show that the Stiefel manifold $V'_{n,k}$ (see Supplement 5.S.7) is an m-dimensional topological manifold; determine m as a function of n and k.

5.P.18. Show that the Grassman manifold $M_{n,k}$ (see Supplement 5.S.7) is an m-dimensional topological manifold; determine m as a function of n and k.

Chapter 6
Metric spaces II

This chapter is a continuation of Chapter 1. We develop further the theory of metric spaces which form the most important class of spaces considered in topology.

In Sections 6.1 and 6.2 we define and study two operations on metric spaces: taking countable products, and formation of function spaces. These operations, together with the operation of taking subspaces defined in Chapter 1, lead from very simple spaces to wide classes of metric and metrizable spaces (cf. Theorem 6.3.9 and Problems 6.P.42, 7.P.56).

The next section is devoted to separable spaces, that is spaces containing a countable dense set. After characterizing this class of spaces and investigating which operations preserve separability we prove that, from the topological point of view, the separable spaces are identical with the subspaces of the Hilbert cube. Towards the end of the section we concern ourselves with totally bounded spaces and show that a metric space is compact if and only if it is complete and totally bounded.

In Section 6.4 we continue our analysis of completeness begun in Chapter 1 (Section 1.9). Among the most important results of the section are Baire's Theorem and Banach's fixed point theorem; they are useful tools in existence proofs for all kinds of mathematical objects and find numerous applications in analysis. In Section 6.4 we also introduce the notion of a completion of a metric space and investigate its properties.

Continua form the topic of Section 6.5 – these are metric spaces which are both connected and compact. We begin the section by showing that no continuum may be expressed as a countable union of pairwise disjoint closed sets, and we then turn our attention to locally connected continua. The main result concerning this class of spaces asserts that every locally connected continuum is pathwise connected and locally pathwise connected. Locally connected continua may also be characterized as the continuous images of the unit interval I (Theorem 6.5.24).

Then, in the next section we study absolute retracts and absolute neighbourhood retracts; these are two classes of spaces with a regular structure, which though defined exclusively in topological terms bear a striking family resemblance to the class of polyhedra.

The last two sections of the chapter contain the outlines of the theory of dimension for separable metric spaces and form an introductory account of this particularly interesting theory.

6.1. Countable products of metric spaces

We begin by generalizing the metric product operation introduced in Section 1.2 to the case of an infinite sequence of spaces.

Assume given a sequence of metric spaces (X_i, ρ_i) where $i = 1, 2, \ldots$ with the property that $\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^2$ converges; we take this to mean that the series considered contains only finitely many terms of the form ∞^2 and that after omitting these terms the series of numerical terms converges. We can equip the set $X = \bigtimes_{i=1}^{\infty} X_i$ with a metric ρ by taking

$$\rho(x, y) = \sqrt{\sum_{i=1}^{\infty} \rho_i(x_i, y_i)^2} \quad \text{for} \quad x = \{x_1, x_2, \ldots\}, \quad y = \{y_1, y_2, \ldots\} \in X.$$

Indeed, since the numerical series appearing under the square root sign is convergent, the function ρ is validly defined. The fact that the metric ρ satisfies the axioms (M1) and (M2) foilows immediately from their being satisfied by the metrics ρ_i for $i = 1, 2, \ldots$ With the aim of proving the triangle inequality let us assume that the sequences $x = \{x_1, x_2, \ldots\}$, $y = \{y_1, y_2, \ldots\}$ and $z = \{z_1, z_2, \ldots\}$ belong to the set X and let us consider the points $x^m = (x_1, x_2, \ldots, x_m)$, $y^m = (y_1, y_2, \ldots, y_m)$, $z^m = (z_1, z_2, \ldots, z_m)$ in the metric product of the spaces (X_i, ρ_i) for $i = 1, 2, \ldots, m$. From the triangle inequality in that product we conclude that

$$\sqrt{\sum_{i=1}^{m} \rho_i(x_i, z_i)^2} \leq \sqrt{\sum_{i=1}^{m} \rho_i(x_i, y_i)^2} + \sqrt{\sum_{i=1}^{m} \rho_i(y_i, z_i)^2}.$$

The right hand side of the inequality above does not exceed $\rho(x, y) + \rho(y, z)$, and hence

$$\sqrt{\sum_{i=1}^{m} \rho_i(x_i, z_i)^2} \leq \rho(x, y) + \rho(y, z) \quad \text{for} \quad m = 1, 2 \ldots;$$

so we have $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

We call the set X equipped with the metric ρ, as defined above, the *metric product* of the spaces (X_i, ρ_i) where $i = 1, 2, \ldots$ and we write $(X, \rho) = \bigtimes_{i=1}^{\infty} (X_i, \rho_i)$ or more briefly $X = \bigtimes_{i=1}^{\infty} X_i$; we emphasize that the metric product of an infinite sequence of spaces (X_i, ρ_i) is only defined when the series $\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^2$ is convergent.

We remark that the Hilbert cube I^{ω} defined in Example 1.2.2 is the metric product of the closed intervals $[0, 1/i]$, where $i = 1, 2, \ldots$

6.1.1. ASSERTION. *If A_i is a metric subspace of the metric space X_i for $i = 1, 2, \ldots$ and the series $\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^2$ is convergent then $\bigtimes_{i=1}^{\infty} A_i$ is a metric subspace of the metric product $\bigtimes_{i=1}^{\infty} X_i$.* ■

6.1.2. ASSERTION. *Let a sequence of metric spaces X_i for $i = 1, 2, \ldots$ be given so that the series $\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^2$ converges and let $X = \bigtimes_{i=1}^{\infty} X_i$. For each natural number i the assignment taking the point $\{x_1, x_2, \ldots\} \in X$ to the point $p_i(\{x_1, x_2, \ldots\}) = x_i \in X_i$ is a non-expansive map from the metric product X to the space X_i.* ■

The map p_i defined in Assertion 6.1.2 is called the *projection of the metric product* $\bigtimes_{i=1}^{\infty} X_i$ *onto the* i^{th}*-factor*; evidently the projection is a continuous map.

In the sequel we shall often make use of the following estimate of distance in the countable metric product (cf. Lemma 1.2.4).

6.1.3. LEMMA. *If* $(X,\rho) = \bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ *and* $x = \{x_1, x_2, \ldots\}$, $y = \{y_1, y_2, \ldots\} \in X$ *then for every natural number* m *such that for all* $i > m$ *the space* X_i *is bounded, we have*

$$\sup\{\rho_i(x_i,y_i) : i = 1,2,\ldots\} \leq \rho(x,y) \leq$$

$$\leq \sqrt{m}\max\{\rho_i(x_i,y_i) : i = 1,2,\ldots,m\} + \sqrt{\sum_{i=m+1}^{\infty} (\operatorname{diam} X_i)^2}. \blacksquare$$

We now study the connection between convergence in the metric product and convergence in the factors of the product.

6.1.4. THEOREM. *If* $(X,\rho) = \bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ *and* $x_n = \{x_n^1, x_n^2, \ldots\} \in X$ *for* $n = 0,1,2,\ldots$ *then* $\lim_n x_n = x_0$ *in the space* X *if and only if* $\lim_n x_n^i = x_0^i$ *in each space* X_i *for* $i = 1,2,\ldots$

PROOF. If $\lim_n x_n = x_0$ then $\lim_n \rho(x_n, x_0) = 0$. In view of Lemma 6.1.3 we infer that $\lim_n[\sup\{\rho_i(x_n^i, x_0^i) : i = 1,2,\ldots\}] = 0$, so that $\lim_n \rho_i(x_n^i, x_0^i) = 0$, or $\lim_n x_n^i = x_0^i$ for $i = 1,2,\ldots$

Conversely, let $\lim_n x_n^i = x_0^i$ so that $\lim_n \rho(x_n^i, x_0^i) = 0$ for $i = 1,2,\ldots$ and let ϵ be any positive real number. It follows from the convergence of the series $\sum_{i=1}^{\infty}(\operatorname{diam} X_i)^2$ that there exists a natural number m such that

$$\sum_{i=m+1}^{\infty} (\operatorname{diam} X_i)^2 < \frac{1}{4}\epsilon^2, \quad \text{that is} \quad \sqrt{\sum_{i=m+1}^{\infty} (\operatorname{diam} X_i)^2} < \frac{1}{2}\epsilon.$$

There also exists an index k such that $\max\{\rho_i(x_n^i, x_0^i) : i = 1,2,\ldots,m\} < \epsilon/(2\sqrt{m})$ for $n \geq k$. By Lemma 6.1.3 we have that $\rho(x_n, x_0) < \epsilon$ for $n \geq k$ and hence that $\lim_n x_n = x_0$. $\blacksquare$

We now prove that the operation of forming the countable metric product preserves connectedness, compactness and completeness.

6.1.5. THEOREM. *If* $(X,\rho) = \bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ *and for each* $i = 1,2,\ldots$ *the space* X_i *is connected, then the metric product is a connected space.*

PROOF. We may assume that $X_i \neq \emptyset$ for $i = 1,2,\ldots$, as otherwise the product X is empty and hence connected. For $i = 1,2,\ldots$ choose an arbitrary point $a_i \in X_i$. For $m = 1,2,\ldots$ the subspace

$$A_m = \{\{x_1, x_2, \ldots\} \in X : x_i = a_i \quad \text{for} \quad i > m\}$$

of the metric product X is isometric with the metric product $\bigtimes_{i=1}^{m}(X_i,\rho_i)$ and hence by Theorems 1.7.3 and 1.7.9 is a connected space. Since $\{a_1, a_2, \ldots\} \in \bigcap_{m=1}^{\infty} A_m$, we infer from Theorem 1.7.5 that $A = \bigcup_{m=1}^{\infty} A_m$ is a connected subspace of X. From Lemma 6.1.3 it follows that for every point $x \in X$ and any positive real number ϵ there is a

point $a \in A$ such that $\rho(x,a) < \epsilon$, and hence A is dense in X. The metric product X is therefore connected by Theorem 1.7.12. ■

6.1.6. THEOREM. *If* $(X,\rho) = \bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ *and for each* $i = 1,2,\ldots$ *the space* X_i *is compact, then the metric product* X *is a compact space.*

PROOF. Consider a sequence of points $x_1 = \{x_1^1, x_1^2, \ldots\}$, $x_2 = \{x_2^1, x_2^2, \ldots\}$, ... of the metric product X. From the sequence $x_1^1, x_2^1, \ldots$ of points of the space X_1 we may choose a subsequence $x_{k_1^1}^1, x_{k_2^1}^1, \ldots$ which is convergent to some point $x_0^1 \in X_1$. Similarly from the sequence $x_{k_1^1}^2, x_{k_2^1}^2, \ldots$ we may extract a subsequence $x_{k_1^2}^2, x_{k_2^2}^2, \ldots$ convergent to some point $x_0^2 \in X_2$. Proceeding by induction we define for $n = 3,4,\ldots$ a subsequence $x_{k_1^n}^n, x_{k_2^n}^n, \ldots$ of the sequence $x_{k_1^{n-1}}^n, x_{k_2^{n-1}}^n, \ldots$ which is convergent to some point $x_0^n \in X_n$. Applying Theorem 6.1.4 and the fact that the sequence $k_1^1, k_2^2, k_3^3, \ldots$ is after omitting its first $n-1$ terms a subsequence of the sequence $k_1^n, k_2^n, \ldots$, we readily see that the point $x_0 = \{x_0^1, x_0^2, \ldots\} \in X$ is the limit of the subsequence $x_{k_1^1}, x_{k_2^2}, x_{k_3^3}, \ldots$ of the sequence $x_1, x_2, \ldots$ ■

6.1.7. THEOREM. *If* $(X,\rho) = \bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ *and for each* $i = 1,2,\ldots$ *the space* X_i *is complete, then the metric product* X *is a complete space.*

PROOF. Consider a sequence of points $x_1 = \{x_1^1, x_1^2, \ldots\}$, $x_2 = \{x_2^1, x_2^2, \ldots\}, \ldots$ in the metric product X satisfying the Cauchy condition. Lemma 6.1.3 implies that for $i = 1,2,\ldots$ the sequence $x_1^i, x_2^i, \ldots$ of points of X_i satisfies the Cauchy condition; there exist therefore points $x_0^1 \in X_1$, $x_0^2 \in X_2, \ldots$ such that $\lim_n x_n^i = x_0^i$ for $i = 1,2,\ldots$ By Theorem 6.1.4 the sequence $x_1, x_2, \ldots$ converges to the point $x_0 = \{x_0^1, x_0^2, \ldots\} \in X$. ■

We conclude from the theorems proved above that the metric product of an infinite sequence of spaces is a natural generalization of the finite metric product studied in Section 1.2. However a serious drawback of the infinite metric product consists in its not being defined for all sequences of spaces but only for sequences $X_1, X_2, \ldots$ for which $\sum_{i=1}^{\infty}(\operatorname{diam} X_i)^2$ converges. Within the context of metric geometry this restriction cannot be dropped (see Problem 6.P.1), but the passage to the topological domain changes the situation entirely. Before introducing the operation of the topological product of a sequence of metric spaces we consider a simple theorem which will be helpful.

6.1.8. THEOREM. *Let* (X,ρ) *be a metric space. For any real number* $a > 0$ *the formula*

$$\sigma(x,y) = \min(a, \rho(x,y)) \quad \textit{for} \quad x,y \in X$$

defines a metric on the set X*; further, the identity map* id_X *is a uniform homeomorphism of the space* (X,ρ) *onto the space* (X,σ).

PROOF. It follows from the corresponding properties of the metric ρ that σ satisfies the first two axioms for a metric space. Let x,y,z be arbitrary points of the set X and let $a_1 = \rho(x,y)$, $a_2 = \rho(y,z)$ and $a_3 = \rho(x,z)$. Since $a \leq \min(2a, a+a_1, a+a_2)$ and $a_3 \leq a_1 + a_2$, we have

$$\min(2a, a+a_1, a+a_2, a_1+a_2) \geq \min(a, a_3).$$

We infer that

$$\begin{aligned}\sigma(x,y)+\sigma(y,z) &= \min(a,a_1)+\min(a,a_2)\\ &= \min(2a,a+a_1,a+a_2,a_1+a_2)\\ &\geq \min(a,a_3)=\sigma(x,z),\end{aligned}$$

and so σ satisfies the triangle inequality. We leave it to the reader to check that id_X is a uniform homeomorphism of the space (X,ρ) onto the space (X,σ). ■

Let the sequence of metric spaces (X_i,ρ_i) for $i=1,2,\ldots$ be given. We equip the set $X=\bigtimes_{i=1}^{\infty}X_i$ with a metric ρ^* by taking for $x=\{x_1,x_2,\ldots\}$, $y=\{y_1,y_2,\ldots\}\in X$

$$\rho^*(x,y)=\sqrt{\sum_{i=1}^{\infty}[\min(1/i,\ \rho_i(x_i,y_i))]^2}.$$

Indeed the space (X,ρ^*) is the metric product of the spaces (X_i,σ_i) for $i=1,2,\ldots$ where σ_i is a metric on the set X_i defined by the formula $\sigma_i(x_i,y_i)=\min(1/i,\rho_i(x_i,y_i))$ for $x_i,y_i\in X_i$. The set X equipped with the metric ρ^* is called the *topological product* of the spaces (X_i,ρ_i) for $i=1,2,\ldots$ (see Supplement 6.S.1).

We denote the topological product symbolically in the same way as the metric product, that is, we write $(X,\rho^*)=\bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ or more briefly $X=\bigtimes_{i=1}^{\infty}X_i$; to avoid ambiguity in the notation we shall always precede the symbols $\bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ or $\bigtimes_{i=1}^{\infty}X_i$ by the words "metric product" or "topological product". We might add that we shall also denote the metric of the topological product by the symbol ρ. The topological product, in contrast to the metric product, is defined for any infinite sequence of metric spaces. We remark that the metric product $(X,\rho)=\bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ is in general different from the topological product $(X,\rho^*)=\bigtimes_{i=1}^{\infty}(X_i,\rho_i)$, because the metrics ρ and ρ^* on the set $X=\bigtimes_{i=1}^{\infty}X_i$ are in general different. The metrics are nevertheless equivalent in the sense defined in Supplement 1.S.16, i.e. the identity map $\mathrm{id}_X\colon(X,\rho)\to(X,\rho^*)$ is a homeomorphism. This follows from the next theorem, which itself is a simple consequence of Theorems 6.1.4 and 6.1.8.

6.1.9. THEOREM. *If* $(X,\rho^*)=\bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ *and* $x_n=\{x_n^1,x_n^2,\ldots\}\in X$ *for* $n=0,1,2,\ldots$ *then* $\lim_n x_n=x_0$ *in the topological product if and only if* $\lim_n x_n^i=x_0^i$ *in each space* X_i *for* $i=1,2,\ldots$ ■

6.1.10. ASSERTION. *If* A_i *is a metric subspace of the metric space* X_i *for* $i=1,2,\ldots$ *then the topological product* $\bigtimes_{i=1}^{\infty}A_i$ *is a metric subspace of the topological product* $\bigtimes_{i=1}^{\infty}X_i$. ■

Two important corollaries concerning continuous maps follow from Theorems 1.5.6 and 6.1.9; we state them as Assertion 6.1.11 and Theorem 6.1.12.

6.1.11. ASSERTION. *For each natural number* i *the assignment taking the point* $\{x_1,x_2,\ldots\}$ *of the topological product* $\bigtimes_{i=1}^{\infty}X_i$ *to the point* $p_i(\{x_1,x_2,\ldots\})=x_i\in X_i$ *is a continuous map.* ■

The map defined in Assertion 6.1.11 is called the *projection of the topological product* $\bigtimes_{i=1}^{\infty}X_i$ *onto the* i^{th}*-factor.*

6.1.12. THEOREM. *A map f from a space Y into the topological product $\bigtimes_{i=1}^{\infty} X_i$ is continuous if and only if the composition $p_i f: Y \to X_i$ is continuous for $i = 1, 2, \ldots$* ■

We now prove analogues to Theorems 1.6.6 and 1.6.21 for countable topological products; we begin with the analogue to the second theorem.

6.1.13. THEOREM. *If A_i for each $i = 1, 2, \ldots$ is a closed subset of the space X_i, then the topological product $\bigtimes_{i=1}^{\infty} A_i$ is a closed subset of the topological product $\bigtimes_{i=1}^{\infty} X_i$.*

PROOF. If for $n = 1, 2, \ldots$ $a_n = \{a_n^1, a_n^2, \ldots\} \in A = \bigtimes_{i=1}^{\infty} A_i \subset X = \bigtimes_{i=1}^{\infty} X_i$ and $\lim_n a_n = x = \{x^1, x^2, \ldots\} \in X$, then by Theorem 6.1.9 we have $\lim_n a_n^i = x^i$ for $i = 1, 2, \ldots$ In view of the fact that A_i is closed in X_i we infer that $x^i \in A_i$ for $i = 1, 2, \ldots$ and so $x \in A$. ■

6.1.14. THEOREM. *If A_i for each $i = 1, 2, \ldots, m$ is an open subset of the space X_i, then the product $A = \bigtimes_{i=1}^{\infty} A_i$, where $A_i = X_i$ for $i > m$, is open in the topological product $\bigtimes_{i=1}^{\infty} X_i$.*

PROOF. We have

$$\bigtimes_{i=1}^{\infty} X_i \setminus \bigtimes_{i=1}^{\infty} A_i = \bigcup_{j=1}^{m} \bigtimes_{i=1}^{\infty} X_i^j,$$

where $X_i^j = X_i$ for $i \neq j$ and $X_j^j = X_j \setminus A_j$, so the fact that the product $\bigtimes_{i=1}^{\infty} A_i$ is open follows from Theorem 6.1.13, from the fact that the union of a finite family of closed sets is closed (Theorem 1.6.19) and from the fact that the complement of a closed set is open (Corollary 1.6.12). ■

In connection with the last theorem we note that if A_i is a non-empty proper subset of the space X_i for $i = 1, 2, \ldots$ then the product $\bigtimes_{i=1}^{\infty} A_i$ is not an open subset of the topological product $\bigtimes_{i=1}^{\infty} X_i$. Indeed, an immediate consequence of the definition of the topological product and of Lemma 6.1.3 is the following.

6.1.15. ASSERTION. *If U is an open subset of the topological product $\bigtimes_{i=1}^{\infty} X_i$, then for every point $x \in U$ there exists a natural number m and sets $A_1, A_2, \ldots$, where A_i is an open set of the space X_i for $i = 1, 2, \ldots, m$ and $A_i = X_i$ for $i > m$, such that $x \in \bigtimes_{i=1}^{\infty} A_i \subset U$.* ■

It is readily seen that the sets described in Theorems 6.1.13 and 6.1.14 do not exhaust all the possible closed and open subsets of the topological product $\bigtimes_{i=1}^{\infty} X_i$.

Since connectedness and compactness are invariant under homeomorphisms (Theorems 1.7.3 and 1.8.2), and since completeness is invariant under uniform homeomorphisms (Theorem 1.9.9), we have from Theorem 6.1.8 and the definition of the topological product, that Theorems 6.1.5, 6.1.6 and 6.1.7 lead to analogous results on topological products:

6.1.16. THEOREM. *The topological product of an infinite sequence of connected spaces is connected.* ■

6.1.17. THEOREM. *The topological product of an infinite sequence of compact spaces is compact.* ■

6.1.18. THEOREM. *The topological product of an infinite sequence of complete spaces is complete.* ■

The topological product $\bigtimes_{i=1}^{\infty} X_i$ when $X_i = X$ for every $i = 1, 2, \ldots$ is called the *topological* $\aleph_0$*-power* of the space X and is denoted $X^{\aleph_0}$. We note that if X contains at least two points then the corresponding metric product is undefined.

6.1.19. EXAMPLE. The *Hilbert cube.* We shall show that the $\aleph_0$-power of the unit interval, that is the space $I^{\aleph_0}$, is homeomorphic to the Hilbert cube I^{ω}. Indeed it follows from Theorems 6.1.4, 6.1.9 and 1.5.6 that the Cartesian product map $\bigtimes_{i=1}^{\infty} f_i: I^{\omega} \to I^{\aleph_0}$ of the maps $f_i: [0, 1/i] \to I$ where $f_i(x) = ix$ for $x \in [0, 1/i]$ when $i = 1, 2, \ldots$ is a homeomorphism. The topological power $I^{\aleph_0}$ is also referred to as the *Hilbert cube.* ■

We deduce from the representation of the Hilbert cube as a topological product an analogue to Tietze's Theorem (3.1.4) for maps that take values in this space.

6.1.20. THEOREM. *Let A be a closed subset of the metric space X. Every continuous map $f: A \to I^{\aleph_0}$ has a continuous extension $f^*: X \to I^{\aleph_0}$.*

PROOF. The Hilbert cube $I^{\aleph_0}$ is the topological product $\bigtimes_{i=1}^{\infty} I_i$, where $I_i = I$ for $i = 1, 2, \ldots$ It follows from Theorem 6.1.12 that for each natural number i the composition $f^i = p_i f: A \to I$ is continuous. By Tietze's Theorem the function f^i has a continuous extension $f^{i*}: X \to I$ for $i = 1, 2, \ldots$ Putting $f^*(x) = \{f^{1*}(x), f^{2*}(x), \ldots\}$ for $x \in X$ we obtain, in view of Theorem 6.1.12, a continuous extension $f^*: X \to I^{\aleph_0}$ of the map f. ■

6.1.21. EXAMPLE. *The Cantor set.* We shall show that the $\aleph_0$-power of the two-point discrete space $D = \{0, 1\}$, that is the space $D^{\aleph_0}$, is homeomorphic to the Cantor set $C \subseteq \mathbf{R}$ defined in Example 4.4.1. Consider the map $f: C \to D^{\aleph_0}$ assigning to each point $r = \sum_{i=1}^{\infty} r_i 3^{-i}$, where r_i is 0 or 2 for $i = 1, 2, \ldots$, the point $f(r) = \{\frac{1}{2}r_1, \frac{1}{2}r_2, \ldots\} \in D^{\aleph_0}$. Evidently f is a bijective map from C onto $D^{\aleph_0}$, and since for all $r = \sum_{i=1}^{\infty} r_i 3^{-i}$, $r' = \sum_{i=1}^{\infty} r'_i 3^{-i} \in C$ we have $r_i = r'_i$ provided $|r - r'| < 3^{-i}$, it must be that the composition $p_i f: C \to D$ is continuous for $i = 1, 2, \ldots$; hence by Theorem 6.1.12, the map f is continuous. By the compactness of C and by Theorem 1.8.15 we conclude that f is a homeomorphism. The topological product $D^{\aleph_0}$ is also referred to as the *Cantor set.* ■

We close this section with a proof of an interesting theorem about topological $\aleph_0$-powers and deduce from it an important corollary concerning the Hilbert cube and the Cantor set.

6.1.22. THEOREM. *For any space X the spaces $X^{\aleph_0}$ and $(X^{\aleph_0})^{\aleph_0}$ are homeomorphic. Similarly the spaces $X^{\aleph_0}$ and $(X^{\aleph_0})^m$ are homeomorphic for $m = 1, 2, \ldots$*

PROOF. The points of the space $(X^{\aleph_0})^{\aleph_0}$ are sequences of the form $x = \{x_1, x_2, \ldots\}$, where $x_i = \{x_i^1, x_i^2, \ldots\}$ is an element of $X^{\aleph_0}$, i.e. $x_i^j \in X$ for $i, j = 1, 2, \ldots$ Let us assign to the point x the sequence $f(x) = \{x_1^1, x_3^1, x_2^2, x_1^3, \ldots\} \in X^{\aleph_0}$. Evidently f is a bijective map from $(X^{\aleph_0})^{\aleph_0}$ onto $X^{\aleph_0}$; from Theorem 6.1.9 it follows that f is a homeomorphism. We leave the proof of the second part of the theorem to the reader. ■

6.1.23. COROLLARY. *The $\aleph_0$-power and all the finite powers of the Hilbert cube $I^{\aleph_0}$ are homeomorphic to $I^{\aleph_0}$.* ■

6.1.24. COROLLARY. *The $\aleph_0$-power and all the finite powers of the Cantor set $D^{\aleph_0}$ are homeomorphic to $D^{\aleph_0}$.* ■

We now show that the Hilbert cube is a continuous image of the Cantor set.

6.1.25. THEOREM. *There exists a continuous map of the Cantor set C onto the Hilbert cube $I^{\aleph_0}$.*

PROOF. By Example 6.1.21 and Corollary 6.1.24 it is enough to find a continuous map F of the space $C^{\aleph_0}$ onto the Hilbert cube $I^{\aleph_0}$. The reader can check without difficulty that, by taking $F(\{x_1, x_2, \ldots\}) = \{f(x_1), f(x_2), \ldots\}$ for $\{x_1, x_2, \ldots\} \in C^{\aleph_0}$ where $f: C \to I$ is the staircase function (see Example 4.4.4), we obtain the desired map F. ■

Exercises

a) Suppose given a sequence of metric spaces (X_i, ρ_i) for $i = 1, 2, \ldots$ and a bijective map φ of the natural numbers onto themselves. Show that the metric product $\bigvee_{i=1}^{\infty} X_{\varphi(i)}$ is defined if and only if the metric product $\bigvee_{i=1}^{\infty} X_i$ is defined and that they are isometric.

b) Suppose given a sequence of metric spaces (X_i, ρ_i) for $i = 1, 2, \ldots$ and a natural number m. Show that the metric product $\bigvee_{i=m+1}^{\infty} X_i$ is defined if and only if the metric product $\bigvee_{i=1}^{\infty} X_i$ is defined and that the products $(\bigvee_{i=1}^{m} X_i) \times (\bigvee_{i=m+1}^{\infty} X_i)$ and $\bigvee_{i=1}^{\infty} X_i$ are isometric.

c) State and prove the analogues of Exercises (a) and (b) for the topological product of metric spaces (cf. Theorems 7.4.30 and 7.4.31).

d) Let a sequence of non-empty metric spaces (X_i, ρ_i) for $i = 1, 2, \ldots$ be given with the property that the series $\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^2$ is convergent and suppose $a_i \in X_i$ for $i = 1, 2, \ldots$ Show that for every natural number m the assignment taking the point $(x_1, x_2, \ldots, x_m) \in \bigvee_{i=1}^{m} X_i$ to the point $(x_1, x_2, \ldots, x_m, a_{m+1}, a_{m+2}, \ldots) \in \bigvee_{i=1}^{\infty} X_i$ is an isometric map of the metric product $\bigvee_{i=1}^{m} X_i$ onto a subset of the metric product $\bigvee_{i=1}^{\infty} X_i$.

e) Observe that the finite metric product $\bigvee_{i=1}^{m} X_i$ is isometric with the countable metric product $\bigvee_{i=1}^{\infty} X_i$ where X_i is a space consisting of one point whenever $i > m$.

f) Show that if the spaces X_i and Y_i are homeomorphic (uniformly homeomorphic) for $i = 1, 2 \ldots$, then the topological products $\bigvee_{i=1}^{\infty} X_i$ and $\bigvee_{i=1}^{\infty} Y_i$ are also homeomorphic (uniformly homeomorphic).

g) Show that the components of the topological product $\bigvee_{i=1}^{\infty} X_i$ coincide with the sets of the form $\bigvee_{i=1}^{\infty} C_i$ where C_i is a component of the space X_i for $i = 1, 2, \ldots$

h) Prove that if $(X, \rho) = \bigvee_{i=1}^{\infty} (X_i, \rho_i)$ and for each $i = 1, 2, \ldots$ the space X_i is pathwise connected, then the metric product X is pathwise connected. Deduce that the topological product of an infinite sequence of pathwise connected spaces is pathwise connected.

6.2. Spaces of maps

We now undertake a more detailed study of the space of maps, which was mentioned several times in Chapter 1. First of all we recall (see Example 1.1.9 and Supplement 1.S.13) that the formula

$$\tilde{\rho}(f,g) = \sup\{\rho(f(x),g(x)) : x \in X\}$$

defines a metric on the set $B(X,Y)$ of all bounded maps from a non-empty set X into the metric space (Y,ρ). We note that when the space (Y,ρ) is bounded, $B(X,Y)$ is the set of all maps from X into Y and so $\tilde{\rho}$ is a metric for the set of all maps of X into Y.

Spaces of maps are extremely useful in topology and its applications. The introduction of a metric into a set of maps allows the use of topological tools in the study of the set; these tools turn out to be particularly effective especially in proofs of the existence of certain maps (see Examples 6.4.4 and 6.4.6 and the proof of Theorem 6.8.18).

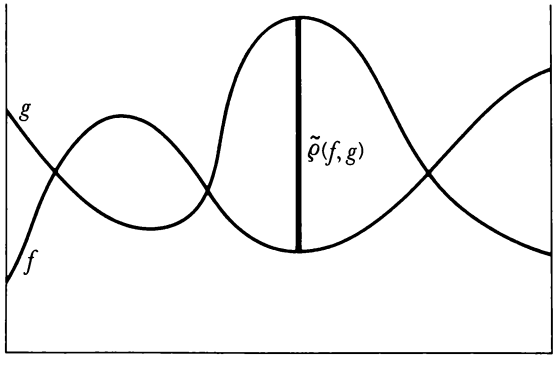


Fig.130. Measuring distance in the space of maps.

A most important case occurs when X is itself a metric space; the space of continuous maps can then also be considered. The set of all continuous maps from the space X into the space Y is denoted by the symbol $C(X,Y)$; its subset $BC(X,Y) = B(X,Y) \cap C(X,Y)$ consisting of bounded maps, is a subspace of the metric space $B(X,Y)$ defined above. Observe that when X is a compact space, we have by Theorem 1.8.2 and Corollary 1.8.11 the inclusion $C(X,Y) \subset B(X,Y)$ so that in this case $\tilde{\rho}$ is a metric for the set of all continuous maps of the space X into the space Y; the same is true when (Y,ρ) is a bounded space. Henceforth the symbols $B(X,Y)$, $C(X,Y)$ and $BC(X,Y)$ will denote not only the appropriate sets of maps but also the corresponding metric spaces as defined above, that is, the respective sets equipped with the metric $\tilde{\rho}$. We remark that the set of all continuous maps of the space X into the space Y is often denoted by the symbol Y^X. However, since this same symbol is used in set theory to denote the set of all maps of the set X into the set Y we have preferred a separate symbol $C(X,Y)$ for the set of continuous maps.

It turns out that convergence in $B(X,Y)$ coincides with uniform convergence.

6.2.1. THEOREM. *Let $f_0, f_1, f_2, \ldots$ be any sequence of elements of $B(X,Y)$. The equation $\lim_n f_n = f_0$ holds if and only if the sequence $f_1, f_2, \ldots$ is uniformly convergent to f_0.*

PROOF. The equation $\lim_n f_n = f_0$ signifies that $\lim_n \tilde{\rho}(f_n, f_0) = 0$, that is, that for every positive real number ϵ there exists an index k such that $\sup\{\rho(f_n(x), f_0(x)) : x \in X\} \leq \epsilon$ for $n \geq k$, or, equivalently that $\rho(f_n(x), f_0(x)) \leq \epsilon$ for $n \geq k$ and for every point $x \in X$. ■

From Theorems 6.2.1 and 1.5.15 we infer the following.

6.2.2. ASSERTION. *The set $BC(X,Y)$ is closed in the space $B(X,Y)$.* ■

The operation of forming the space of maps does not preserve either connectedness or compactness (see Problem 6.P.5 and Exercise (d); compare Theorem 6.2.9) but it does preserve completeness.

6.2.3 LEMMA. *If a sequence $f_1, f_2, \ldots$ of elements of the space $B(X,Y)$ satisfies the Cauchy condition for the metric $\tilde{\rho}$ and if for every point $x \in X$ the closure of the set $\{f_n(x) : n = 1,2,\ldots\}$ in the space Y is complete, then the sequence $f_1, f_2, \ldots$ is convergent.*

PROOF. It is easy to see that for each point $x \in X$ the sequence $f_1(x), f_2(x), \ldots$ satisfies the Cauchy condition for the metric ρ. In view of the completeness of the closure of the set $\{f_n(x) : n = 1,2,\ldots\}$ the sequence $f_1(x), f_2(x), \ldots$ is convergent; denote its limit by $f_0(x)$. We have thus defined a map f_0 of the set X into the space Y. It remains to prove that $f_0 \in B(X,Y)$ and that $f_0 = \lim_n f_n$. Of course it suffices to show that the sequence $f_1, f_2, \ldots$ converges uniformly to f_0. Let ϵ be any positive real number. There exists an index k such that $\tilde{\rho}(f_n, f_{n'}) < \epsilon/2$ for $n, n' \geq k$. We thus have $\rho(f_n(x), f_{n'}(x)) < \epsilon/2$ for all $n, n' \geq k$ and for every point $x \in X$. Now $\lim_n f_n(x) = f_0(x)$, so there exists an index $n'(x) \geq k$ such that $\rho(f_{n'(x)}(x), f_0(x)) < \epsilon/2$. From the triangle inequality we obtain

$$\rho(f_n(x), f_0(x)) \leq \rho(f_n(x), f_{n'(x)}(x)) + \rho(f_{n'(x)}(x), f_0(x)) < \epsilon,$$

for every $n \geq k$ and for every point $x \in X$; in other words, the sequence $f_1, f_2, \ldots$ uniformly converges to f_0. ■

Applying Lemma 6.2.3, Theorem 1.9.12 and Assertion 6.2.2 we obtain

6.2.4. THEOREM. *If Y is a complete space, then both spaces $B(X,Y)$ and $BC(X,Y)$ are complete.* ■

The notion of a bounded map of a space X into a space Y is not topological, as it depends on the choice of metric in the range space. It follows from Theorem 6.1.8 that the metric ρ on the space Y may be replaced by an equivalent metric (i.e. one leading to the same notion of convergence, see Supplement 1.S.16) σ under which all maps from X into Y will be bounded. At a first glance it may seem that this fact can be used, as in the case of the countable product, to define a "topological" operation of forming

a space of continuous maps from X into Y, namely the space $C(X,Y)$ with metric $\tilde{\sigma}$. It turns out however that convergence in the space $C(X,Y)$ in general depends on the choice of a bounded metric σ equivalent to ρ. Thus the proposed operation of forming a "topological" space of maps is not topological in character (see Exercise (a) and Problem 6.P.4).

We now prove an interesting theorem relating the space X and the space of bounded continuous maps of the space X into the real line $\mathbf{R}$ (cf. Problems 6.P.41 and 6.P.42).

6.2.5. THEOREM. *Every non-empty metric space X is isometric to a subset of the space of maps $BC(X,\mathbf{R})$.*

PROOF. Consider a non-empty metric space (X,ρ) and let σ denote the metric on the set $BC(X,\mathbf{R})$ determined by the usual metric on the real line, that is, we set $\sigma(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$ for $f,g \in BC(X,\mathbf{R})$. Fix a point $x_0 \in X$ and assign to each point $a \in X$ the function $f_a\colon X \to \mathbf{R}$ defined by the formula

$$f_a(x) = \rho(x,a) - \rho(x,x_0) \quad \text{for} \quad x \in X.$$

By the triangle inequality $|f_a(x)| \leq \rho(x_0,a)$ for each $x \in X$, so $f_a \in B(X,\mathbf{R})$. Using Example 1.3.7 we infer that $f_a \in BC(X,\mathbf{R})$. We shall show that

$$\sigma(f_a,f_b) = \rho(a,b) \quad \text{for all} \quad a,b \in X,$$

which will complete the proof of the theorem.

Observe first of all that for each $x \in X$ we have

$$f_a(x) - f_b(x) = \rho(x,a) - \rho(x,x_0) - \rho(x,b) + \rho(x,x_0) \leq \rho(b,a).$$

In view of the symmetry law for metrics, it follows that $|f_a(x) - f_b(x)| \leq \rho(a,b)$, and so

$$\sigma(f_a,f_b) \leq \rho(a,b).$$

But $f_a(b) - f_b(b) = \rho(b,a) - \rho(b,x_0) + \rho(b,x_0) = \rho(b,a)$, so we also have

$$\rho(a,b) \leq \sigma(f_a,f_b).$$

Hence the equation $\sigma(f_a,f_b) = \rho(a,b)$ holds. ■

We conclude this section with a very important condition for the compactness of subsets in a space of maps; this is known as Ascoli's Theorem. We begin by introducing the notion of equicontinuity of maps. Let F be a family of maps of a metric space X into a metric space Y; we shall use the same symbol ρ to denote the metrics in both spaces X and Y. We say that the maps in the family F are *equicontinuous* if for each point $x \in X$ and for every positive real number ϵ there is a positive real number δ such that for every map $f \in F$ if $x' \in X$ and $\rho(x,x') < \delta$ then $\rho(f(x),f(x')) < \epsilon$.

We now show that equicontinuous maps defined on a compact space are in some sense uniformly equicontinuous.

6.2.6. LEMMA. *Let F be a family of maps from a non-empty compact metric space X into a metric space Y. If the maps in F are equicontinuous, then for every positive real number ϵ there exists a positive real number δ such that for every map $f \in F$ if $x, x' \in X$ and $\rho(x,x') < \delta$ then $\rho(f(x), f(x')) < \epsilon$.*

PROOF. For each point $z \in X$ there exists a positive real number δ_z with the property that if $z' \in B(z;\delta_z)$, then $\rho(f(z), f(z')) < \frac{1}{2}\epsilon$ for each map $f \in F$. The sets $B(z,\delta_z)$ for $z \in X$ form an open covering of the space X. Let δ be the Lebesgue number of this covering (see Lemma 1.8.13). Thus for any pair of points $x, x' \in X$ such that $\rho(x,x') < \delta$ there exists a point $z \in X$ with the property that $x, x' \in B(z;\delta_z)$ and hence

$$\rho(f(x), f(x')) \leq \rho(f(x), f(z)) + \rho(f(z), f(x')) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

for each map $f \in F$. ■

The proof of Ascoli's Theorem will be preceded by two further lemmas.

6.2.7. LEMMA. *For any pair of metric spaces X, Y with $X \neq \emptyset$, the assignment taking the function $f \in BC(X,Y)$ and the point $x \in X$ to the point $f(x) \in Y$ is a continuous map of the metric product $BC(X,Y) \times X$ into the space Y.*

PROOF. The assertion is a consequence of the inequality

$$\rho(f(x), f'(x')) \leq \rho(f(x), f(x')) + \rho(f(x'), f'(x'))$$

and of a straightforward calculation which we leave to the reader. ■

6.2.8. LEMMA. *Let $f_1, f_2, \ldots$ be a sequence of maps of a non-empty metric space X into a metric space Y. Suppose that for every point a of a countable set A in the space X the closure of the set $\{f_j(a) : j = 1,2,\ldots\}$ in the space Y is compact, then there exists an increasing sequence of natural numbers $k_1, k_2, \ldots$ such that the sequence $f_{k_1}(a), f_{k_2}(a), \ldots$ converges for every $a \in A$.*

PROOF. Arrange the elements of the set A in a sequence $a_1, a_2, \ldots$, possibly with repetitions, and let $Y_i = \mathrm{cl}\{f_j(a_i) : j = 1,2,\ldots\} \subset Y$ for $i = 1,2,\ldots$ By Theorem 6.1.17 the topological product $\bigtimes_{i=1}^{\infty} Y_i$ is a compact space and so the sequence $x_1 = \{f_1(a_1), f_1(a_2), \ldots\}$, $x_2 = \{f_2(a_1), f_2(a_2), \ldots\}, \ldots$ of points in this product contains a convergent subsequence $x_{k_1}, x_{k_2}, \ldots$ By Theorem 6.1.9 it follows that the sequence $k_1, k_2, \ldots$ fulfils the statement of the lemma. ■

6.2.9. THEOREM (Ascoli). *Let F be a family of continuous maps from a non-empty compact metric space X into a metric space Y. The closure of F in the space $C(X,Y)$ is compact if and only if the maps in F are equicontinuous and for each point $x \in X$ the closure of the set $\{f(x) : f \in F\}$ in the space Y is compact.*

PROOF. Consider a set $F \subset C(X,Y)$ with compact closure. From Lemma 6.2.7 and Theorems 1.8.2, 1.8.4 and 1.8.3 it follows that for every point $x \in X$ the closure of the set $\{f(x) : f \in F\}$ in the space Y is compact. Suppose that the maps in F are not equicontinuous. Thus there is a point $x \in X$ and a positive real number ϵ and there

exists a sequence $x_1, x_2, \ldots$ of points of the space X and a sequence $f_1, f_2, \ldots$ of maps in F with the property that

$$\rho(x, x_n) < 1/n \quad \text{and} \quad \rho(f_n(x), f_n(x_n)) \geq \epsilon \quad \text{for} \quad n = 1, 2, \ldots$$

Using the compactness of the space $\operatorname{cl} F$ we can find an increasing sequence of natural numbers $k_1, k_2, \ldots$ such that the sequence $f_{k_1}, f_{k_2}, \ldots$ converges to a map $f_0 \in C(X, Y)$. From the continuity of metrics (Example 1.3.7) and from Lemma 6.2.7 it follows that $\rho(f_0(x), f_0(x)) \geq \epsilon > 0$, so we have reached a contradiction. This completes the proof of the necessity of the stated conditions.

Now consider a set $F \subset C(X, Y)$ satisfying the stated conditions. To complete the proof it is enough to show that every sequence $g_1, g_2, \ldots$ of maps belonging to $\operatorname{cl} F$ contains a convergent subsequence. By Lemma 1.8.10 for $i = 1, 2, \ldots$ there exists a finite sequence of points $a_1^i, a_2^i, \ldots, a_{n_i}^i$ of the space X with the property that

$$B(a_1^i; 1/i) \cup B(a_2^i; 1/i) \cup \ldots \cup B(a_{n_i}^i; 1/i) = X;$$

evidently the set $A = \{a_j^i : i = 1, 2, \ldots, \ j = 1, 2, \ldots, n_i\}$ is countable. For each $n = 1, 2, \ldots$ choose a map $f_n \in F$ with $\tilde{\rho}(f_n, g_n) < 1/n$, and then using Lemma 6.2.8 consider an increasing sequence of natural numbers $k_1, k_2, \ldots$ such that the sequence $f_{k_1}(a), f_{k_2}(a), \ldots$ converges for every $a \in A$.

We shall show that the sequence $f_{k_1}, f_{k_2}, \ldots$ satisfies the Cauchy condition in the space $C(X, Y)$. Let ϵ be any positive real number. By Lemma 6.2.6 there exists a natural number m such that if $x, x' \in X$ and $\rho(x, x') < 1/m$ then $\rho(f_{k_n}(x), f_{k_n}(x')) < \frac{1}{3}\epsilon$ for $n = 1, 2, \ldots$ There also exists an index k such that $\rho(f_{k_n}(a_j^m), f_{k_{n'}}(a_j^m)) < \frac{1}{6}\epsilon$ whenever $n, n' \geq k$ and $j = 1, 2, \ldots, n_m$. Now consider an arbitrary point $x \in X$. There exists a number $j \leq n_m$ such that $x \in B(a_j^m; 1/m)$; for $n, n' \geq k$ we have

$$\begin{aligned} \rho(f_{k_n}(x), f_{k_{n'}}(x)) &\leq \rho(f_{k_n}(x), f_{k_n}(a_j^m)) + \rho(f_{k_n}(a_j^m), f_{k_{n'}}(a_j^m)) + \rho(f_{k_{n'}}(a_j^m), f_{k_{n'}}(x)) \\ &< \frac{1}{3}\epsilon + \frac{1}{6}\epsilon + \frac{1}{3}\epsilon = \frac{5}{6}\epsilon. \end{aligned}$$

We infer that $\tilde{\rho}(f_{k_n}, f_{k_{n'}}) < \epsilon$ for $n, n' \geq k$, and so the sequence $f_{k_1}, f_{k_2}, \ldots$ satisfies the Cauchy condition in the space $C(X, Y)$.

From Lemma 6.2.3, Assertion 6.2.2 and Theorem 1.9.11 it follows that the sequence $f_{k_1}, f_{k_2}, \ldots$ converges. Now $\tilde{\rho}(f_{k_n}, g_{k_n}) < 1/k_n \leq 1/n$ for $n = 1, 2, \ldots$, so the subsequence $g_{k_1}, g_{k_2}, \ldots$ of the sequence $g_1, g_2, \ldots$ also converges. ■

Exercises

a) Let X be the discrete space of cardinality $\aleph_0$. Find equivalent bounded metrics ρ, ρ' for the real line $\mathbf{R}$ which determine the usual notion of convergence in $\mathbf{R}$, but for which the metrics $\tilde{\rho}$ and $\tilde{\rho}'$ on $C(X, \mathbf{R})$ are not equivalent.

b) Show that if the metrics ρ and ρ' on a space Y are bounded and uniformly equivalent (see Supplement 1.S.16), i.e. the identity map is a uniform homeomorphism

of (Y,ρ) and (Y,ρ'), then for every non-empty space X the metrics $\tilde{\rho}$ and $\tilde{\rho'}$ on the space $C(X,Y)$ are uniformly equivalent.

c) Check that if the space (X,ρ) is bounded, then the isometry from X onto a subspace of $BC(X,\mathbf{R})$ may be defined more simply than in the proof of Theorem 6.2.5 by assigning to each point $a \in X$ the map $f_a\colon X \to \mathbf{R}$ defined by the formula $f_a(x) = \rho(x,a)$.

d) Let ρ be the usual metric on the unit interval I. Using Example 1.5.14 show that the spaces $C(I,I)$ and $B(I,I)$ equipped with the metric $\tilde{\rho}$ are not compact.

e) Let X be a non-empty compact metric space and Y a closed and bounded subspace of the Euclidean space $\mathbf{R}^m$. Show that for every real number $c \geq 0$ the subspace of the space $C(X,Y)$ consisting of all Lipschitz maps with constant c is compact.

f) Let $f_1, f_2, \ldots$ be a sequence of maps from a non-empty compact metric space X into the Euclidean space $\mathbf{R}^m$. Show that if the maps $f_1, f_2, \ldots$ are equicontinuous and for each point $x \in X$ the set $\{f_i(x) : i = 1,2,\ldots\}$ is bounded, then the sequence $f_1, f_2, \ldots$ contains a subsequence which is uniformly convergent to a continuous map $f\colon X \to \mathbf{R}^m$.

6.3. Separable spaces

A metric space X is said to be *separable* if X contains a countable dense set, that is, if there exists a set $A \subset X$ with $\operatorname{card} A \leq \aleph_0$ such that $\operatorname{cl} A = X$. Thus the real line $\mathbf{R}$ and the unit interval I are separable, since the set of rational numbers Q and the set $Q \cap I$ are countable dense subsets of $\mathbf{R}$ and I respectively. A discrete space is separable if and only if it is countable; this is because proper subsets of discrete spaces cannot be dense. In particular the set of real numbers with the discrete metric is not a separable space.

We begin our study of separable spaces by verifying that separability is a topological concept. In fact we prove more, namely the following.

6.3.1. THEOREM. *If f is a continuous map of a separable space onto a space Y, then the space Y is also separable.*

PROOF. Let A be a countable dense set in X. Evidently the set $f(A) \subset Y$ is countable and from the continuity of f it follows that this set is dense in Y. ■

We proceed to a theorem containing two useful characterizations of separability (see Supplement 6.S.3). We precede this by introducing the important concept of a base of a space. A family $\mathcal{B} = \{V_s\}_{s\in S}$ consisting of open sets of a metric space X will be called a *base of the space* X, if every non-empty open set $U \subset X$ may be expressed as a union of some collection of members of $\mathcal{B}$, that is, there exists a set $S(U) \subset S$ such that $U = \bigcup_{s\in S(U)} V_s$. It is easy to check that a family $\mathcal{B}$ of subsets of the space X is a base of the space if and only if $\mathcal{B}$ consists of open subsets of X and for every point $x \in X$ and for every neighbourhood U of x there exists a set $V \in \mathcal{B}$ such that $x \in V \subset U$. We recall that a family of subsets $\{U_t\}_{t\in T}$ of a metric space X is called a covering of the space when $X = \bigcup_{t\in T} U_t$; if all the sets U_t are open, we say that the covering $\{U_t\}_{t\in T}$ is open.

6.3.2. LEMMA. *If, for $n = 1, 2, \ldots$, $\mathcal{B}_n$ is an open covering of the space X by sets of diameter less than $1/n$, then the family $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a base of the space X.*

PROOF. Consider any open set $U \subset X$ and a point $x \in U$. There exists a natural number n such that the open ball $B(x; 1/n)$ is contained in U. Now $\mathcal{B}_n$ is a covering of the space X, so the point x belongs to some element V of the covering. It follows from the inequality $\operatorname{diam} V < 1/n$ that $V \subset U$. Thus $V \in \mathcal{B}$ and $x \in V \subset U$. ■

6.3.3. THEOREM. *For every metric space X the following conditions are equivalent:*

(1) *X is separable,*

(2) *X has a countable base,*

(3) *every open covering of the space X has a countable subfamily which is a covering of the space X.*

PROOF. (1) $\Rightarrow$ (2). Let A be a countable dense subset of the space X. For each $n = 1, 2, \ldots$ let $\mathcal{B}_n$ denote the family of all open balls with centres belonging to the set A and with radii $1/3n$. We infer from the density of A that $\mathcal{B}_n$ is a covering of the space X. Since $\operatorname{card} \mathcal{B}_n \leq \aleph_0$ the union $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is countable. From Lemma 6.3.2 it follows that $\mathcal{B}$ is a base of the space X.

(2) $\Rightarrow$ (3). Let $\{V_n\}$, where n runs through the positive integers, be a countable base of the space X, and let $\{U_t\}_{t \in T}$ be any open covering of the space. Consider the set M consisting of all natural numbers n for which an index $t \in T$ exists with $V_n \subset U_t$ and assign to every $n \in M$ some index $t(n)$ such that $V_n \subset U_{t(n)}$. We shall show that the countable subfamily $\{U_{t(n)}\}_{n \in M}$ of $\{U_t\}_{t \in T}$ is a covering of the space. Indeed for every point $x \in X$ there exists an index $t_0 \in T$ such that $x \in U_{t_0}$ and a natural number n_0 such that $x \in V_{n_0} \subset U_{t_0}$; of course $n_0 \in M$ and so $x \in V_{n_0} \subset U_{t(n_0)} \subset \bigcup_{n \in M} U_{t(n)}$.

(3) $\Rightarrow$ (1). For $n = 1, 2, \ldots$ let us consider the covering of X consisting of all open balls of radius $1/n$ and choose a countable subfamily $\mathcal{A}_n$ which is a covering of the space. Let A be a set obtained by selecting one point from each element of the family $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. Of course $\operatorname{card} A \leq \aleph_0$. We shall check that the set A is dense in X. Let us consider an arbitrary point $x \in X$. For each $n = 1, 2, \ldots$ there exists a ball $K_n \in \mathcal{A}_n$ containing x; let a_n be the point of A selected from K_n. We have $\rho(x, a_n) \leq \operatorname{diam} K_n \leq 2/n$, so $x = \lim_n a_n$, i.e. x is a limit point of the set A. ■

We remark that from the Borel-Lebesgue Theorem (1.8.12) and from the equivalence of conditions (1) and (3) of Theorem 6.3.3 the following theorem may be deduced (cf. Exercise (e)).

6.3.4. THEOREM. *Every compact metric space is separable.* ■

However, there are no inclusions among the classes of separable spaces, connected spaces and complete spaces. The space of rational numbers is an example of a separable space which is neither complete nor connected. The discrete space of cardinality $\mathfrak{c}$ is a non-separable complete space. We give an example of a non-separable, connected (and complete) space which will be of use to us in the next chapter.

6.3.5. EXAMPLE. *The hedgehog with* $\mathbf{m}$ *spikes.* Consider an arbitrary set T of cardinality $\mathbf{m} \geq 1$. We define an equivalence relation R in the Cartesian product $I \times T$ by taking $(x_1, t_1)R(x_2, t_2)$ if and only if either $(x_1, t_1) = (x_2, t_2)$, or $x_1 = 0 = x_2$. It is readily verified that the formula

$$\rho([(x_1, t_1)], [(x_2, t_2)]) = \begin{cases} |x_1 - x_2|, & \text{if } t_1 = t_2, \\ x_1 + x_2, & \text{if } t_1 \neq t_2, \end{cases}$$

defines a metric on the set of equivalence classes of R.

For a fixed $\mathbf{m}$ the metric space so constructed does not depend (up to isometry) on the choice of the set T; we denote by $J(\mathbf{m})$ this metric space and call it the *hedgehog with* $\mathbf{m}$ *spikes.* It is easy to see that for each $t \in T$ the map j_t of the unit interval I into the space $J(\mathbf{m})$ defined by the formula $j_t(x) = [(x, t)]$ for $x \in I$, is an isometry onto the subspace $j_t(I) \subset J(\mathbf{m})$. Now $J(\mathbf{m}) = \bigcup_{t \in T} j_t(I)$ and $\bigcap_{t \in T} j_t(I) = [(0, t_0)]$, where t_0 is any element of T, so by Theorem 1.7.5 it follows that the space $J(\mathbf{m})$ is connected. Since the set $\{j_t(1/2) : t \in T\}$ is a discrete subspace of the hedgehog $J(\mathbf{m})$ and has cardinality $\mathbf{m}$, the space $J(\mathbf{m})$ is not separable for $\mathbf{m} > \aleph_0$. We leave to the reader the verification that $J(\mathbf{m})$ is complete. It is also easily checked that $J(\mathbf{m})$ is homeomorphic to the metric cone over a discrete space of cardinality $\mathbf{m}$. ■

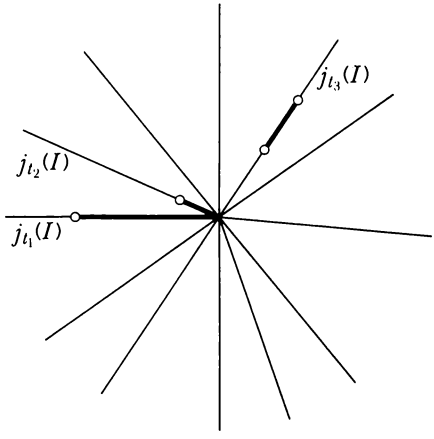


Fig.131. Measuring distance in the hedgehog with $\mathbf{m}$ spikes.

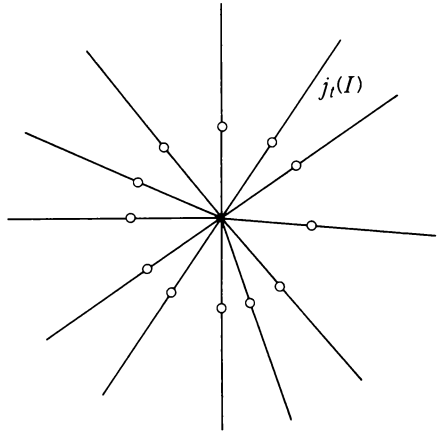


Fig.132. The hedgehog with $\mathbf{m}$ spikes contains a discrete subspace of cardinality $\mathbf{m}$.

We next study the behaviour of separability under various operations on spaces.

6.3.6. THEOREM. *If X is a separable metric space and A is a subset of X, then the subspace A is separable.*

PROOF. Let $\{V_n\}$, for n running through $1, 2, \ldots$, be a countable base of the space X. We infer from the form taken by the open subsets of a subspace (Theorem 1.6.5) that the family $\{A \cap V_n\}$, where n runs through the natural numbers, is a base of the subspace A and so the subspace has a countable base. ■

6.3.7. THEOREM. *The metric product of a finite number of separable metric spaces and also the metric and the topological products of an infinite sequence of separable metric spaces are all separable.*

PROOF. Both the case of a finite metric product and of a countable topological product reduces to the case of a countable metric product by an application respectively of Theorem 6.3.6 (cf. Exercise (e) of Section 6.1) and of Theorems 6.3.1 and 6.1.8. So it is enough to show that if $(X,\rho) = \bigtimes_{i=1}^{\infty}(X_i,\rho_i)$ and the spaces X_i are separable for $i = 1,2,\ldots$, then the metric product X is separable. We may of course assume that the spaces X_i are non-empty.

Choose for each $i = 1,2,\ldots$ a dense countable subset A_i of X_i, then arrange the elements of A_i as a sequence $a_1^i, a_2^i, \ldots$, possibly with repetitions. The subset A of the metric product X consisting of all points of the form $\{a_{n_1}^1, a_{n_2}^2, \ldots, a_{n_m}^m, a_1^{m+1}, a_1^{m+2}, \ldots\}$, where $m, n_1, n_2, \ldots, n_m$ are arbitrary natural numbers is countable. We shall show that the set A is dense in X.

Let $x = \{x_1, x_2, \ldots\}$ be an arbitrary point of X and ϵ any positive real number. The convergence of the series $\sum_{i=1}^{\infty}(\operatorname{diam} X_i)^2$ implies the existence of a natural number m such that $\sum_{i=m+1}^{\infty}(\operatorname{diam} X_i)^2 < \frac{1}{4}\epsilon^2$, that is $\sqrt{\sum_{i=m+1}^{\infty}(\operatorname{diam} X_i)^2} < \frac{1}{2}\epsilon$. Since A_i is dense in X_i it follows that there are natural numbers $n_1, n_2, \ldots, n_m$ for which $\rho_i(x_i, a_{n_i}^i) < \epsilon/(2\sqrt{m})$ for $i = 1,2,\ldots m$. Using Lemma 6.1.3 we infer that the point $a = \{a_{n_1}^1, a_{n_2}^2, \ldots, a_{n_m}^m, a_1^{m+1}, a_1^{m+2}, \ldots\} \in A$ satisfies the equation $\rho(x,a) < \epsilon$. Since x was arbitrary we have $\operatorname{cl} A = X$. ■

6.3.8. THEOREM. *If X is a non-empty compact metric space and Y a separable space, then the space $C(X,Y)$ is separable.*

PROOF. Let $\{V_n\}$, for n running through $1,2\ldots$, be a countable base of the space Y. By Lemma 1.8.10 for each $k = 1,2\ldots$ there exists a finite covering $\{B_i^k : i = 1,2,\ldots,m_k\}$ of the space X consisting of balls of radius $1/k$. For each natural number k and each finite sequence $n_1, n_2, \ldots, n_{m_k}$ consisting of m_k natural numbers we consider the set

$$(*) \qquad \{g \in C(X,Y) : g(B_i^k) \subset V_{n_i} \quad \text{for} \quad i = 1,2,\ldots,m_k\}.$$

Let $A \subset C(X,Y)$ be a set obtained by selecting one map from each non-empty set of the form $(*)$. Evidently $\operatorname{card} A \leq \aleph_0$. We shall show that the set A is dense in $C(X,Y)$.

Let f be any member of $C(X,Y)$ and ϵ any positive real number. Sets of the form $f^{-1}(V_n)$ where $\operatorname{diam} V_n < \epsilon$ form an open covering $\mathcal{U}$ of the space X. Let λ be the Lebesgue number of the covering $\mathcal{U}$ (see Lemma 1.8.13), i.e. a positive number λ with the property that any subset of X of diameter less than λ is contained in some member of $\mathcal{U}$. Fix a natural number k such that $2/k < \lambda$ and consider the covering $\{B_i^k : i = 1,2,\ldots,m_k\}$ of the space X. For each $i \leq m_k$ the set B_i^k is contained in some element of $\mathcal{U}$, that is, in a set of the form $f^{-1}(V_{n_i})$, where $\operatorname{diam} V_{n_i} < \epsilon$. The set $(*)$ corresponding to the natural number k and the sequence $n_1, n_2, \ldots, n_{m_k}$ is non-empty since it contains the map f. Accordingly some map g must have been selected from $(*)$ in the construction of A. Since the sets B_i^k for $i = 1,2,\ldots,m_k$ are a covering of X and for each $x \in B_i^k$ we have $f(x), g(x) \in V_{n_i}$, it follows that $\tilde{\rho}(f,g) \leq \epsilon$. We infer that $\operatorname{cl} A = C(X,Y)$. ■

We remark that it is not possible to drop the hypothesis of compactness of X in the last theorem. For, the space $C(\mathbf{N},D)$, where $\mathbf{N}$ is the space of natural numbers

and $D = \{0,1\}$ is the two-point discrete space, is an uncountable discrete space and hence is not separable. Similarly, it is not possible to replace $C(X,Y)$ by $B(X,Y)$ in the statement of Theorem 6.3.8; we leave it to the reader to provide an appropriate example.

We prove now an important theorem asserting that from the topological point of view the separable metric spaces coincide with the subspaces of the Hilbert cube $I^{\aleph_0}$ (cf. Problem 6.P.42). This is often put more succinctly by saying that the Hilbert cube is universal for the separable metric spaces (cf. Theorem 7.4.42).

6.3.9. THEOREM (Urysohn). *Every separable metric space is homeomorphic to some subspace of the Hilbert cube* $I^{\aleph_0}$.

PROOF. Consider an arbitrary non-empty separable metric space (X,ρ); by Theorem 6.1.8 we may assume that $\operatorname{diam} X \leq 1$. Arrange the elements of some countable dense subset A of X into a sequence $a_1, a_2, \ldots$ and associate with each point $x \in X$ the point $f(x) = \{\rho(x,a_1), \rho(x,a_2), \ldots\} \in I^{\aleph_0}$. We have thus defined a map $f: X \to I^{\aleph_0}$. It follows from the continuity of metrics (Example 1.3.7) and from Theorem 6.1.12 that the map f is continuous. To complete the proof it is enough to show that for every sequence of points $x_n \in X$, where $n = 0,1,2\ldots$, if $\lim_n f(x_n) = f(x_0)$ then $\lim x_n = x_0$, since by Corollary 1.5.7 this implication combined with the reverse implication, arising from the continuity of f, proves that f is a homeomorphism of the space X onto the subspace $f(X)$ of the Hilbert cube $I^{\aleph_0}$.

Let ϵ be any positive real number and a_i a point of A satisfying $\rho(x_0,a_i) < \frac{1}{3}\epsilon$. As $\lim_n f(x_n) = f(x_0)$, we have by Theorem 6.1.9 that $\lim_n \rho(x_n,a_i) = \rho(x_0,a_i)$. There therefore exists an index k such that $\rho(x_n,a_i) < \rho(x_0,a_i) + \frac{1}{3}\epsilon$ for $n \geq k$. Thus we have

$$\rho(x_n,x_0) \leq \rho(x_n,a_i) + \rho(a_i,x_0) < 2\rho(x_0,a_i) + \frac{1}{3}\epsilon < \epsilon$$

for $n \geq k$ and this proves that $\lim_n x_n = x_0$. ■

6.3.10. COROLLARY. *Every separable metric space has cardinality* $\leq \mathfrak{c}$. ■

The compactness of the Hilbert cube and Theorems 6.3.6, 6.3.4 and 6.3.9 together imply:

6.3.11. COROLLARY. *A metric space X is separable if and only if it is homeomorphic to a subspace of a compact metric space.* ■

Yet another interesting corollary follows from Theorem 6.3.9.

6.3.12. COROLLARY. *For every separable metric space X there exists a continuous map from a subset A of the Cantor set onto the space X. If moreover X is a compact space, there exists a continuous map from a closed subset of the Cantor set onto the space X.*

PROOF. By appeal to Theorem 6.3.9 we may assume that X is a subspace of the Hilbert cube $I^{\aleph_0}$. By Theorem 6.1.25 there exists a continuous map $F: C \to I^{\aleph_0}$ of the Cantor set onto the Hilbert cube. It is easy to see that the restriction of the map F to the set $A = F^{-1}(X) \subset C$ is a continuous map of A onto the space X. If X is a compact space, A is closed by Theorems 1.8.4 and 1.6.24. ■

We might add that if X is a non-empty compact space then there also exists a continuous map of the whole of the Cantor set onto X (see Problem 6.P.39).

We devote the concluding part of this section to totally bounded metric spaces; as Theorem 6.3.15 below shows, the notion of total boundedness is intimately related to the notion of separability (see Supplement 6.S.4).

We call a metric space X *totally bounded* if for every positive number ϵ there exists a finite covering of the space by sets of diameter less than ϵ. Of course every totally bounded space is bounded (see Lemma 1.4.9); the example of an infinite discrete space shows that the converse fails. However, it is easy to check that every bounded subset of the real line, or more generally of an m-dimensional Euclidean space, is totally bounded. Since every open interval and the real line are homeomorphic, we see that total boundedness is not a topological notion; it is, however, a notion of uniform topology (see Exercise (f)).

From the definition of total boundedness we obtain the following.

6.3.13. ASSERTION. *If X is a totally bounded metric space and A is a subset of X, then the subspace A is totally bounded.* ■

The metric product of a finite number of totally bounded spaces and also the metric and the topological products of an infinite sequence of totally bounded spaces are all totally bounded (see Exercise (g)).

6.3.14. LEMMA. *Every totally bounded space is separable.*

PROOF. Let X be a totally bounded space. For each $n = 1, 2, \ldots$ take a finite covering $\mathcal{A}_n$ of the space X consisting of sets of diameter less than $\frac{1}{3}n$. The generalized open balls

$$B(A; 1/3n) = \{x : \rho(x, A) < 1/3n\}, \quad \text{where} \quad A \in \mathcal{A}_n,$$

have diameter less than $1/n$ and constitute an open covering $\mathcal{B}_n$ of the space X (see Assertion 1.6.30). By Lemma 6.3.2 the union $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a base of the space X. Since $\operatorname{card} \mathcal{B}_n < \aleph_0$ for $n = 1, 2, \ldots$, we have $\operatorname{card} \mathcal{B} \leq \aleph_0$, that is, the space X is separable. ■

6.3.15. THEOREM. *A metric space is separable if and only if it is homeomorphic to a totally bounded space.*

PROOF. In view of Lemma 6.3.14 and Theorem 6.3.1 it is enough to show that every separable metric space is homeomorphic to a totally bounded space. This follows from Theorem 6.3.9 and Assertion 6.3.13, since Lemma 1.8.10 in fact asserts that every compact metric space, and in particular the Hilbert cube $I^{\aleph_0}$, is totally bounded. ■

The last theorem may be phrased differently to say that a space X with metric ρ is separable if and only if there exists a metric ρ' equivalent to ρ such that the space (X, ρ') is totally bounded. We then say that the metric ρ' is a *totally bounded metric on the space X*. Of course there may exist several totally bounded metrics on a space X, but they will all be equivalent.

In Chapter 1 we showed that every compact metric space is totally bounded (Lemma 1.8.10) and complete (Theorem 1.9.11). We show that these two properties together characterize compactness.

6.3.16. THEOREM. *A metric space is compact if and only if it is complete and totally bounded.*

PROOF. It is enough to show that if a metric space X is complete and totally bounded, then it is compact. Consider an arbitrary sequence of points $x_n \in X$ where $n = 1, 2, \ldots$; we prove that a convergent subsequence exists. Since X is complete it suffices to prove that a subsequence of $\{x_n\}$ may be selected which satisfies the Cauchy condition.

For each $n = 1, 2, \ldots$ we consider a finite covering $\mathcal{A}_n$ of the space X by sets of diameter less than $1/n$. Naturally a subsequence $x_{k_1^1}, x_{k_2^1}, \ldots$ may be selected from the sequence $x_1, x_2, \ldots$ so that all its terms lie in one and the same member of $\mathcal{A}_1$, and hence form a set of diameter less than 1. Similarly a subsequence $x_{k_1^2}, x_{k_2^2}, \ldots$ may be selected from the sequence $x_{k_1^1}, x_{k_2^1}, \ldots$ all of whose terms lie in one and the same member of $\mathcal{A}_2$ and so form a set of diameter less than $1/2$. Continuing inductively for $n = 3, 4, \ldots$ we define a subsequence $x_{k_1^n}, x_{k_2^n}, \ldots$ of the sequence $x_{k_1^{n-1}}, x_{k_2^{n-1}}, \ldots$ whose terms form a set of diameter less than $1/n$. Consider now the subsequence $x_{k_1^1}, x_{k_2^2}, x_{k_3^3}, \ldots$ of the sequence $x_1, x_2, \ldots$ Since the sequence $k_1^1, k_2^2, k_3^3, \ldots$ after dropping its first $n-1$ terms is a subsequence of the sequence $k_1^n, k_2^n, \ldots$, we have $\lim_n \operatorname{diam}\{x_{k_m^m} : m = n, n+1, \ldots\} = 0$, from which it obviously follows that $x_{k_1^1}, x_{k_2^2}, x_{k_3^3}, \ldots$ is a Cauchy sequence. ■

Exercises

a) Give an example of a countable base of the m-dimensional Euclidean space $\mathbf{R}^m$, where $m = 1, 2, \ldots$ Note that the space $\mathbf{R}^m$ has infinitely many different bases, some of them uncountable. Check that the hedgehog with $\mathfrak{m}$ spikes for $\mathfrak{m} \geq \aleph_0$ has a base of cardinality $\mathfrak{m}$ but has no bases of cardinality less than $\mathfrak{m}$.

b) Show that both the cardinality of the family of all open subsets and the cardinality of the family of all closed subsets of a separable metric space does not exceed $\mathfrak{c}$. Define a family of cardinality greater than $\mathfrak{c}$ consisting of dense subsets of the real line without interior points.

c) Observe that every family of pairwise disjoint, non-empty, open subsets of a separable space has cardinality $\leq \aleph_0$ (cf. Problem 6.P.7). Deduce that the isolated points of any subset of a separable metric space form a countable set.

d) Show that if X and Y are separable metric spaces then the set $C(X, Y)$ of all continuous maps of X into Y has cardinality $\leq \mathfrak{c}$; cf. Theorem 6.3.8 and the remark following it. (Hint: See Exercise (i) of Section 1.6).

e) Deduce Theorem 6.3.4 from Lemmas 1.8.10 and 6.3.2.

f) Show that if f is a uniformly continuous map of a totally bounded space X onto a metric space Y, then Y is also totally bounded.

g) Check that the metric product of a finite number of totally bounded spaces and also the metric and the topological products of an infinite sequence of totally bounded spaces are all totally bounded.

h) Show that a metric space X is totally bounded if and only if for each positive real number ϵ there exists a finite subset A of the space X such that $B(A;\epsilon) = X$, that is, a finite subset $A \subset X$ with the property that for every $x \in X$ there is $a \in A$ with $\rho(x,a) < \epsilon$. Deduce from this and the final section of the proof of Theorem 6.3.16 that the space X is totally bounded if and only if each sequence of points of the space contains a subsequence satisfying the Cauchy condition.

6.4. Complete spaces and completions

We introduced the notion of completeness in Section 1.9 and proved a few simple theorems on complete spaces. In Sections 6.1 and 6.2 we showed that the countable product of complete spaces and the space of maps with values in a complete space are both complete. In this section we continue our study of the class of complete spaces.

We begin with a variant of Cantor's Theorem (1.8.9).

6.4.1. THEOREM (Cantor). *If X is a complete space and $X \supset F_1 \supset F_2 \supset \ldots$, where $\emptyset \neq F_n = \operatorname{cl} F_n$ for $n = 1,2,\ldots$ and $\lim_n \operatorname{diam} F_n = 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

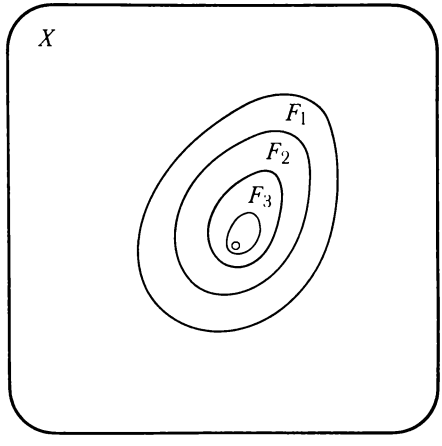


Fig.133. In a complete space every decreasing sequence of non-empty closed sets with diameters shrinking to zero intersects in one point (cf. Theorem 6.4.1).

PROOF. Choose points $x_n \in F_n$ for $n = 1,2,\ldots$ arbitrarily. Note that all the terms of the sequence $\{x_n\}$ with indices greater than k also lie in F_k; since $\lim_k \operatorname{diam} F_k = 0$, the sequence $\{x_n\}$ satisfies the Cauchy condition. But X is complete, so $\{x_n\}$ converges. Let $x = \lim_n x_n$. As each F_n is closed we have $x \in F_n$ for $n = 1,2,\ldots$ and thus $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. ■

Evidently the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

We shall now prove Baire's Theorem, an important result in view of its numerous applications in topology and analysis. Applying the theorem to an appropriately selected space one can give existence proofs for a variety of mathematical objects (the procedure is known as the *category method*; see Supplement 6.S.5). By way of example

we show below how to apply the theorem to a space of maps in order to deduce the existence of a continuous function $\varphi: \mathbf{R} \to \mathbf{R}$ which is not differentiable at any point.

6.4.2. THEOREM (Baire). *If X is a complete space and the sets $B_1, B_2, \ldots$ are closed in X and have no interior points then their union $B = \bigcup_{n=1}^{\infty} B_n$ has no interior points.* *

PROOF. We have to show that for every non-empty open set U of the space X the set $U \setminus B$ is non-empty.

Since the set B_1 has no interior points the difference $U \setminus B_1$ is non-empty. As this difference is open there is an open subset U_1 of the space X such that the closed set $F_1 = \operatorname{cl} U_1$ satisfies the conditions

$$F_1 \subset U \setminus B_1 \quad \text{and} \quad \operatorname{diam} F_1 \leq 1.$$

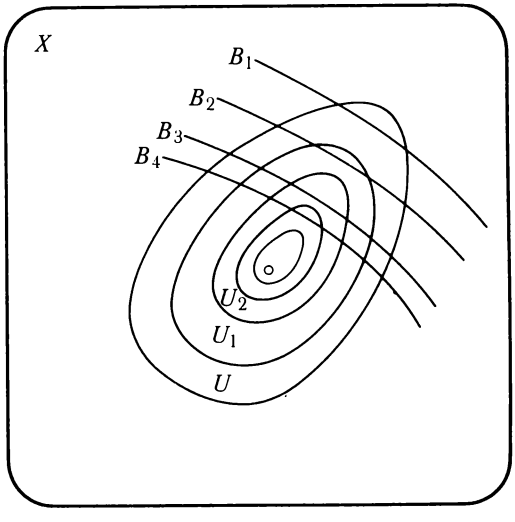


Fig.134. In a complete space the union of a countable number of closed sets without interior points has no interior points (see Theorem 6.4.2).

Similarly, using the fact that $U_1 \setminus B_2$ is non-empty and open we may find a non-empty open subset U_2 of the space X such that the set $F_2 = \operatorname{cl} U_2$ satisfies the conditions

$$F_2 \subset U_1 \setminus B_2 \quad \text{and} \quad \operatorname{diam} F_2 \leq 1/2.$$

For $n = 1, 2, \ldots$ we may inductively define a sequence U_n of non-empty open subsets of the space X such that the sets $F_n = \operatorname{cl} U_n$ satisfy the conditions

$$U \supset F_1 \supset F_2 \supset \ldots, \quad F_n \cap B_n = \emptyset \quad \text{and} \quad \operatorname{diam} F_n \leq 1/n \quad \text{for} \quad n = 1, 2, \ldots.$$

By Cantor's Theorem (6.4.1) the set $F = \bigcap_{n=1}^{\infty} F_n$ is non-empty; since $F \subset U$ and $F \cap B = (\bigcap_{n=1}^{\infty} F_n) \cap (\bigcup_{n=1}^{\infty} B_n) \subset \bigcup_{n=1}^{\infty} (F_n \cap B_n) = \emptyset$ we have $U \setminus B \neq \emptyset$. ■

* Since each B_i is closed, this is equivalent to demanding that each B_i is nowhere dense; see 6.S.5. Note of the Translator.

The next corollary embraces an often used dual version of Baire's Theorem.

6.4.3. COROLLARY. *If X is a complete space and the sets $G_1, G_2, \ldots$ are open and dense in the space X, then the intersection $G = \bigcap_{n=1}^{\infty} G_n$ is a dense set.* ■

We now give the promised example of an application of Baire's Theorem.

6.4.4. EXAMPLE. *A non-differentiable continuous function.* We shall define a function $\psi: I \to \mathbf{R}$ which is not differentiable at any point of the unit interval I. It is then easy to use this function to define a continuous function $\varphi: \mathbf{R} \to \mathbf{R}$ which is not differentiable at any point of the real line $\mathbf{R}$.

Let X be the function space $C(I, \mathbf{R})$ with metric σ specified by the formula $\sigma(f,g) = \sup\{|f(r) - g(r)| : r \in I\}$. By Theorem 6.2.4 the space X is complete. For each function $f \in C(I, \mathbf{R})$ each $r \in I$ and each positive real number $s \leq \frac{1}{2}$ at least one of the following two numbers is well defined:

$$\frac{|f(r+s) - f(r)|}{s}, \quad \text{or} \quad \frac{|f(r-s) - f(r)|}{s};$$

let $D(f,r,s)$ denote the greater of the two in the case when both are defined (or either one in case they are equal), otherwise let $D(f,r,s)$ denote the only one which is defined. Furthermore, let $D(f,r,s) = 0$ for $f \in C(I, \mathbf{R})$ when $r \in I$ and $s \in (\frac{1}{2}, 1)$. For $n = 1, 2, \ldots$ put

$$G_n = \bigcup_{a>n} \bigcup_{s<1/n} \{f \in X : D(f,r,s) \geq a \text{ for all } r \in I\}.$$

Of course our objective is to show that $\bigcap_{n=1}^{\infty} G_n \neq 0$, since every function $\psi \in \bigcap_{n=1}^{\infty} G_n$ has the desired property. By Corollary 6.4.3 it is enough to show that the sets G_n are open and dense in the space X (see Exercise (m) of Section 7.5).

We begin by showing that the sets G_n are dense in X. Let g be any element of X and ϵ any positive real number. We define a function $f \in G_n$ such that $\sigma(f,g) < \epsilon$. First of all using Heine's Theorem (1.8.14) we partition the interval I by means of points $0 = r_0 < r_1 < \ldots < r_m = 1$ such that $\operatorname{diam} g([r_{i-1}, r_i]) < \frac{1}{4}\epsilon$ for $i = 1, 2, \ldots, m$. Next for $i = 1, 2, \ldots, m$ choose points $a_i \in [r_{i-1}, r_i]$ with $r_i - a_i \leq [1/(n+1)]\,|g(r_i) - g(r_{i-1})|$ and consider the function $f_0 \in X$ defined by the formula

$$f_0(r) = \begin{cases} g(r_{i-1}), & \text{for } r \in [r_{i-1}, a_i], \\ g(r_{i-1}) + \dfrac{g(r_i) - g(r_{i-1})}{r_i - a_i}(r - a_i), & \text{for } r \in [a_i, r_i]. \end{cases}$$

It is easy to see that the graph of the function f_0 is a broken line consisting of line segments parallel to the axis of abscissae and of line segments lying on lines with equation of the form $x^2 = bx^1 + c$ where $|b| \geq n + 1$. Moreover $\sigma(f_0, g) < \frac{1}{2}\epsilon$, since $\operatorname{diam} f_0([r_{i-1}, r_i]) < \frac{1}{4}\epsilon$ for $i = 1, 2, \ldots, m$. The required function f may be obtained from the function f_0 by replacing each segment of the graph of f_0 parallel to the axis of abscissae by 'teeth' of height $\frac{1}{2}\epsilon$ with edges lying on straight lines that also have equations of the form $x^2 = bx^1 + c$ with $|b| \geq n + 1$. The graph of the function f is thus also a broken line. Let d denote the minimum length of the projections of the straight

segments of this broken line onto the axis of abscissae; evidently $d > 0$. The reader will easily verify that $D(f, r, s) \geq n + 1$ for each $r \in I$ when $s = \min[1/(n+1), d/2]$; and so $f \in G_n$. Of course $\sigma(f, g) < \epsilon$.

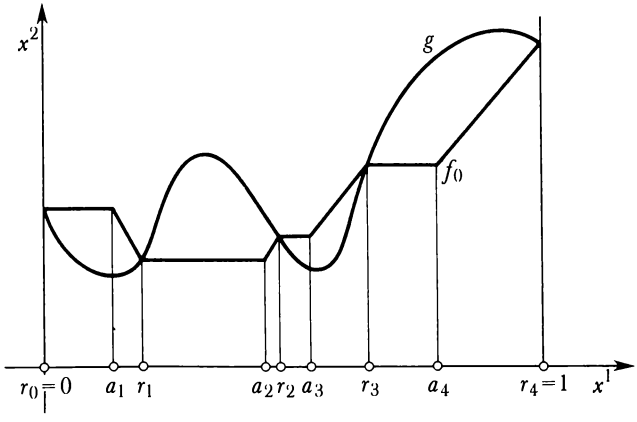


Fig.135. To show that G_n is dense in X we first define an auxiliary function f_0 whose graph is a broken line (Example 6.4.4).

Now we show that the sets G_n are open. Consider any function $f \in G_n$ and fix $a > n$ and $s < 1/n$ such that $D(f, r, s) \geq a$ for each $r \in I$. The reader will easily verify that if $\sigma(f, g) < \epsilon = s(a - n)/4$ then $D(g, r, s) \geq (a + n)/2$ for each $r \in I$ and so the ball $B(f; \epsilon)$ is contained in the set G_n.

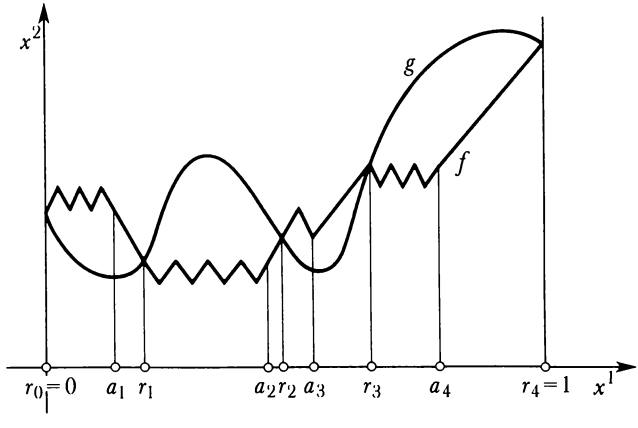


Fig.136. The function f is obtained from f_0 by replacing each horizontal segment of the graph of f_0 by steep "teeth" (Example 6.4.4).

We note that the application of Baire's Theorem leads to "quantitative" results; in proving the existence of a singular function $\psi: I \to \mathbf{R}$ we showed that the set of functions with the singular property under discussion is dense in the space $C(I, \mathbf{R})$. ■

The next theorem, like Baire's Theorem, finds wide application in analysis. It is one of several fixed point theorems and so asserts the existence of a certain point.

Applied to an appropriately chosen function space it provides the existence of functions satisfying certain conditions, for example the existence of a solution of a differential equation. By way of illustration we shall show below how to obtain from it an implicit function theorem; another application is given in Problem 6.P.16. Recall that a map f of a metric space into itself is called contractive (see Supplement 1.S.8) if f is a Lipschitz map with constant $c < 1$, that is, if there exists a number $c \in [0,1)$ such that $\rho(f(x), f(y)) \leq c\rho(x,y)$ for any two points $x, y \in X$.

6.4.5. THEOREM (Banach). *Every contractive map f of a complete space X into itself has exactly one fixed point.*

PROOF. Let f be a Lipschitz map with constant $c \in [0,1)$. Take an arbitrary point $x \in X$ and consider the sequence $x_1 = f(x)$, $x_2 = f(x_1) = f^2(x)$, $x_3 = f(x_2) = f^3(x), \ldots$ of images under successive iteration of the map f. It is not difficult to check that $\rho(f^n(x), f^{n+1}(x)) \leq c^n \rho(x, f(x))$, so for $n' \geq n \geq k$ we have

$$\begin{aligned}\rho(x_n, x_{n'}) &= \rho(f^n(x), f^{n'}(x)) \\ &\leq \rho(f^n(x), f^{n+1}(x)) + \rho(f^{n+1}(x), f^{n+2}(x)) + \ldots + \rho(f^{n'-1}(x), f^{n'}(x)) \\ &\leq (c^n + c^{n+1} + \ldots + c^{n'-1})\rho(x, f(x)) \\ &\leq \frac{c^k}{1-c}\rho(x, f(x)).\end{aligned}$$

Because $c < 1$, the inequality $\rho(x_n, x_{n'}) \leq [c^k/(1-c)]\rho(x, f(x))$ just obtained shows that the sequence $x_1, x_2, \ldots$ satisfies the Cauchy condition. We prove that the limit $x_0 \in X$ of the sequence is a fixed point of the map f. It follows from the continuity of f that $\lim_n f(x_n) = f(x_0)$, but since $f(x_n) = x_{n+1}$ we also have $\lim_n f(x_n) = \lim_n x_{n+1} = x_0$. Thus $f(x_0) = x_0$. To prove the uniqueness of the fixed point x_0, observe that if $f(y_0) = y_0$ then $\rho(x_0, y_0) = \rho(f(x_0), f(y_0)) \leq c\rho(x_0, y_0)$ so $\rho(x_0, y_0) = 0$, that is $x_0 = y_0$. ■

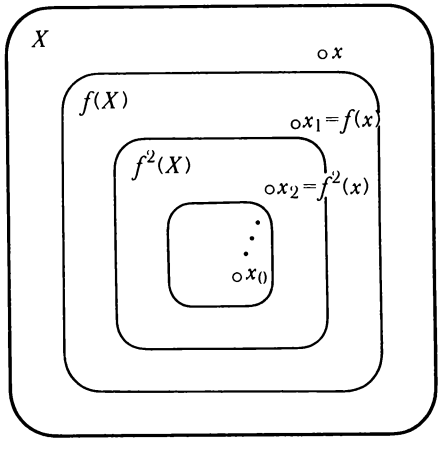


Fig.137. The fixed point of a contractive map f is the limit of the sequence $f(x), f^2(x), f^3(x), \ldots$ (Theorem 6.4.5).

We now give the promised example of an application of Banach's fixed point theorem.

6.4.6. EXAMPLE. *Implicit function theorem.* We prove that for every continuous function $\Phi \in C(K, \mathbf{R})$ defined on a square K centered on $(x_0^1, x_0^2) \in \mathbf{R}^2$ which has a continuous partial derivative Φ'_{x^2} in K and satisfies the two conditions $\Phi(x_0^1, x_0^2) = 0$ and $\Phi'_{x^2}(x_0^1, x_0^2) \neq 0$, there exists a neighbourhood U of the point x_0^1 and an interval $B = [x_0^2 - a, x_0^2 + a]$, for some $a > 0$, with the property that $U \times B \subset K$ and there is a unique continuous function $\varphi\colon U \to B$ such that $\varphi(x_0^1) = x_0^2$ and $\Phi(x^1, \varphi(x^1)) = 0$ for every $x^1 \in U$.

Consider the function $\Psi \in C(K, \mathbf{R})$ defined by the formula

$$\Psi(x^1, x^2) = x^2 - x_0^2 - \frac{\Phi(x^1, x^2)}{\Phi'_{x^2}(x_0^1, x_0^2)};$$

evidently $\Psi(x_0^1, x_0^2) = 0 = \Psi'_{x^2}(x_0^1, x_0^2)$. By possibly shrinking the square K we may assume that $K = A \times B$ where $A = [x_0^1 - a, x_0^1 + a]$, $B = [x_0^2 - a, x_0^2 + a]$ with $a > 0$ and that for every point $(x^1, x^2) \in K$ the inequality $|\Psi'_{x^2}(x^1, x^2)| < 1/2$ is satisfied. The set $U = \operatorname{int}\{x^1 \in A : |\Psi(x^1, x_0^2)| < \frac{1}{2}a\}$ is a neighbourhood of the point x_0^1. First of all, observe that for $x^1 \in A$ and $x_1^2, x_2^2 \in B$ the mean value theorem tells us that

$$(*) \qquad |\Psi(x^1, x_1^2) - \Psi(x^1, x_2^2)| = |\Psi'_{x^2}(x^1, x^2)||x_1^2 - x_2^2| < \frac{1}{2}|x_1^2 - x_2^2|$$

for some $x^2 \in [x_1^2, x_2^2]$.

Now consider the function space $C(U, B)$ with metric σ defined by the formula $\sigma(f, g) = \sup\{|f(r) - g(r)| : r \in U\}$ and also the closed subspace $X = \{f \in C(U, B) : f(x_0^1) = x_0^2\}$. By Theorems 6.2.4 and 1.9.12 the space X is complete. Associate with each function $f \in X$ the function $F(f)$ defined by the formula

$$[F(f)](x^1) = x_0^2 + \Psi(x^1, f(x^1)) \quad \text{for} \quad x^1 \in U.$$

By $(*)$ we have

$$\begin{aligned} |[F(f)](x^1) - x_0^2| &= |\Psi(x^1, f(x^1)| \\ &\leq |\Psi(x^1, f(x^1)) - \Psi(x^1, x_0^2)| + |\Psi(x^1, x_0^2)| \\ &< \frac{1}{2}|f(x^1) - x_0^2| + \frac{1}{2}a \leq \frac{1}{2}a + \frac{1}{2}a = a, \end{aligned}$$

and of course $[F(f)](x_0^1) = x_0^2$, so $F(f) \in X$. We have thus specified a map F of the space X into itself. Applying $(*)$ a second time, we infer for every $f, g \in X$ and every $x^1 \in U$ that

$$\begin{aligned} |[F(f)](x^1) - [F(g)](x^1)| &= |\Psi(x^1, f(x^1)) - \Psi(x^1, g(x^1))| \\ &\leq \frac{1}{2}|f(x^1) - g(x^1)| \leq \frac{1}{2}\sigma(f, g), \end{aligned}$$

hence $\sigma(F(f), F(g)) \leq \frac{1}{2}\sigma(f, g)$, so that F is a contractive map. By Banach's fixed point theorem there exists a continuous function $\varphi \in X$ such that $F(\varphi) = \varphi$. The reader may easily verify that φ has the required properties. The uniqueness of φ follows from the uniqueness of the fixed point of F. ■

We shall now define for any metric space X its completion $\tilde{X}$, that is a complete space which contains a dense subspace isometric to X. It will actually turn out that the space $\tilde{X}$ is uniquely determined up to isometry. The enlargement of a set by the addition of points which in some sense are missing from the original set is a frequent process in mathematics and leads to many interesting notions (see Supplement 6.S.6). We should add that the operation of completion may also be defined on pseudometric spaces (see Supplements 1.S.1 and 6.S.6).

6.4.7. THEOREM. *For any metric space X there exists up to isometry exactly one complete space $\tilde{X}$ which contains a dense subspace isometric to X. Furthermore, if X is totally bounded, then $\tilde{X}$ is compact.*

PROOF. We infer from Theorems 6.2.5 and 6.2.4 that there exists an isometry of X onto a subspace of the complete space $BC(X, \mathbf{R})$; for $\tilde{X}$ we may therefore take the closure of the image of the space X in $BC(X, \mathbf{R})$ under the isometry.

We now need to show that $\tilde{X}$ is unique. Of course it is enough to check that if Y and Z are complete spaces and A and B are dense subsets of Y and Z respectively, then for every isometry $f: A \to B$ there exists an isometry $F: Y \to Z$ such that $F(y) = f(y)$ for every $y \in A$. Let y be any point of the space Y; consider a sequence $a_1, a_2, \ldots$ of points of A converging to y. By Theorem 1.9.2 the sequence satisfies the Cauchy condition, hence the sequence $b_1, b_2, \ldots$ where $b_n = f(a_n)$ for $n = 1, 2, \ldots$ also satisfies the Cauchy condition and converges to some point $z \in Z$. The point z does not depend on the choice of sequence $a_1, a_2, \ldots$ For, if $a'_1, a'_2, \ldots$ is any sequence of points of A converging to y then the sequence $b_1, b'_1, b_2, b'_2, \ldots$, where $b'_n = f(a'_n)$ for $n = 1, 2, \ldots$, is convergent and so $\lim_n b'_n = \lim_n b_n$. It follows from this that, in particular, if $y \in A$ then $F(y) = f(y)$. We define the map $F: Y \to Z$ by assigning to the point $y \in Y$ the point $z \in Z$ as determined above. Using the completeness of Y it is easy to verify that $F(Y) = Z$. Since A is dense in the space Y and B is dense in the space Z, it follows immediately that F is an isometry.

We are left to prove that if X is totally bounded then the space $\tilde{X}$ is compact. By Theorem 6.3.16 it is enough to show that $\tilde{X}$ is totally bounded. Consider an arbitrary positive real number ϵ and a finite covering $A_1, A_2, \ldots, A_k$ of the space X by sets of diameter less than ϵ. The closures of the images of these sets in $\tilde{X}$ have diameters less than ϵ and form a finite covering of the space $\tilde{X}$ in view of the density of the image of X in $\tilde{X}$. ■

The space $\tilde{X}$ which satisfies the conclusion of Theorem 6.4.7 is called the *completion of the space* X. Of course X may be regarded as being a dense subspace of its completion $\tilde{X}$.

A more intuitive construction of $\tilde{X}$, though requiring longer calculation, is sketched in Problem 6.P.20.

We now define the $\mathcal{G}_\delta$-sets which arise in a natural way in the course of our study of complete spaces. We say that a subset A of a metric space is a *$\mathcal{G}_\delta$-set* in X if A may be expressed as the intersection of a countable number of open sets of X. The complements of $\mathcal{G}_\delta$-sets are called *$\mathcal{F}_\sigma$-sets*. Evidently a subset A of a metric space X is an $\mathcal{F}_\sigma$-set in X if and only if it may be expressed as the union of a countable number of

closed sets of X. It is easy to see that the set of rational numbers is an $\mathcal{F}_\sigma$-set in $\mathbf{R}$ and that the set of irrational numbers is a $\mathcal{G}_\delta$-set in $\mathbf{R}$ (see Exercise (c)).

Starting with the open sets and taking alternately countable intersections and countable unions of sets previously defined, we obtain increasingly wider classes of subsets of a given space; the open sets are followed by the $\mathcal{G}_\delta$-sets, these in turn by countable unions of $\mathcal{G}_\delta$-sets (known as the *$\mathcal{G}_{\delta\sigma}$-sets*) etc. Similarly starting with the closed sets we obtain the class of $\mathcal{F}_\sigma$-sets, then the class of countable intersections of $\mathcal{F}_\sigma$-sets (known as the *$\mathcal{F}_{\sigma\delta}$-sets*) etc. Both hierarchies are dual to each other. More detailed information is provided in Supplement 6.S.7; here we shall only be concerned with the $\mathcal{G}_\delta$-sets and the $\mathcal{F}_\sigma$-sets.

6.4.8. THEOREM. *The union (intersection) of two $\mathcal{G}_\delta$-sets ($\mathcal{F}_\sigma$-sets) is again a $\mathcal{G}_\delta$-set (an $\mathcal{F}_\sigma$-set). A countable union (intersection) of $\mathcal{F}_\sigma$-sets ($\mathcal{G}_\delta$-sets) is again an $\mathcal{F}_\sigma$-set (a $\mathcal{G}_\delta$-set).*

PROOF. Consider two $\mathcal{G}_\delta$-sets G and H in X. Let $G = \bigcap_{n=1}^{\infty} G_n$ and $H = \bigcap_{n=1}^{\infty} H_n$, where G_n and H_n are open for $n = 1, 2, \ldots$ Since $G \cup H = \bigcap_{n,m=1}^{\infty}(G_n \cup H_m)$, we see that $G \cup H$ is a $\mathcal{G}_\delta$-set. It follows from De Morgan's Laws that the intersection of two $\mathcal{F}_\sigma$-sets is again an $\mathcal{F}_\sigma$-set. The second part of the theorem is obvious. ■

6.4.9. THEOREM. *Every closed subset A of a metric space X is a $\mathcal{G}_\delta$-set in X.*

PROOF. In view of Theorem 1.4.10 the formula $f(x) = \rho(x, A)$ for $x \in X$ defines a continuous function $f: X \to \mathbf{R}_+$. But the set A is closed, so by Assertion 1.6.7 we have $A = f^{-1}(0)$. We then also have

$$A = f^{-1}(0) = f^{-1}\left(\bigcap_{n=1}^{\infty}[0, 1/n)\right) = \bigcap_{n=1}^{\infty} f^{-1}([0, 1/n)),$$

and by Theorem 1.6.24 it follows that A is a $\mathcal{G}_\delta$-set in X. ■

We now prove a theorem on the extension of maps with values in a complete metric space. We begin by defining the oscillation of a map in a context which is more general than that considered in Supplement 1.S.10. Suppose given a map $f: A \to Y$ defined on a dense subset A of a metric space X taking values in a metric space Y. The *oscillation of the map f at the point $x \in X$* is the number

$$o_f(x) = \inf\{\operatorname{diam} f(A \cap B(x; \delta)) : \delta > 0\}.$$

For any positive real number ϵ the set $\{x \in X : o_f(x) < \epsilon\}$ is open. Indeed, if $o_f(x_0) < \epsilon$ then there is a number $\delta_0 > 0$ such that $\operatorname{diam} f(A \cap B(x_0; \delta_0)) < \epsilon$, but then for every point $x \in B(x_0; \delta_0)$ taking $\delta = \delta_0 - \rho(x_0, x) > 0$ the inclusion $f(A \cap B(x; \delta)) \subset f(A \cap B(x_0; \delta_0))$ holds, so $\operatorname{diam} f(A \cap B(x; \delta)) < \epsilon$ whence it follows that $o_f(x) < \epsilon$. We conclude that the set $B = \{x \in X : o_f(x) = 0\}$ is a $\mathcal{G}_\delta$-set in X; of course, if $f: A \to Y$ is a continuous map then $A \subset B$.

6.4.10. LEMMA. *Let A be a dense subset of a metric space X. For every continuous map $f: A \to Y$, where Y is a complete space, there exists a continuous extension $f^*: B \to Y$, where $B = \{x \in X : o_f(x) = 0\}$.*

PROOF. By Cantor's Theorem (6.4.1) for each $x \in B$ the intersection $\bigcap_{n=1}^{\infty} \operatorname{cl} f(A \cap B(x; 1/n))$ is a singleton in the space Y. Denote by $f^*(x)$ the unique point of the intersection. From the definition it follows that $f^*(x) = f(x)$ when $x \in A$. It remains to show that the map $f^*: B \to Y$ thus defined is continuous. Consider any point $x_0 \in B$ and any positive real number ϵ. From the definition of the set B follows the existence of a positive real number δ such that $\operatorname{diam} f(A \cap B(x_0; \delta)) < \epsilon$. For each $x \in B(x_0; \delta)$ there exists a natural number n such that $B(x_0; 1/n) \cup B(x; 1/n) \subset B(x_0; \delta)$; but then the inclusion

$$\operatorname{cl} f(A \cap B(x_0; 1/n)) \cup \operatorname{cl} f(A \cap B(x; 1/n)) \subset \operatorname{cl} f(A \cap B(x_0; \delta))$$

holds. Now, if $x \in B$, the points $f^*(x_0)$ and $f^*(x)$ belong to the set on the left-hand side of the inclusion, while the diameter of the right-hand side does not exceed ϵ, hence $f^*(x) \in B(f^*(x_0); \epsilon)$ for each $x \in B(x_0; \delta)$. We deduce that f^* is a continuous map. ■

Two theorems are immediate consequences of the lemma just proved.

6.4.11. THEOREM. *Let A be a dense subset of a metric space X. For every continuous map $f: A \to Y$, where Y is a complete space, there exists a $\mathcal{G}_\delta$-set B in X containing A and a continuous extension $f^*: B \to Y$.* ■

6.4.12. THEOREM. *Let A be a dense subset of a metric space X. For every uniformly continuous map $f: A \to Y$, where Y is a complete space, there exists a continuous extension $f^*: X \to Y$.* ■

6.4.13. COROLLARY. *Let X be any metric space and $\tilde{X}$ its completion. For every uniformly continuous map $f: X \to Y$, where Y is a complete space, there exists a continuous extension $\tilde{f}: \tilde{X} \to Y$.* ■

We now prove an important theorem on the extension of homeomorphisms between subspaces of complete spaces.

6.4.14. THEOREM (Lavrentiev). *Let X and Y be complete spaces and let $A \subset X$ and $C \subset Y$ be any subspaces. For each homeomorphism $f: A \to C$ there is a homeomorphism $f^*: B \to D$ such that $f^*(x) = f(x)$ for each $x \in A$, where $A \subset B \subset X$ and $C \subset D \subset Y$ and B and D are $\mathcal{G}_\delta$-sets.*

PROOF. Note first of all that by Theorem 6.4.9 we may assume that $\operatorname{cl} A = X$ and $\operatorname{cl} C = Y$ because, as may easily be checked, any $\mathcal{G}_\delta$-set in a subspace where the subspace is also a $\mathcal{G}_\delta$-set, is itself a $\mathcal{G}_\delta$-set in the original space. Let $g: C \to A$ be the inverse map of f. By Theorem 6.4.11 there exist continuous extensions $f_0^*: B_0 \to Y$ and $g_0^*: D_0 \to X$ for the respective maps f and g defined on $\mathcal{G}_\delta$-sets B_0 and D_0. Notice that

$$(*) \quad f_0^*(x) \in D_0 \quad \text{implies} \quad g_0^* f_0^*(x) = x, \quad \text{and} \quad g_0^*(y) \in B_0 \quad \text{implies} \quad f_0^* g_0^*(y) = y.$$

For, taking any sequence $x_1, x_2, \ldots$ of points of A converging to x, we have by the continuity of f_0^* and g_0^* and the identity $gf = \mathrm{id}_A$ that

$$g_0^* f_0^*(x) = \lim_n g_0^* f_0^*(x_n) = \lim_n gf(x_n) = \lim_n x_n = x.$$

The proof of the second part of (*) is similar. Since the preimage of a $\mathcal{G}_\delta$-set under a continuous map is a $\mathcal{G}_\delta$-set, the sets

$$B = B_0 \cap f_0^{*-1}(D_0) \quad \text{and} \quad D = D_0 \cap g_0^{*-1}(B_0)$$

are $\mathcal{G}_\delta$-sets; of course $A \subset B \subset X$ and $C \subset D \subset Y$. Consider the restrictions $f^* = f_0^* | B$ and $g^* = g_0^* | D$. To complete the proof it is enough to prove that $f^*(B) = D$ and $g^*(D) = B$, since by (*) it then follows that g^* is the inverse map of f^*. Consider any point $x \in B$. By the definition of B we have $f_0^*(x) \in D_0$; hence, using (*), we infer that $g_0^* f_0^*(x) = x \in B \subset B_0$, and so $f_0^*(x) \in D_0 \cap g_0^{*-1}(B_0) = D$, that is, $f^*(B) \subset D$. Now consider any point $y \in D$. By the definition of D we have $g_0^*(y) \in B_0$; hence, again using (*), we infer that $f_0^* g_0^*(y) = y \in D \subset D_0$, and hence $g_0^*(y) \in B_0 \cap f_0^{*-1}(D_0) = B$ and so $y = f_0^* g_0^*(y) = f^* g_0^*(y) \in f^*(B)$, that is, $D \subset f^*(B)$. Thus we have $f^*(B) = D$. The proof that $g^*(D) = B$ is similar. ■

The reader has doubtless noticed that in some of the theorems on complete spaces – for instance Baire's Theorem or Lavrentiev's Theorem – completeness is the only assumption of a metric character, that is to say, the only assumption which refers to a fixed metric on the space considered. In contrast, in some of the other theorems there are additional assumptions referring to the specific metric: for instance in Cantor's Theorem, the assumption that the diameter of the sets F_n tends to zero, or in Banach's fixed point theorem, the assumption that f is a contractive map. Theorems of the first type are true also for, say, the open interval $(0,1)$, which, though not complete, is nevertheless homeomorphic to a complete space, namely the real line $\mathbf{R}$. This observation leads us to define an important class of metric spaces: we say that the space X with metric ρ is *completely metrizable* if there is a metric ρ' on the set X equivalent to ρ such that the space (X, ρ') is complete. We then call ρ' a *complete metric on the space* X. Of course there may exist several complete metrics on a space X, but they will all be equivalent. It is easy to see that a space X is completely metrizable if and only if it is homeomorphic to a complete metric space (see Supplement 6.S.4). It transpires that Baire's Theorem and Lavrentiev's Theorem are true in completely metrizable spaces. The same applies to Theorem 6.4.11. Let us look at Baire's Theorem by way of illustration. Let (X, ρ) be a completely metrizable space and let $B_1, B_2, \ldots$ be a sequence of closed sets with no interior points. Consider a complete metric ρ' on the space X. The sets $B_1, B_2, \ldots$ are closed and have no interior points in the space (X, ρ'), since "being closed" and "not having interior points" are topological properties and depend only on convergence in X and that is unaltered when ρ is replaced by an equivalent metric ρ'. Baire's Theorem applied to the complete space (X, ρ') implies that the union $B = \bigcup_{n=1}^{\infty} B_n$ has no interior points in that space. Of course B has no interior points in the space (X, ρ). We thus see that the extension of the class of complete spaces to the class of completely metrizable spaces extends in an essential way the range of applicability of some of the theorems. We remark that complete metrizability is inherited by subspaces which are $\mathcal{G}_\delta$-sets in the original space (see Exercise (f); cf. Problem 6.P.21). Evidently Theorem 6.1.18 implies that the topological product of countably many completely metrizable spaces is completely metrizable. On the other hand, it follows from Theorem 6.2.4 and Problem 6.P.4 that for X a non-empty compact metric space and Y

a completely metrizable space the space of maps $C(X,Y)$ is completely metrizable. We should also add that completely metrizable spaces may be characterized as the absolute $\mathcal{G}_\delta$-sets; viz. it may be shown that a metric space X is completely metrizable if and only if for any space Y every subspace homeomorphic to X is a $\mathcal{G}_\delta$-set in Y (see Problem 6.P.22).

Exercises

a) Show that if in a space X every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of non-empty closed sets with diameters tending to zero has non-empty intersection, then the space X is complete.

b) Check that the union of two sets both of which have no interior points and one of which is closed, has no interior points. Give an example of two subsets of the real line both of which have no interior points but whose union is the whole line.

c) Deduce from Baire's Theorem that the set of irrational numbers is not an $\mathcal{F}_\sigma$-set in the real line and that the set of rational numbers is not a $\mathcal{G}_\delta$-set in the real line. Notice that the space of rational numbers is not completely metrizable (see Problem 1.P.34).

d) Show that the map $f^*: X \to Y$ of Theorem 6.4.12 and the map $\tilde{f}: \tilde{X} \to T$ of Corollary 6.4.13 are uniformly continuous.

e) Show that every open subspace of a metric space X is homeomorphic to a closed subspace of the metric product $X \times \mathbf{R}$. Deduce that if X is completely metrizable and U is an open set of X then the subspace U is completely metrizable. (Hint: Consider the set $\{(x,t) \in X \times \mathbf{R} : t\rho(x, X\backslash U) = 1\}$.)

f) Show that if X is completely metrizable and A is a $\mathcal{G}_\delta$-set in X then the subspace A is completely metrizable. (Hint: Express A as the countable intersection of open sets $U_1, U_2, \ldots$, consider the map which assigns to a point a of the subspace A the point $\{a, a, \ldots\}$ in the topological product $\bigtimes_{n=1}^{\infty} U_n$ and use the previous exercise; cf. Problem 6.P.21.)

g) Give an example of a subspace of the plane in which the union of every sequence of closed sets with no interior points is a set without interior points, but which is not completely metrizable. (Hint: Consider the subspaces which contain a dense subspace that is completely metrizable.)

6.5. Continua

We recall that a metric space X is called a continuum if X is both compact and connected. The unit interval, the cube I^m and the sphere S^{m-1} for $m = 2, 3, \ldots$, and also the Hilbert cube $I^{\aleph_0}$ are continua, but the Euclidean space $\mathbf{R}^m$ is not a continuum because it is not compact. It follows from Theorem 6.3.4 that every continuum is separable and so by Theorem 6.3.9 every continuum is homeomorphic to some continuum lying in the Hilbert cube $I^{\aleph_0}$. In Section 1.8 we proved that if f is a continuous map of a continuum X onto a space Y, then the space Y is also a continuum (see Corollary 1.8.18); also we proved that the metric product of a finite number of continua is a

continuum (see Corollary 1.8.19). Theorems 6.1.5, 6.1.6, 6.1.16 and 6.1.17 imply the following.

6.5.1. THEOREM. *Both the metric and the topological product of an infinite sequence of continua is a continuum.* ■

We note two simple assertions concerning continua; the first follows from Theorem 1.7.5 and from the fact that a finite union of compact sets is compact, the other from Corollary 1.7.19 and Theorem 1.8.3.

6.5.2. ASSERTION. *If $X = \bigcup_{i=1}^{m} X_i$ where each of the subspaces X_i is a continuum for $i = 1, 2, \ldots, m$, and $\bigcap_{i=1}^{m} X_i \neq \emptyset$, then the space X is a continuum.* ■

6.5.3. ASSERTION. *The components of a compact metric space are continua.* ■

It turns out that the components of a compact space may be characterized as the minimal sets which are intersections of open-and-closed sets. This provides the opportunity of introducing the notion of a quasi-component.

Let X be any metric space. A non-empty subset of the space X which is an intersection of sets that are open-and-closed and is inclusion-minimal with respect to this property is called a *quasi-component* of the space X. In other words a non-empty set $K \subset X$ is a quasi-component of the space X if K is the intersection of a family of sets that are open-and-closed and for every open-and-closed set $W \subset X$ satisfying $K \cap W \neq \emptyset$ we have the inclusion $K \subset W$.

Evidently the quasi-components of a space are closed sets. It is easily checked that the quasi-components are pairwise disjoint and that every metric space is the union of its quasi-components.

6.5.4. ASSERTION. *Every component is contained in a quasi-component.* ■

In general the components are distinct from the quasi-components, that is to say, a quasi-component might be a union of a family of components (see Exercise (c)); however, for compact spaces we do have the following.

6.5.5. THEOREM. *If X is a compact space, then the components and the quasi-components of X coincide.*

PROOF. It suffices to prove that the quasi-components are connected. Suppose that the quasi-component K of the space X is the disjoint union of two closed sets $A, B \subset X$ with $A \neq \emptyset$. We shall show that in fact $B = \emptyset$. Say $K = \bigcap_{t \in T} W_t$ where the sets W_t are open-and-closed in X. Applying Theorem 1.6.27 pick open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The subspace $F = X \setminus (U \cup V)$ of X is compact (see Theorem 1.8.3) and the family $\{U_t\}_{t \in T}$ where $U_t = F \setminus W_t$ for $t \in T$ covers F. It follows from the Borel-Lebesgue Theorem (1.8.12) that there exists a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that

$$F = \bigcup_{i=1}^{m} U_{t_i} = \bigcup_{i=1}^{m} (F \setminus W_{t_i}) = F \setminus \bigcap_{i=1}^{m} W_{t_i}.$$

So the open-and-closed set $W = \bigcap_{i=1}^{m} W_{t_i}$ is disjoint from F, and $W \subset U \cup V$. Now $\operatorname{cl} U \cap V \subset \operatorname{cl}(X \setminus V) \cap V = \emptyset$, so

$$\operatorname{cl}(U \cap W) \subset \operatorname{cl} U \cap W = \operatorname{cl} U \cap W \cap (U \cup V) = U \cap W;$$

whence we deduce that the set $U \cap W$ is open-and-closed. Since $\emptyset \neq A \subset K \cap (U \cap W)$, the inclusion $K \subset U \cap W$ follows and so $B \subset K \cap V \subset U \cap V \cap W = \emptyset$. ■

We now show that no continuum may be decomposed into countably many disjoint non-empty closed sets; thus the property used to define connectedness may be significantly strengthened within the realm of continua. We add that the assumption of compactness is essential here, as there do exist connected spaces which can be expressed as a countable union of pairwise disjoint non-empty closed sets (see Problem 6.P.28).

The proof of the promised theorem is preceded by two lemmas.

6.5.6. LEMMA. *If A is a proper, non-empty closed subset of the continuum X, then for each component S of the subspace A we have $S \cap \operatorname{bd} A \neq \emptyset$, where $\operatorname{bd} A$ denotes the boundary of the set A in the space X.*

PROOF. Suppose that $S \cap \operatorname{bd} A = \emptyset$ and consider the family $\{W_t\}_{t \in T}$ of all open-and-closed subsets of the subspace A which contains the component S; it follows from the last theorem that $S = \bigcap_{t \in T} W_t$. The subspace $\operatorname{bd} A$ of the space X is compact and the family $\{U_t\}_{t \in T}$ where $U_t = \operatorname{bd} A \setminus W_t$ for $t \in T$ is an open covering of it. There therefore exists a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $\operatorname{bd} A = \bigcup_{i=1}^{m} U_{t_i}$. The set $W = \bigcap_{i=1}^{m} W_{t_i}$ is open-and-closed in A and disjoint from $\operatorname{bd} A$; let U be an open subset of the space X satisfying $W = U \cap A$ (see Theorem 1.6.5). It follows from the equations $A = \operatorname{int} A \cup \operatorname{bd} A$ and $W \cap \operatorname{bd} A = \emptyset$ that $W = U \cap \operatorname{int} A$ is open in X. Since W is also closed in X and non-empty we have $W = X$. We infer that $\operatorname{bd} A = \emptyset$ which, according to Corollary 1.6.31, contradicts the connectedness of X. ■

6.5.7. LEMMA. *If the continuum X is the union of pairwise disjoint closed sets X_1, $X_2, \ldots$ at least two of which are non-empty, then for each natural number n there exists a continuum $C \subset X$ such that $C \cap X_n = \emptyset$ and such that at least two sets of the sequence $C \cap X_1, C \cap X_2, \ldots$ are non-empty.*

PROOF. When $X_n = \emptyset$ we may take $C = X$; so assume that $X_n \neq \emptyset$. Choose a natural number m distinct from n for which $X_m \neq \emptyset$ and disjoint open sets $U, V \subset X$ with $X_n \subset U$, $X_m \subset V$. Let x be an arbitrary point of X_m and let C be the component of the space $\operatorname{cl} V$ which contains the point x. Evidently C is a continuum, $C \cap X_n = \emptyset$ and $C \cap X_m \neq \emptyset$. Now $C \cap \operatorname{bd} \operatorname{cl} V \neq \emptyset$ by the previous lemma and $X_m \subset \operatorname{int} \operatorname{cl} V$, so there exists a natural number m' distinct from m such that $C \cap X_{m'} \neq \emptyset$. ■

6.5.8. THEOREM (Sierpiński). *No continuum can be represented as a union of countably many pairwise disjoint closed sets at least two of which are non-empty.*

PROOF. Suppose that the continuum X can in fact be represented as a union $\bigcup_{n=1}^{\infty} X_n$ where the sets X_n are closed and pairwise disjoint and at least two are non-empty. It follows from Lemma 6.5.7 that there is a decreasing sequence of non-empty continua $C_1 \supset C_2 \supset \ldots$ contained in X such that $C_n \cap X_n = \emptyset$ for $n = 1, 2, \ldots$ We

thus have $\bigcap_{n=1}^{\infty} C_n = (\bigcap_{n=1}^{\infty} C_n) \cap (\bigcup_{n=1}^{\infty} X_n) = \emptyset$ which contradicts Cantor's Theorem (1.8.9). ■

We now introduce the notion of local connectedness and, after some straightforward remarks regarding amongst other things its relationship with the notion of local pathwise connectedness introduced in Section 3.4, we return again to the theory of continua and concern ourselves with locally connected continua.

We say that the space X is *locally connected at the point* x if for every neighbourhood U of the point x there is a connected set $C \subset U$ with $x \in \operatorname{int} C$; if the space is locally connected at every point then we say that the space is *locally connected* (see Supplement 6.S.8). The Euclidean space $\mathbf{R}^m$, the cube I^m and the sphere S^m are locally connected for $m = 0, 1, 2, \ldots$ The space X described in Example 3.1.22 is not locally connected at points of the form $(0, x^2)$ for $-1 \leq x^2 \leq 1$. It is easily checked that local connectedness is a topological property.

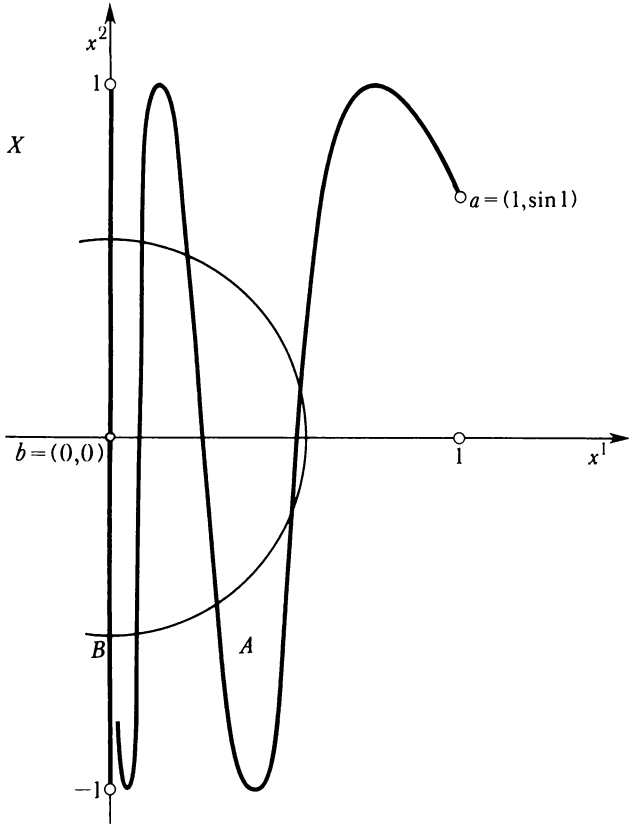


Fig.138. The space X is not locally connected at the point b, as this point does not lie in the interior of any small connected set (see Example 3.1.22).

6.5.9. THEOREM. *A space X is locally connected if and only if for each open set $U \subset X$ the components of the subspace U are open subsets of the space X.*

PROOF. Let U be an open subset of a locally connected space X; consider any component S of the space U. By the local connectedness of X for each point $x \in S$ there exists a connected set $C \subset U$ such that $x \in \operatorname{int} C$. By the connectedness of C we have $x \in \operatorname{int} C \subset C \subset S$ and so S is an open subset of the space X.

Now suppose that the space X satisfies the condition of the theorem and consider an arbitrary point $x \in X$ and its neighbourhood U. Let C be the component of the subspace $U \subset X$ which contains x; since C is a connected set, $C \subset U$ and $x \in C = \operatorname{int} C$, the space X is locally connected. ■

6.5.10. COROLLARY. *A space X is locally connected if and only if it has a base consisting of regions.* ■

We recall that a space X is locally pathwise connected at the point x if for every neighbourhood U of the point x there is a neighbourhood V of the point which is contained in U such that for any pair of points $x', x'' \in V$ there is a path from x' to x'' lying in U; if the space is locally pathwise connected at every point then we say that the space is locally pathwise connected (see Supplement 6.S.8).

6.5.11. THEOREM. *If a space X is locally pathwise connected at the point x, then it is locally connected at that point.*

PROOF. Consider an arbitrary neighbourhood U of the point x. Let $V \subset U$ be a neighbourhood of x with the property that for any two points $x', x'' \in V$ there exists a path from x' to x'' in U. For each point $x' \in V$ choose in U a path $d_{x'}\colon I \to U$ from x to x'. From Theorem 1.7.5 it follows that the set $C = \bigcup_{x' \in V} d_{x'}(I)$ is connected, as obviously $x \in \bigcap_{x' \in V} d_{x'}(I)$. Since $C \subset U$ and $x \in V \subset \operatorname{int} C$, the space X is locally connected at the point x. ■

6.5.12. COROLLARY. *Every locally pathwise connected space is locally connected.* ■

We leave the reader to find an example of a locally connected space which is not locally pathwise connected (see Exercise (j)); to clarify the situation we hasten to add that every locally connected continuum is locally pathwise connected (see Corollary 6.5.20).

We now return to the theory of continua. We begin with Sierpiński's criterion characterizing local connectedness within the realm of compact spaces (see Supplement 6.S.9).

6.5.13. THEOREM (Sierpiński). *A compact metric space X is locally connected if and only if for every positive real number ϵ there exists a finite covering of the space consisting of continua of diameter less than ϵ.*

PROOF. Consider a locally connected compact space X; let ϵ be an arbitrary positive real number. From Corollary 6.5.10 follows the existence of an open covering $\{U_t\}_{t \in T}$ of the space X consisting of connected sets of diameter less than ϵ. By the Borel-Lebesgue Theorem (1.8.12) there exists a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $X = U_{t_1} \cup U_{t_2} \cup \ldots \cup U_{t_m}$. The subspaces $C_i = \operatorname{cl} U_{t_i}$ of the space X are for $i = 1, 2, \ldots, m$ continua of diameter less than ϵ and constitute a covering of the space X.

Now assume that a given compact space satisfies the condition considered in the theorem. Let x be any point of X and let U be a neighbourhood of x. Fix a positive real number ϵ such that $B(x;\epsilon) \subset U$ and consider a finite covering $\{C_i : i = 1,2,\ldots,m\}$ of the space X by continua of diameter less than ϵ. We may assume, renumbering the elements of the covering if necessary, that $x \in C_i$ for $i = 1,2,\ldots,k$ and $x \notin C_i$, for $i = k+1, k+2, \ldots, m$. It follows from Theorem 1.7.5 that the set $C = C_1 \cup C_2 \cup \ldots \cup C_k$ is connected; evidently, $C \subset B(x;\epsilon) \subset U$. Now $X \setminus C \subset C_{k+1} \cup C_{k+2} \cup \ldots C_m$, so it is also true that $\operatorname{cl}(X \setminus C) \subset C_{k+1} \cup C_{k+2} \cup \ldots \cup C_m$, whence it follows that $x \in X \setminus \operatorname{cl}(X \setminus C) = \operatorname{int} C$. We have thus shown that the space X is locally connected. ■

Theorems 6.5.13, 1.8.2, 1.8.14 and Corollary 1.8.18 imply the following theorem (cf. Exercise (e)).

6.5.14. THEOREM. *If f is a continuous map of a compact, locally connected space X onto a space Y, then the space Y is also compact and locally connected.* ■

We shall now prove a deep theorem of Mazurkiewicz and Moore asserting that every pair of points of a region lying in a locally connected, complete space may be joined by an arc, that is by a set homeomorphic to the unit interval I (see Supplement 6.S.10). The idea of the proof is straightforward: it relies on joining the pair of points considered by successively narrower sets the intersection of which is the required arc. A precise construction of these sets - defined as unions of small regions making up chains joining the points considered - nevertheless requires a certain amount of care. We begin by introducing the notion of a chain and then prove two lemmas on chains. The second of these lemmas is a basic step in the construction of the desired arc.

Suppose given a space X, a subset U and two points $x, y \in U$. We call a finite sequence of regions $V_1, V_2, \ldots, V_k$ of the space X a *chain in U linking the points x and y*, if $x \in V_1, y \in V_k, V_i \cap V_{i+1} \neq \emptyset$ for $i = 1,2,\ldots,k-1$ and $V_i \subset U$ for $i = 1,2,\ldots,k$; the elements of the chain will be called *links*. We call the chain $V_1, V_2, \ldots, V_k$ linking the points x and y *simple* if $x \notin V_i$ for $i > 1$ and $y \notin V_i$ for $i < k$ and $V_i \cap V_j \neq 0$ if and only if $|i-j| \leq 1$. Observe that any chain $V_1, V_2, \ldots, V_k$ linking the points x and y contains a subsequence which is a simple chain linking the points x and y. In fact it is enough to choose from among the subsequences which form a chain linking the points x and y one which is of minimal length.

6.5.15. LEMMA. *Let U be an arbitrary region in a locally connected space X and let $\mathcal{V}$ be an open covering of the set U. For any two points $x, y \in U$ there exists a simple chain in U linking the points x and y such that the closure of each link of the chain lies in some element of the covering $\mathcal{V}$.*

PROOF. Let $\mathcal{W}$ be the family of all regions W of the space X for which there exists an element $V \in \mathcal{V}$ with $\operatorname{cl} W \subset V$. Denote by A the set of all points $z \in U$ for which there exists a chain in U linking the points x and z whose links are members of $\mathcal{W}$. Evidently the set A is open in the subspace U of X. We shall show that $U \cap \operatorname{cl} A \subset A$, that is, that A is also closed in U. Let a' be any point of $U \cap \operatorname{cl} A$ and let $a' \in V \in \mathcal{V}$. Consider a neighbourhood W' of the point a' such that $\operatorname{cl} W' \subset V$; by Theorem 6.5.9 the component W of the set W' which contains the point a' belongs to the family $\mathcal{W}$. Since $W \cap \operatorname{cl} A \neq \emptyset$, there exist a point $a \in W \cap A$. Appending the set W as the final

link to any chain in U which links the points x and a and whose elements belong to $\mathcal{W}$, yields a similar chain linking the points x and a'. Hence $a' \in A$; and thus A is an open-and-closed subset of U. Since $x \in A$ we have $A \neq \emptyset$ and from the connectedness of U it follows that $A = U$. There thus exists a chain in U linking the points x and y whose elements belong to $\mathcal{W}$; it remains only to select from it a simple chain. ■

6.5.16. LEMMA. *Let V be any region in a locally connected space X. For any two points $x, y \in V$ there exists in V a sequence of simple chains $V_1^n, V_2^n, \ldots, V_{k_n}^n$ $(n = 1, 2, \ldots)$ linking the points x and y such that:*

(1) $\operatorname{diam} V_i^n \leq 1/n$ *for* $i = 1, 2, \ldots, k_n$ *and* $n = 1, 2, \ldots$,

(2) *a link $V_{j(i)}^{n-1}$ may be associated with the link V_i^n for each $i = 1, 2, \ldots, k_n$ and each $n > 1$ in such a way that $j(i_1) \leq j(i_2)$ whenever $i_1 \leq i_2$.*

PROOF. The existence of the chain $V_1^1, V_2^1, \ldots, V_{k_1}^1$ is guaranteed by Lemma 6.5.15 when for U we take the set V and for $\mathcal{V}$ the covering of V consisting of all open balls of radius $1/2$. Suppose that the chain $V_1^{n-1}, V_2^{n-1}, \ldots, V_{k_{n-1}}^{n-1}$ has already been defined. We now construct the chain $V_1^n, V_2^n, \ldots, V_{k_n}^n$.

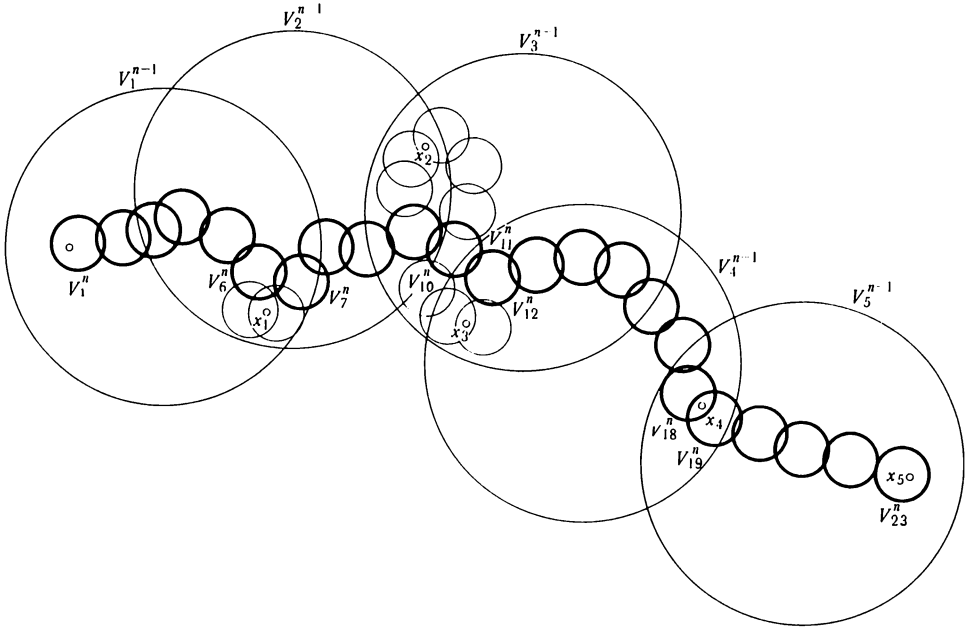


Fig.139. Construction of the chain $V_1^n, V_2^n, \ldots, V_{k_n}^n$ (see proof of Lemma 6.5.16).

For each $i < k_{n-1}$ let us choose an arbitrary point $x_i \in V_i^{n-1} \cap V_{i+1}^{n-1}$ and in addition let us put $x_0 = x$ and $x_{k_{n-1}} = y$. Applying Lemma 6.5.15, taking for U the set V_i^{n-1}, for $\mathcal{V}$ the covering of V_i^{n-1} consisting of all open balls of radius $1/2n$ and for x and y the points x_{i-1} and x_i, we obtain for $i = 1, 2, \ldots, k_{n-1}$ a chain $\mathcal{U}_i$ in V_i^{n-1} linking the points x_{i-1} and x_i whose links have diameter less than $1/n$ and have closures lying in V_i^{n-1}.

Listing the links of the chains $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{k_n-1}$ consecutively we obtain a chain linking the points x and y. Select from it a simple chain $V_1^n, V_2^n, \ldots, V_{k_n}^n$ linking the points x and y and for each link V_i^n denote by $j(i)$ the index of the chain $\mathcal{U}_{j(i)}$ from which V_i^n arose; evidently $\operatorname{cl} V_i^n \subset V_{j(i)}^{n-1}$. It is readily observed that $j(i_1) \leq j(i_2)$ whenever $i_1 \leq i_2$; the proof of the lemma is thus complete. ■

6.5.17. THEOREM (Mazurkiewicz, Moore). *For any two distinct points x and y of an arbitrary region V contained in a locally connected, complete space X there exists a homeomorphism $h\colon I \to L \subset V$ of the unit interval onto a subspace L of the space X such that $h(0) = x$ and $h(1) = y$.*

PROOF. Pick simple chains $V_1^n, V_2^n, \ldots, V_{k_n}^n$ for $n = 1, 2, \ldots$ satisfying the conclusions of Lemma 6.5.16. Proceeding by induction we construct for $n = 1, 2, \ldots$ coverings $\mathcal{I}_n = \{I_1^n, I_2^n, \ldots, I_{k_n}^n\}$ of the unit interval I by closed intervals such that $0 \in I_1^n$, $1 \in I_{k_n}^n$ and the right-hand endpoint of I_i^n is the left-hand endpoint of I_{i+1}^n for $i = 1, 2, \ldots, k_n - 1$ and such that the following two conditions are met:

$$I_i^n \subset I_{j(i)}^{n-1} \quad \text{for} \quad i = 1, 2, \ldots, k_n \quad \text{and} \quad n > 1,$$

$$\operatorname{diam} I_i^n = \operatorname{diam} I_{i'}^n \quad \text{whenever} \quad j(i) = j(i').$$

We obtain the covering $\mathcal{I}_1$ by dividing the interval into k_1 contiguous closed subintervals of equal length numbered consecutively from left to right. Suppose that the covering $\mathcal{I}_{n-1} = \{I_1^{n-1}, I_2^{n-1}, \ldots, I_{k_{n-1}}^{n-1}\}$ has already been defined; we obtain the covering $\mathcal{I}_n$ by dividing each interval I_j^{n-1} for $j = 1, 2, \ldots, k_{n-1}$ into as many contiguous and consecutive closed subintervals of equal length as there are members in the set $\{i \leq k_n : j(i) = j\}$; this set is non-empty because the chain $V_1^{n-1}, V_2^{n-1}, \ldots, V_{k_{n-1}}^{n-1}$ is simple.

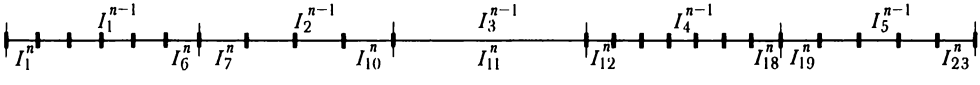


Fig.140. Construction of the covering $\{I_1^n, I_2^n, \ldots, I_{k_n}^n\}$ (see the proof of Theorem 6.5.17).

For $n = 1, 2, \ldots$ let d_n denote the length of the longest interval belonging to the covering $\mathcal{I}_n$; we show that $\lim_n d_n = 0$. Suppose otherwise, then there exists a natural number m and an index $i_m \leq k_m$ such that the interval $I_{i_m}^m$ belongs to the covering $\mathcal{I}_n$ for each $n \geq m$; that is, for each $n \geq m$ there exists an index $i_n \leq k_n$ such that $I_{i_n}^n = I_{i_m}^m$. Since $I_{i_m}^m = I_{i_{m+1}}^{m+1} = I_{i_{m+2}}^{m+2} = \ldots$, we have $V_{i_m}^m \supset V_{i_{m+1}}^{m+1} \supset V_{i_{m+2}}^{m+2} \supset \ldots$ By Cantor's Theorem for complete spaces (6.4.1) the set $\bigcap_{k=0}^{\infty} \operatorname{cl} V_{i_{m+k}}^{m+k}$ consists of exactly one point; denote the point by x_0. It is not difficult to check that $x_0 \in \operatorname{cl} V_{i_p-1}^p \cap \operatorname{cl} V_{i_p+1}^p$, where $p = m + 1$ and to simplify notation we take $V_0^p = \{x\}$ and $V_{k_p+1}^p = \{y\}$; we have thus obtained a contradiction to the assumption that the chain $V_1^m, V_2^m, \ldots, V_{k_m}^m$ is simple.

For each point $t \in I$ and each $n = 1, 2, \ldots$ let $A_n(t) = \{i \leq k_n : t \in I_i^n\}$; evidently $1 \leq \operatorname{card} A_n(t) \leq 2$. Let us observe that if $i \in A_n(t)$ for $n > 1$ then $j(i) \in A_{n-1}(t)$, and so the closed sets $F_1(t), F_2(t), \ldots$ of the space X defined as $F_n(t) = \bigcup\{\operatorname{cl} V_i^n : i \in A_n(t)\}$

form a decreasing sequence. Since $\lim_n \operatorname{diam} F_n(t) = 0$, we have by Cantor's Theorem for complete spaces (6.4.1) that the set $\bigcap_{n=1}^{\infty} F_n(t)$ consists of exactly one point of X. Denote the point by $h(t)$; of course, $h(t) \in V$. We have thus defined a map $h: I \to L = h(I) \subset V$. Observe that h is a continuous map, since for every $t \in I$ the set $\operatorname{int}(\bigcup\{I_i^n : i \in A_n(t)\})$ is a neighbourhood of the point t in the space I and its image lies in $F_n(t)$ so has diameter not greater than $2/n$. Moreover the map h is injective, because for $t \neq t'$ there exists a natural number $n > 1$ such that $t \in I_i^n$, $t' \in I_{i'}^n$ and $|j(i) - j(i')| > 3$, whence – setting up the additional notation $V_0^n = V_{k_n+1}^n = V_0^{n-1} = V_{k_{n-1}+1}^{n-1} = \emptyset$ – we have

$$\begin{aligned} F_n(t) \cap F_n(t') &\subset (\operatorname{cl} V_{i-1}^n \cup \operatorname{cl} V_i^n \cup \operatorname{cl} V_{i+1}^n) \cap (\operatorname{cl} V_{i'-1}^n \cup \operatorname{cl} V_{i'}^n \cup \operatorname{cl} V_{i'+1}^n) \\ &\subset (V_{j(i)-1}^{n-1} \cup V_{j(i)}^{n-1} \cup V_{j(i)+1}^{n-1}) \cap (V_{j(i')-1}^{n-1} \cup V_{j(i')}^{n-1} \cup V_{j(i')+1}^{n-1}) = \emptyset, \end{aligned}$$

so that $h(t) \neq h(t')$. Thus from Theorem 1.8.15 we infer that h is a homeomorphism. We leave to the reader to check that $h(0) = x$ and $h(1) = y$. ■

6.5.18. COROLLARY. *Every locally connected complete space is locally pathwise connected.* ■

6.5.19. COROLLARY. *Every connected and locally connected complete space is pathwise connected and locally pathwise connected.* ■

6.5.20. COROLLARY. *Every locally connected continuum is pathwise connected and locally pathwise connected.* ■

It follows from Corollary 1.8.18 and Theorem 6.5.14 that every metric space which is the image of the unit interval under a continuous map is a locally connected continuum; hence by Theorem 6.5.17 for any two distinct points x, y in such a space there exists an arc with endpoints x and y. We thus arrive at two assertions stating that pathwise connectedness and local pathwise connectedness are equivalent to "arcwise connectedness" and "local arcwise connectedness" (cf. Supplement 3.S.3).

6.5.21. ASSERTION. *A space X is pathwise connected if and only if for any pair of distinct points $x, y \in X$ there exists an arc L in X with endpoints x and y.* ■

6.5.22. ASSERTION. *A space X is locally pathwise connected at the point x if and only if for every neighbourhood U of the point x there is a neighbourhood V of the point which is contained in U such that for any pair of distinct points $x', x'' \in V$ there is an arc L in U with endpoints x' and x''.* ■

Before closing this section we give an interesting characterization of locally connected continua. We show that they are images of the unit interval under continuous maps (cf. Example 4.4.5). We begin with a lemma asserting that locally pathwise connected compact spaces are in some sense uniformly locally pathwise connected.

6.5.23. LEMMA. *If a metric space X is compact and locally pathwise connected, then for every positive real number ϵ there exists a positive real number δ such that for any pair of points $x, y \in X$ satisfying the inequality $\rho(x, y) < \delta$ there is a path d in X from x to y with the property that* $\operatorname{diam} d(I) < \epsilon$.

PROOF. For every point $z \in X$ there exists a neighbourhood V_z of z with the property that for every pair of points $x, y \in V_z$ there exists a path d from x to y in $B(z; \frac{1}{3}\epsilon)$; of course, $\operatorname{diam} d(I) \leq \operatorname{diam} B(x; \frac{1}{3}\epsilon) < \epsilon$. The sets V_z for $z \in X$ form an open covering of the space X. Let δ be the Lebesgue number of the covering (see Lemma 1.8.13). Then, whenever $x, y \in X$ and $\rho(x, y) < \delta$, there exists a point $z \in X$ for which $x, y \in V_z$. ■

6.5.24. THEOREM (Hahn, Mazurkiewicz). *A metric space X is a non-empty locally connected continuum if and only if it is the image of the unit interval I under a continuous map.*

PROOF. It is enough to show that for every locally connected continuum X containing more than one point there exists a continuous map $f: I \to X$ of the unit interval onto the space X. It follows from Corollary 6.3.12 that there exists a continuous map $f_0: A \to X$ from a closed subset A of the Cantor set C onto the space X. Let $a_0 = \inf A$ and $b_0 = \sup A$. Now $a_0, b_0 \in A$ (see Lemma 1.6.22) and $a_0 \neq b_0$, so the set $[a_0, b_0] \setminus A$ is an open subset of the real line and hence its components are open intervals (see Theorem 1.10.4); arrange them into a sequence (a_1, b_1), $(a_2, b_2), \ldots$ (cf. Exercise (c) of Section 6.3).

Assume first that this sequence is infinite – we then have $\lim_i (b_i - a_i) = 0$. From the uniform continuity of the map f_0 (see Theorem 1.8.14) and from Lemma 6.5.23 we infer that for $n = 1, 2, \ldots$ a natural number i_n may be found with the property that for $i \geq i_n$ there exists a path d in X from $f_0(a_i)$ to $f_0(b_i)$ such that $\operatorname{diam} d(I) < 1/n$. Without loss of generality we may assume that $i_1 \leq i_2 \leq \ldots$. For each natural number $i < i_1$ let $f_i: [a_i, b_i] \to X$ be an arbitrary continuous map such that $f_i(a_i) = f_0(a_i)$ and $f_i(b_i) = f_0(b_i)$; its existence is assured by Corollary 6.5.20. Now for i satisfying $i_n \leq i \leq i_{n+1}$ where $n = 1, 2, \ldots$ let $f_i: [a_i, b_i] \to X$ be a continuous map for which $\operatorname{diam} f_i([a_i, b_i]) < 1/n$ and $f_i(a_i) = f_0(a_i)$ and $f_i(b_i) = f_0(b_i)$. Setting

$$f(r) = \begin{cases} f_0(a_0), & \text{if } r \in [0, a_0], \\ f_0(r), & \text{if } r \in A, \\ f_i(r), & \text{if } r \in [a_i, b_i], \text{ for } i = 1, 2, \ldots, \\ f_0(b_0), & \text{if } r \in [b_0, 1], \end{cases}$$

we obtain a map $f: I \to X$ of the interval I onto the space X. We leave it to the reader to check the continuity of f.

In the case when the sequence of components of the set $[a_0, b_0] \setminus A$ is finite and its last term is (a_m, b_m) the proof runs as above with $m+1$ replacing i_1 and using only the maps $f_1, f_2, \ldots, f_m$. ■

Exercises

a) Show that if the metric space X is connected then for any pair of points $x, y \in X$ and any real number $\epsilon > 0$ there exists a sequence of points $x_0, x_1, \ldots, x_k \in X$ such that $x_0 = x$, $x_k = y$ and $\rho(x_{j-1}, x_j) < \epsilon$ for $j = 1, 2, \ldots, k$. Give an example of a space which is not connected but which satisfies the above condition and prove that every compact metric space satisfying the condition is a continuum.

b) Observe that if $X = \bigcup_{n=1}^{\infty} X_n$, where each of the subspaces X_n is a continuum for $n = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$ and $\lim_n \operatorname{diam} X_n = \emptyset$, then the space X is a continuum. Check that the assumption about the diameters of the subspaces X_n may not be omitted.

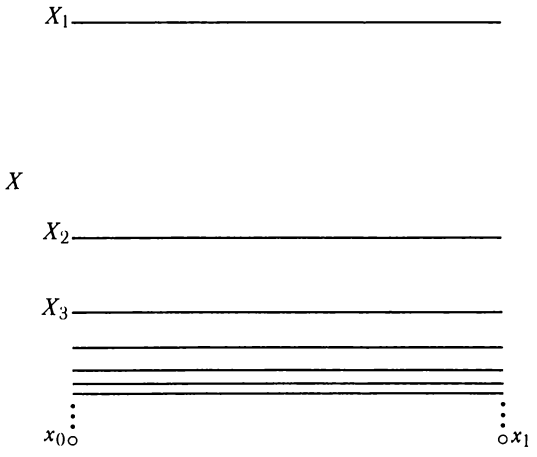


Fig.141. The set $\{x_0, x_1\}$ is a quasi-component of the space X; the quasi-component $\{x_0, x_1\}$ is a union of the components $\{x_0\}$ and $\{x_1\}$ of the space X (see Exercise (c)).

c) Give an example of a space whose components are distinct from the quasi-components and show that the components and quasi-components of any locally connected space coincide. (Hint: Consider the subspace of the plane given by $X = \{x_0, x_1\} \cup \bigcup_{n=1}^{\infty} X_n$, where $x_0 = (0,0)$, $x_1 = (1,0)$ and X_n is the line segment with endpoints $(0, 1/n)$ and $(1, 1/n)$ for $n = 1, 2, \ldots$)

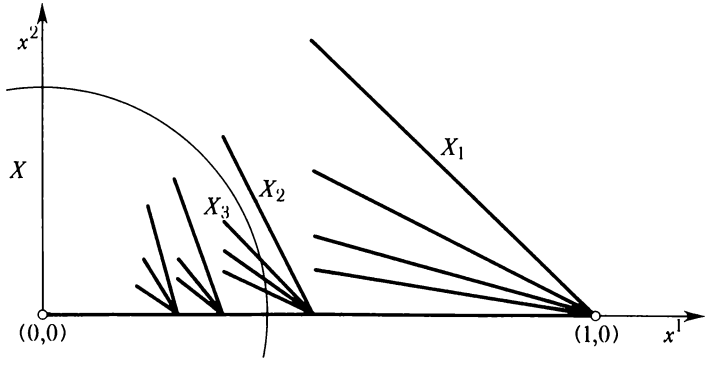


Fig.142. The space X is locally connected at the origin, but the origin does not have small connected neighbourhoods (see Exercise d).

d) Give an example of a space X which contains a point x with the property that X is locally connected at x but there exists a neighbourhood U of x such that no neighbourhood of the point contained in U is connected. (Hint: Consider the subspace of the plane given by $X = (I \times \{0\}) \cup \bigcup_{n=1}^{\infty} X_n$, where X_n for $n = 1, 2, \ldots$ is the union of the line segments with endpoints $(1/n, 0)$ and $(1/(n+1), 1/k(n+1))$ where $k = 1, 2, \ldots$)

e) Give an example of a continuous map of the real line onto a space which is not locally connected (cf. Theorem 6.5.14).

f) Show that if a metric space X is locally connected and separable then it has a countable base consisting of regions.

g) Prove that if $X = \bigcup_{n=1}^{\infty} X_n$, where each of the subspaces X_n is a locally connected continuum for $n = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$ and $\lim_n \operatorname{diam} X_n = 0$, then the space X is a locally connected continuum.

h) Show that a space X is locally pathwise connected at a point x if and only if for every neighbourhood U of the point x there is a pathwise connected set $C \subset U$ such that $x \in \operatorname{int} C$.

i) Prove that every connected, locally pathwise connected space is pathwise connected.

j) Give an example of a locally connected space which is not locally pathwise connected. (Hint: Partition the set of rational numbers Q into two disjoint, dense sets Q_1 and Q_2 and consider the subspace of the plane $X = (Q_1 \times \mathbf{R}) \cup (P \times Q_1) \cup (Q_2 \times Q_2)$, where P denotes the set of irrational numbers. Prove that the projection of any continuum $C \subset X$ onto the axis of abscissae is a singleton. For this purpose observe that for every number $q \in Q_1$ the set $\{r \in \mathbf{R} : (r, q) \in C\}$ has empty interior.)

k) Show that a space X is locally pathwise connected if and only if it has a base consisting of pathwise connected sets. Observe that if X is a locally pathwise connected separable metric space, then it has a countable base consisting of pathwise connected sets.

l) Prove that if a compact metric space is locally connected then for every positive real number ϵ there exists a finite covering of the space consisting of locally connected continua of diameter less than ϵ. (Hint: Apply Theorem 6.5.24.)

6.6. Absolute retracts and absolute neighbourhood retracts

Recall that a continuous map $r\colon X \to A$ of a metric space X onto a subspace A is called a retraction if $r(a) = a$ for every $a \in A$, that is, if $r|A = \mathrm{id}_A$. Associated with the notion of a retraction are the notions of retract and neighbourhood retract, which were introduced in Section 3.1: a set $A \subset X$ is a retract of the space X if there exists a retraction $r\colon X \to A$; similarly, a closed set $A \subset X$ is a neighbourhood retract of the space X if A is the retract of some open set $U \subset X$ containing A. A retract of a space X is always a closed set (Theorem 3.1.14) and so a retract of a space X is also a neighbourhood retract of the space.

The notions of retract and neighbourhood retract allow us to single out two classes of compact metric spaces which are noteworthy, both on account of their regular structure and because they are closely linked with the class of polyhedra (see Theorems 6.6.14 and 6.6.20 and Supplement 6.S.18). A compact metric space X is called an ***absolute retract*** if every homeomorphic image of X which lies in a compact metric space Y is a retract of the space Y. A compact metric space is called an ***absolute neighbourhood retract*** if every homeomorphic image of X which lies in a compact metric space Y is a neighbourhood retract of the space Y. Evidently every absolute retract is an absolute neighbourhood retract. It follows immediately from the definitions that the two notions

just introduced are indeed topological notions (see Supplement 6.S.14).

It transpires that absolute retracts may be characterized as the compact metric spaces for which the analogue of Tietze's Theorem holds: the characterization is contained in the next two theorems (see Supplement 6.S.15).

6.6.1. THEOREM. *If Y is an absolute retract, then for every continuous map $f: A \to Y$ defined on a closed subset A of a metric space X there exists a continuous extension $f^*: X \to Y$.*

PROOF. By Theorems 6.3.4 and 6.3.9 we may assume that the space Y is a retract of the Hilbert cube $I^{\aleph_0}$. According to Theorem 6.1.20 the composition $g = i_Y f: A \to I^{\aleph_0}$, where $i_Y: Y \to I^{\aleph_0}$ is the inclusion map, has a continuous extension $g^*: X \to I^{\aleph_0}$. The composition $f^* = rg^*: X \to Y$ of g^* with the retraction r of the Hilbert cube $I^{\aleph_0}$ onto Y is a continuous extension of the map f. ■

6.6.2. THEOREM. *If a compact metric space Y has the property that for each continuous map $f: A \to Y$ defined on a closed subset A of a compact metric space X there exists a continuous extension $f^*: X \to Y$, then Y is an absolute retract.*

PROOF. We consider an arbitrary compact metric space X, a subspace A homeomorphic to Y and a homeomorphism $f: A \to Y$; of course A is a closed subset of X. The composition $f^{-1}f^*: X \to A$, where $f^*: X \to Y$ is a continuous extension of the map f is a retraction, so the set A is a retract of the space X. ■

Similar reasoning yields the next two theorems which are analogues of Theorems 6.6.1 and 6.6.2 for absolute neighbourhood retracts (see Supplement 6.S.15).

6.6.3. THEOREM. *If Y is an absolute neighbourhood retract then for every continuous map $f: A \to Y$ defined on a closed subset A of a metric space X there exists a set U, open in X and containing A, and a continuous extension $f^*: U \to Y$.* ■

6.6.4. THEOREM. *If a compact metric space Y has the property that for each continuous map $f: A \to Y$ defined on a closed subset A of a compact metric space X there exists a set U, open in X and containing A, and a continuous extension $f^*: U \to Y$ then Y is an absolute neighbourhood retract.* ■

6.6.5. THEOREM. *The m-dimensional unit cube I^m, the m-dimensional closed unit ball $\bar{B}^m$ and the m-dimensional unit simplex Δ^m are absolute retracts for $m = 0, 1, 2, \ldots$*

PROOF. By Corollaries 1.10.10 and 2.1.9 the ball $\bar{B}^m$ and the simplex Δ^m are homeomorphic to the cube I^m and from Corollary 3.1.6 and Theorem 6.6.2 it follows that I^m is an absolute retract. ■

We infer from Theorems 6.1.20 and 6.6.2 the following.

6.6.6. THEOREM. *The Hilbert cube $I^{\aleph_0}$ is an absolute retract.* ■

From Theorems 3.1.19 and 6.6.4 we deduce as follows.

6.6.7. THEOREM. *The $(m-1)$-dimensional sphere S^{m-1} is an absolute neighbourhood retract for $m = 1, 2, \ldots$* ■

From Theorem 6.6.3 and Borsuk's homotopy extension theorem (3.2.8) we have the following result.

6.6.8. THEOREM. *If Y is an absolute neighbourhood retract then every pair (X, A), where A is a closed subset of the metric space X, has the homotopy extension property relative to Y.* ■

The next result is a consequence of Theorems 6.6.1–6.6.4.

6.6.9. THEOREM. *A retract (neighbourhood retract) of any absolute retract (absolute neighbourhood retract) is an absolute retract (absolute neighbourhood retract).* ■

The following important characterization of absolute retracts and absolute neighbourhood retracts, which appeals to the universality property of the Hilbert cube, comes from Theorems 6.3.4, 6.3.9, 6.6.6 and 6.6.9.

6.6.10. THEOREM. *A compact metric space is an absolute retract (absolute neighbourhood retract) if and only if it is homeomorphic to a retract (neighbourhood retract) of the Hilbert cube.* ■

Using Theorem 6.6.10 we now prove that absolute retracts have the fixed point property.

6.6.11. LEMMA. *The Hilbert cube $I^{\aleph_0}$ has the fixed point property.*

PROOF. Suppose there exists a continuous map $f: I^{\aleph_0} \to I^{\aleph_0}$ such that $f(x) \neq x$ for every $x \in I^{\aleph_0}$. The compactness of the Hilbert cube implies the existence of a positive real number ϵ such that $\rho(x, f(x)) \geq \epsilon$ for every $x \in I^{\aleph_0}$. Let m be a natural number satisfying the inequality $\sum_{i=m+1}^{\infty}(1/i)^2 < \epsilon^2/4$. The map taking the point $x = \{x_1, x_2, \ldots\} \in I^{\aleph_0}$ to the point $p(x) = \{x_1, x_2, \ldots, x_m, 0, 0, \ldots\} \in I^{\aleph_0}$ is a continuous map of $I^{\aleph_0}$ onto the subspace $A = \{\{x_1, x_2, \ldots\} \in I^{\aleph_0} : x_{m+1} = x_{m+2} = \ldots = 0\}$ with the property that $\rho(x, p(x)) < \epsilon/2$ for each $x \in I^{\aleph_0}$. We consider the composition $g = p(f|A): A \to A$. For each $x \in A$ we have

$$\rho(x, g(x)) \geq \rho(x, f(x)) - \rho(f(x), g(x)) = \rho(x, f(x)) - \rho(f(x), pf(x)) > \epsilon/2,$$

so the function g does not have a fixed point. We thus have a contradiction to Brouwer's Theorem (3.1.16) since the space A is homeomorphic to the ball $\bar{B}^m$. ■

From Theorem 6.6.10, Assertion 3.1.17 and Lemma 6.6.11 we obtain the following.

6.6.12. THEOREM. *Every absolute retract has the fixed point property.* ■

As the example of the sphere S^0 shows, Theorem 6.6.12 is not true for absolute neighbourhood retracts. We also remark that there exist spaces with the fixed point property which are not absolute retracts (see Problem 3.P.5 and Exercise (a)).

We now prove a theorem on the union of absolute retracts and absolute neighbourhood retracts; we will make use of it in the course of proving the theorem which asserts that polyhedra are absolute neighbourhood retracts.

6.6.13. THEOREM. *Let X be a compact metric space and let X_1, X_2 be closed subsets such that $X_1 \cup X_2 = X$; moreover, let $X_0 = X_1 \cap X_2$. If the subspaces X_0, X_1, X_2 are*

absolute retracts (absolute neighbourhood retracts) then the space X is also an absolute retract (absolute neighbourhood retract).

PROOF. It is enough to show that if X is a subset of a compact metric space Y then X is a retract of Y (a neighbourhood retract of Y). Consider the closed subspaces Y_0, Y_1, Y_2 of Y defined by

$$Y_0 = \{y \in Y : \rho(y, X_1) = \rho(y, X_2)\},$$
$$Y_1 = \{y \in Y : \rho(y, X_1) \leq \rho(y, X_2)\},$$
$$Y_2 = \{y \in Y : \rho(y, X_1) \geq \rho(y, X_2)\},$$

where ρ is the metric on Y. Evidently $Y_1 \cup Y_2 = Y$, $Y_1 \cap Y_2 = Y_0$, the set X_0 is contained in Y_0 and for $i = 1, 2$ we have $X_i \cap Y_0 = X_0$.

We first consider the case when X_0, X_1 and X_2 are absolute retracts. There then exists a retraction $r_0 \colon Y_0 \to X_0$ of the space Y_0 onto X_0. By Corollary 1.6.29 the map $r_i \colon X_i \cup Y_0 \to X_i$, where $i = 1, 2$, defined by the formula

$$r_i(y) = \begin{cases} y, & \text{for } y \in X_i, \\ r_0(y), & \text{for } y \in Y_0 \end{cases}$$

is continuous, and since $X_i \cup Y_0$ is a closed subset of Y_i, it follows by Theorem 6.6.1 that the map r_i has a continuous extension $R_i \colon Y_i \to X_i$. It is readily checked that

$$r(y) = \begin{cases} R_1(y), & \text{for } y \in Y_1, \\ R_2(y), & \text{for } y \in Y_2 \end{cases}$$

defines a retraction $r \colon Y \to X$ of the space Y onto the set X.

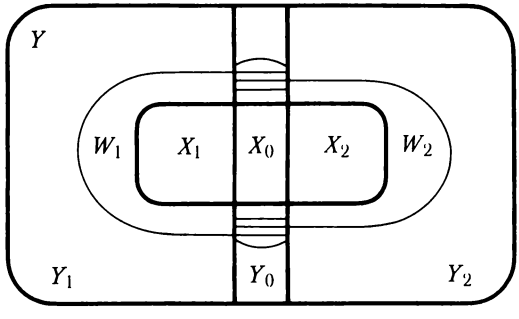


Fig.143. If the space X is a union of closed subspaces X_1 and X_2 and if $X_0 = X_1 \cap X_2$ as well as X_1 and X_2 are absolute neighbourhood retracts, then the space X is also an absolute neighbourhood retract (see Theorem 6.6.13).

We pass now to the case when X_0, X_1 and X_2 are absolute neighbourhood retracts. For some open set W_0 of Y_0 which contains X_0 there exists a retraction $r_0' \colon W_0 \to X_0$. By Theorem 1.6.27 applied to the closed sets $A = X_0$ and $B = Y_0 \backslash W_0$ of the space Y_0 there exists a set U_0 open in Y_0 such that $X_0 \subset U_0$ and $\operatorname{cl} U_0 \subset W_0$, where the symbol cl denotes closure in the space Y. The map $r_i' \colon X_i \cup \operatorname{cl} U_0 \to X_i$, where $i = 1, 2$, defined by the formula

$$r_i'(y) = \begin{cases} y, & \text{for } y \in X_i, \\ r_0'(y), & \text{for } y \in \operatorname{cl} U_0 \end{cases}$$

is continuous, and since $X_i \cup \operatorname{cl} U_0$ is closed in Y_i it follows from Theorem 6.6.3 that the map r'_i has a continuous extension $R'_i: W_i \to X_i$ defined on some set W_i open in Y_i and containing the union $X_i \cup \operatorname{cl} U_0$. Applying Theorem 1.6.27 we can find a set U_i open in Y_i and containing X_i such that $\operatorname{cl} U_i \subset W_i$ and $\operatorname{cl} U_i \cap Y_0 \subset U_0$. It is readily checked that

$$r'(y) = \begin{cases} R'_1(y), & \text{for } y \in \operatorname{cl} U_1, \\ R'_2(y), & \text{for } y \in \operatorname{cl} U_2 \end{cases}$$

defines a retraction $r': \operatorname{cl} U_1 \cup \operatorname{cl} U_2 \to X$. To complete the proof it remains to show that the interior U in Y of the set $\operatorname{cl} U_1 \cup \operatorname{cl} U_2$ contains X.

Suppose that $X \cap (Y \setminus U) \neq \emptyset$; then there exists a sequence $y_1, y_2, \ldots$ of points of $Y \setminus (\operatorname{cl} U_1 \cup \operatorname{cl} U_2)$ convergent to a point $y_0 \in X$. One of the sets Y_i contains infinitely many terms of this sequence. Without loss of generality we may suppose that $y_n \in Y_1$ for $n = 1, 2, \ldots$ and so $y_n \in Y_1 \setminus U_1$ for $n = 1, 2, \ldots$ Since $Y_1 \setminus U_1$ is closed in Y it follows that $y_0 \in X \cap (Y_1 \setminus U_1) = X_1 \setminus U_1 = \emptyset$, so our assumption leads to a contradiction; that is, the inclusion $X \subset U$ holds. ■

6.6.14. THEOREM. *Every polyhedron is an absolute neighbourhood retract.*

PROOF. We need to show that the underlying space $|K|$ of any simplicial complex K is an absolute neighbourhood retract. We proceed by induction on the number k of simplices in K. If $k = 1$, then $|K| = \emptyset$ is an absolute neighbourhood retract. Assume that the underlying space of any simplicial complex K consisting of k simplices is an absolute neighbourhood retract. We consider a complex K consisting of $k + 1$ simplices. Let Δ be the simplex of maximal dimension in K. Then $K_0 = K \setminus \{\Delta\}$ is a simplicial complex and by the inductive hypothesis the underlying space $|K_0|$ is an absolute neighbourhood retract. Since $|K| = |K_0| \cup \Delta$ and $|K_0| \cap \Delta = \operatorname{bd} \Delta$ it follows by Theorems 6.6.7 and 6.6.13 that $|K|$ is an absolute neighbourhood retract. This completes the proof. ■

Manifolds are also absolute neighbourhood retracts but a proof of this fact lies beyond the scope of this book (see Supplements 6.S.14 and 6.S.17).

We now introduce the notion of local contractibility, the local analogue of contractibility introduced in Section 3.2. The notion plays a fundamental role in the theory of retracts. We begin with an auxiliary definition. Let A be a subset of the space X. We say that the set A is *contractible to the point x_0 in the space X*, if the inclusion map $i_A: A \to X$ is homotopic to the constant map $c: A \to X$ where $c(A) = \{x_0\}$. Thus contractibility of a space X signifies that the space X is contractible in itself to a point x_0. A space X is *locally contractible at a point x* if for each neighbourhood U of x there is a neighbourhood V of the point which is contained in U such that the set V is contractible in U to the point x; if a space is locally contractible at every point, we say that the space is *locally contractible* (see Supplement 6.S.8). It follows immediately from the definition that local contractibility is a topological notion. The local analogue of Assertion 3.2.10 follows.

6.6.15. ASSERTION. *Every locally contractible space is locally pathwise connected.* ■

We leave it to the reader to give an example of a locally pathwise connected space which is not locally contractible (see Exercise (g)).

6.6.16. LEMMA. *The Hilbert cube $I^{\aleph_0}$ is contractible and also locally contractible.*

PROOF. Let $x_0 = \{x_1^0, x_2^0, \ldots\}$ be an arbitrary point of the Hilbert cube. By Assertion 6.1.11 and Theorem 6.1.12 the formula

$$h(\{x_1, x_2, \ldots\}, s) = \{(1-s)x_1 + sx_1^0, (1-s)x_2 + sx_2^0, \ldots\},$$

where $\{x_1, x_2, \ldots\} \in I^{\aleph_0}$ and $s \in I$, defines a homotopy $h\colon I^{\aleph_0} \times I \to I^{\aleph_0}$ between the identity map $\mathrm{id}\colon I^{\aleph_0} \to I^{\aleph_0}$ and the constant map $c\colon I^{\aleph_0} \to I^{\aleph_0}$ where $c(I^{\aleph_0}) = \{x_0\}$, and so the Hilbert cube is contractible.

Consider now a point $x = \{x_1, x_2, \ldots\} \in I^{\aleph_0}$ and its neighbourhood $U \subset I^{\aleph_0}$. It follows from Assertion 6.1.15 that there exists a natural number m and a positive real number ϵ such that the topological product $A = \bigtimes_{i=1}^{\infty} A_i$ where $A_i = [x_i - \epsilon, x_i + \epsilon]$ for $i = 1, 2, \ldots, m$ and $A_i = I$ for $i > m$ is contained in U. The set A is homeomorphic to $I^{\aleph_0}$ hence is contractible in itself to the point x. But $x \in \operatorname{int} A$ by Theorem 6.1.14, so the Hilbert cube is locally contractible because the set $V = \operatorname{int} A$ is contractible to the point x in the space U. ■

6.6.17. LEMMA. *A retract (a neighbourhood retract) of a contractible (locally contractible) space is also a contractible (locally contractible) space.*

PROOF. We consider first a contractible space X and a retract A of X. Let $h_0\colon X \times I \to X$ be a homotopy between the identity map $\mathrm{id}\colon X \to X$ and the constant map $c_0\colon X \to X$ where $c_0(X) = \{x_0\}$ and let $r\colon X \to A$ be a retraction of X onto A. Take

$$h(x, s) = rh_0(x, s) \quad \text{for } x \in A, \quad s \in I.$$

This defines a homotopy $h\colon A \times I \to A$ between the identity map $\mathrm{id}\colon A \to A$ and the constant map $c\colon A \to A$ where $c(A) = \{r(x_0)\}$, and so A is a contractible space.

Now consider a locally contractible space X and a neighbourhood retract A of X. Let U_0 be an open set of X containing A such that there is a retraction $r\colon U_0 \to A$. Take an arbitrary point x_0 of the subspace A and let U be a neighbourhood of x_0 in the subspace A. The set $r^{-1}(U)$ is open in the subspace $U_0 \subset X$ and hence also in the space X. Since X is locally contractible and $x_0 \in r^{-1}(U)$ there exists a neighbourhood W of x_0 in the space X contained in $r^{-1}(U)$ which is contractible to the point x_0 in the space $r^{-1}(U)$. Let $h_0\colon W \times I \to r^{-1}(U)$ be a homotopy between the inclusion map $i_W\colon W \to r^{-1}(U)$ and the constant map $c_0\colon W \to r^{-1}(U)$ where $c_0(W) = \{x_0\}$. The set $V = A \cap W \subset U$ is a neighbourhood of x_0 in the space A. Take

$$h(x, s) = rh_0(x, s) \quad \text{for } x \in V, \quad s \in I.$$

This defines a homotopy $h\colon V \times I \to U$ between the inclusion map $i_V\colon V \to U$ and the constant map $c\colon V \to U$ where $c(V) = \{x_0\}$, and so A is a locally contractible space. ■

Theorem 6.6.10 and Lemmas 6.6.16 and 6.6.17 imply the following.

6.6.18. THEOREM. *Absolute retracts are contractible and locally contractible; absolute neighbourhood retracts are locally contractible.* ■

In the realm of finite-dimensional compact spaces, that is, in the realm of compact subspaces of the Euclidean spaces (see Sections 6.7 and 6.8), the converse of Theorem 6.6.18 is also true. It may indeed be proved that every locally contractible compact space of finite dimension is an absolute neighbourhood retract: the proof lies outside the scope of this book (see Supplement 6.S.17). Similarly every contractible, locally contractible compact space of finite dimension is an absolute retract. The latter claim is a consequence of the converse of the second part of Theorem 6.6.18 and of the following important theorem which characterizes the difference between absolute retracts and absolute neighbourhood retracts by reference to the internal properties of spaces.

6.6.19. THEOREM. *A compact metric space is an absolute retract if and only if it is an absolute neighbourhood retract and is contractible.*

PROOF. It is enough to prove that if an absolute neighbourhood retract Y is contractible then it is an absolute retract. Without loss of generality we may assume that Y is a subspace of the Hilbert cube $I^{\aleph_0}$. Consider any homotopy $h: Y \times I \to Y$ between a constant map $c: Y \to Y$, where $c(Y) = \{x\}$ and the identity map $\mathrm{id}: Y \to Y$. Now the map c has a continuous extension $c^*: I^{\aleph_0} \to Y$, hence by Theorem 6.6.8 the identity map $\mathrm{id}: Y \to Y$ also has a continuous extension $r = \mathrm{id}^*: I^{\aleph_0} \to Y$. Evidently r is a retraction of $I^{\aleph_0}$ onto Y, hence Y is an absolute retract by Theorem 6.6.10. ■

We close this section by proving an interesting theorem connecting absolute neighbourhood retracts with polyhedra. We prove in fact that any absolute neighbourhood retract is homotopically dominated by some polyhedron (see Supplement 6.S.18). The theorem plays a significant role in algebraic topology since it follows from it that many of the properties of absolute neighbourhood retracts which are studied in that branch of topology, in particular homotopy properties, are similar to the corresponding properties of polyhedra.

6.6.20. THEOREM. *For each absolute neighbourhood retract X there exists a polyhedron which homotopically dominates X.*

PROOF. Without loss of generality we may suppose that X is a subspace of the Hilbert cube $I^{\aleph_0}$. We consider an open set $U \subset I^{\aleph_0}$ containing X for which there exists a retraction $r: U \to X$. From the compactness of X, Assertion 1.6.7 and Theorem 1.8.16 follows the existence of a positive real number ϵ with the property that $\rho(x, I^{\aleph_0} \setminus U) \geq \epsilon$ for each $x \in X$; evidently $B(X; \epsilon) \subset U$.

For each natural number m consider the continuous map $g_m: I^{\aleph_0} \to I^m$ defined by the formula

$$g_m(\{x_1, x_2, \ldots\}) = (x_1, x_2, \ldots, x_m), \quad \text{where} \quad \{x_1, x_2, \ldots\} \in I^{\aleph_0},$$

and consider the sets

$$X_m = g_m(X) \subset I^m, \quad W_m = B(X_m; 1/m) \subset I^m, \quad U_m = g_m^{-1}(W_m) \subset I^{\aleph_0}.$$

It is easily checked that $X \subset U_m \subset B(X; \epsilon_m) \subset I^{\aleph_0}$, where

$$\epsilon_m = \frac{1}{m} + 2\sqrt{\sum_{i=m+1}^{\infty} 1/i^2}.$$

Now $\lim_m \epsilon_m = 0$, so there exists a natural number n such that $\epsilon_n < \epsilon$; whence

$$X \subset U_n \subset B(X;\epsilon_n) \subset B(x;\epsilon) \subset U \subset I^{\aleph_0}.$$

By Corollary 2.4.9 there exists a simplicial complex of diameter less than $1/n$ whose underlying space is the cube I^n. The union of all the simplices in the complex which meet X_n form a polyhedron Z satisfying the inclusions $X_n \subset Z \subset W_n$. We prove that this polyhedron homotopically dominates X.

Since $g_n^{-1}(W_n) = U_n \subset U$ we have that for each point $(x_1, x_2, \ldots, x_n) \in Z$ the point $\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$ belongs to U. Hence the formula

$$f(x_1, x_2, \ldots, x_n) = r(\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}), \quad \text{where} \quad (x_1, x_2, \ldots, x_n) \in Z,$$

defines a continuous map $f: Z \to X$. The restriction $g = g_n | X: X \to Z$ is also a continuous map.

Now, the inclusion $g_n^{-1}(W_n) \subset U$ implies that for each point $x = \{x_1, x_2, \ldots\} \in X$ and for each real number $s \in I$ the point $\{x_1, x_2, \ldots, x_n, sx_{n+1}, sx_{n+2}, \ldots\}$ belongs to U, so the formula

$$h(x,s) = r(\{x_1, x_2, \ldots, x_n, sx_{n+1}, sx_{n+2}, \ldots\}), \quad \text{for } x = \{x_1, x_2, \ldots\} \in X \text{ and } s \in I,$$

defines a continuous map $h: X \times I \to X$. But for each $x \in X$ we have $h(x,0) = fg(x)$ and $h(x,1) = x$, so $fg \simeq \mathrm{id}_X$, that is the polyhedron Z homotopically dominates the space X. ■

Exercises

a) Prove that a neighbourhood retract of a locally connected space is also locally connected. Deduce that the space described in Example 3.1.22 is not a retract of the square I^2.

b) Show that the components of an absolute neighbourhood retract are themselves absolute neighbourhood retracts.

c) Show that the metric and the topological products of an infinite sequence of absolute retracts are absolute retracts.

d) Show that the metric product of a finite number of absolute neighbourhood retracts is an absolute neighbourhood retract. Check that the Cantor set $D^{\aleph_0}$ is not an absolute neighbourhood retract.

e) Let X be a compact metric space and let X_1, X_2 be closed subsets such that $X = X_1 \cup X_2$; moreover, let $X_0 = X_1 \cap X_2$. Prove that if the space X and the subspace X_0 are absolute retracts (absolute neighbourhood retracts) then the subspaces X_1 and X_2 of the space X are also absolute retracts (absolute neighbourhood retracts).

f) Prove that if X is locally contractible and A is an open subset of the space X, then the subspace A is locally contractible.

g) Give an example of a closed subspace of the plane which is locally pathwise connected but not locally contractible.

h) Give an example of a compact space which is not an absolute neighbourhood retract but is homotopically dominated by a polyhedron.

6.7. The dimension of separable metric spaces

For any separable metric space X we may define the *dimension of the space* X, denoted $\operatorname{ind} X$, which is an integer greater than or equal to -1, or the symbol ∞. This dimension is defined by recursion (cf. Supplement 6.S.22):

(D1) $\operatorname{ind} X = -1$ *if and only if* $X = \emptyset$,

(D2) $\operatorname{ind} X \leq n$, *where* $n = 0, 1, 2, \ldots$, *if for every* $x \in X$ *and every open* $V \subset X$ *containing* x, *there exists an open* $U \subset X$ *such that* $x \in U \subset V$ *and* $\operatorname{ind} \operatorname{bd} U \leq n - 1$,

(D3) $\operatorname{ind} X = n$ *if* $\operatorname{ind} X \leq n$ *and* $\operatorname{ind} X \leq n - 1$ *does not hold*,

(D4) $\operatorname{ind} X = \infty$, *if* $\operatorname{ind} X \neq n$ *for* $n = -1, 0, 2, \ldots$

It follows from the definition that the dimension is a topological invariant; that is, if two spaces X and Y are homeomorphic, then $\operatorname{ind} X = \operatorname{ind} Y$. Indeed, only notions which reduce to the notion of an open set occur in the definition, and since under a homeomorphism $h: X \to Y$ open subsets of X and Y and open subsets of the subspaces $A \subset X$ and $h(A) \subset Y$ correspond, it must be that $\operatorname{ind} X = \operatorname{ind} Y$. This line of argument can be turned into a proper proof by using induction on the dimension of the space X (see Exercise (a)).

In the interest of brevity we assume for every integer n the relations $n \leq \infty$ and $n + \infty = \infty + n = \infty + \infty = \infty$.

We note that the condition (D2) may be equivalently reformulated to read: $\operatorname{ind} X \leq n$, where $n \geq 0$, provided that for every point $x \in X$ and any closed set $B \subset X$ omitting the point x, there exists an open set $U \subset X$ such that $x \in U$, $U \cap B = \emptyset$ and $\operatorname{ind} \operatorname{bd} U \leq n - 1$.

A separable metric space X with $\operatorname{ind} X = n$ is called *n-dimensional*; instead of "0-dimensional" and "1-dimensional", etc. we shall usually write "zero-dimensional", "one-dimensional", etc.

6.7.1. EXAMPLE. The space of irrational numbers P, regarded as a subspace of the real line $\mathbf{R}$, is zero-dimensional. Indeed, for any point $x \in P$ and for any neighbourhood $V \subset P$ of x, there exists an interval (a, b) with rational endpoints, such that $x \in P \cap (a, b) \subset V$; since $U = P \cap (a, b)$ is an open-and-closed subset of P, it follows by Corollary 1.6.31 that $\operatorname{ind} P = 0$ (see Exercise (b)).

Similarly, the space of rational numbers $Q \subset \mathbf{R}$ is zero-dimensional. More generally, if a separable metric space has power less than $\mathfrak{c}$ then $\operatorname{ind} X = 0$; for, if $x \in X$ is arbitrary and V is a neighbourhood of x, then there exists a positive number ϵ such that $B(x; \epsilon) \subset V$ and $\rho(x, y) \neq \epsilon$ for all $y \in X$. Now taking $U = B(x; \epsilon)$ we have $x \in U \subset V$ and $\operatorname{ind} \operatorname{bd} U = -1$, and so $\operatorname{ind} X = 0$. ■

6.7.2. EXAMPLE. Of course $\operatorname{ind} \mathbf{R}^0 = \operatorname{ind} S^0 = \operatorname{ind} I^0 = 0$. For any point x of the space $\mathbf{R}^1$, of the sphere S^1, or of the interval I^1, and for any neighbourhood V of x, there exists an open set U such that $x \in U \subset V$ and $\operatorname{card} \operatorname{bd} U \leq 2$, hence $\operatorname{ind} \mathbf{R}^1 \leq 1$, $\operatorname{ind} S^1 \leq 1$ and $\operatorname{ind} I^1 \leq 1$. Since a zero-dimensional space with at least two points is not connected, we have $\operatorname{ind} \mathbf{R}^1 = 1$, $\operatorname{ind} S^1 = 1$ and $\operatorname{ind} I^1 = 1$.

For every point x either of the Euclidean space $\mathbf{R}^m$, or the sphere S^m, or the cube I^m, where $m = 2, 3, \ldots$ and for any neighbourhood V of x, there exists an open set U such that $x \in U \subset V$ and such that the boundary $\operatorname{bd} U$ is homeomorphic to S^{m-1} or I^{m-1}. We deduce by induction that $\operatorname{ind} \mathbf{R}^m \leq m$, $\operatorname{ind} S^m \leq m$ and $\operatorname{ind} I^m \leq m$. In the next section we shall prove that $\operatorname{ind} \mathbf{R}^m = m$, $\operatorname{ind} S^m = m$ and $\operatorname{ind} I^m = m$ for $m = 2, 3, \ldots$; the proof is much harder than for the inequalities just established. ■

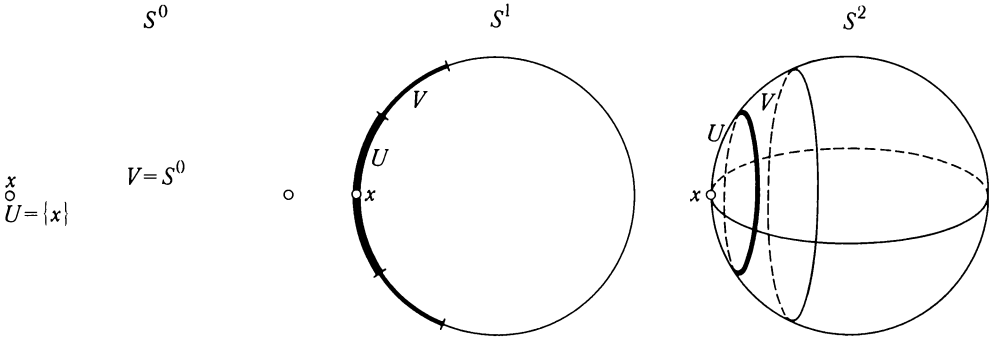


Fig.144. $\operatorname{bd} U$ is empty, so $\operatorname{ind} S^0 \leq 0$.

$\operatorname{bd} U$ is homeomorphic to S^0, so $\operatorname{ind} S^1 \leq 1$.

$\operatorname{bd} U$ is homeomorphic to S^1, so $\operatorname{ind} S^2 \leq 2$.

We can reformulate the condition (D2) which characterizes separable metric spaces X satisfying the inequality $\operatorname{ind} X \leq n$ for $n \geq 0$, by referring to the concept of a base.

6.7.3. THEOREM. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n$ *for* $n \geq 0$ *if and only if X has a countable base $\mathcal{B}$ with the property that* $\operatorname{ind} \operatorname{bd} U \leq n - 1$ *for every* $U \in \mathcal{B}$.

PROOF. From the definition of a base it follows immediately that if X has a base $\mathcal{B}$ with the stated property then $\operatorname{ind} X \leq n$. So let us consider a separable metric space X satisfying the inequality $\operatorname{ind} X \leq n$ for $n \geq 0$. For $i = 1, 2, \ldots$ and for any point $x \in X$ let us select an open set $U_{i,x} \subset X$ such that

$$x \in U_{i,x} \subset B(x; 1/2i) \quad \text{and} \quad \operatorname{ind} \operatorname{bd} U_{i,x} \leq n - 1.$$

The family $\mathcal{A}_i = \{U_{i,x}\}_{x \in X}$ for $i = 1, 2, \ldots$ forms an open covering of the space X, so by Theorem 6.3.3 we can choose a countable subcovering $\mathcal{B}_i$ from $\mathcal{A}_i$. It follows from Lemma 6.3.2 that the union $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a base of the space X. Evidently the base $\mathcal{B}$ is countable and for each $U \in \mathcal{B}$ it is the case that $\operatorname{ind} \operatorname{bd} U \leq n - 1$. ■

It turns out that dimension is a monotonic function.

6.7.4. THEOREM. *For any subspace A of a separable metric space X the inequality* $\operatorname{ind} A \leq \operatorname{ind} X$ *holds.*

PROOF. The theorem is obvious when $\operatorname{ind} X = \infty$. Assume that $\operatorname{ind} X < \infty$. We argue by induction on $\operatorname{ind} X$. If $\operatorname{ind} X = -1$ the theorem is true. We suppose the theorem has been proved true on the assumption that the spaces considered have dimension not exceeding $n - 1$. Consider a space X for which $\operatorname{ind} X = n$, a subspace

A and a neighbourhood V in the space A of a point x lying in A. From Theorem 1.6.5 it follows that there exists an open set V' of the space X satisfying $V = A \cap V'$. Since $\operatorname{ind} X \leq n$, there exists an open set $U' \subset X$ such that

$$x \in U' \subset V' \quad \text{and} \quad \operatorname{ind} \operatorname{bd} U' \leq n-1.$$

The set $U = A \cap U'$ is open in the space A and satisfies $x \in U \subset V$. The boundary $\operatorname{bd}_A U$ of the set U in the space A may be expressed in the form $A \cap \operatorname{cl}(A \cap U') \cap \operatorname{cl}(A \setminus U')$, where the symbol cl denotes closure in the space X (cf. Problem 1.P.28). This boundary is thus a subspace of the space $\operatorname{bd} U' = \operatorname{cl} U' \cap \operatorname{cl}(X \setminus U')$, and so, by the inductive hypothesis, we have $\operatorname{ind} \operatorname{bd}_A U \leq n-1$. From condition (D2) it follows that $\operatorname{ind} A \leq n = \operatorname{ind} X$. ■

We proceed now to the proof of two important theorems of dimension theory, namely the separation theorem and the sum theorem. We shall first prove these theorems when the dimension is zero, and then, arguing by induction, when the dimension is an arbitrary n. Such a proof structure reflects the inductive nature of the definition of dimension.

6.7.5. THEOREM. *Let X be a separable metric space satisfying $\operatorname{ind} X \leq 0$. For every pair of disjoint, closed sets $A, B \subset X$, there exist disjoint, open sets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$ and $\operatorname{bd} U = \operatorname{bd} V = \emptyset$.*

PROOF. For any point $x \in X$ we choose an open-and-closed set $U_x \subset X$ such that $x \in U_x$ and

$$A \cap U_x = \emptyset \quad \text{or} \quad B \cap U_x = \emptyset.$$

By Theorem 6.3.3 the covering $\{U_x\}_{x \in X}$ of X contains a countable covering $\{U_{x_i}\}_{i=1}^{\infty}$. The open sets

$$V_i = U_{x_i} \setminus \bigcup_{j<i} U_{x_j} \subset U_{x_i}, \quad \text{for} \quad i = 1, 2, \ldots,$$

are pairwise disjoint and form a covering of the space X. Let

$$U = \bigcup \{V_i : A \cap V_i \neq \emptyset\} \quad \text{and} \quad V = \bigcup \{V_i : A \cap V_i = \emptyset\}.$$

It may readily be checked that the sets U, V are open and disjoint and that $A \subset U$ and $B \subset V$. Since $U \cup V = X$, the sets U, V are open-and-closed; we thus have $\operatorname{bd} U = \operatorname{bd} V = \emptyset$. ■

6.7.6. THEOREM. *If the separable metric space X is the union of a countable sequence $F_1, F_2, \ldots$ of closed sets with $\operatorname{ind} F_i \leq 0$ for $i = 1, 2, \ldots$, then $\operatorname{ind} X \leq 0$.*

PROOF. Consider an arbitrary point $x \in X$ and a neighbourhood $V \subset X$ of x. Applying Theorem 1.6.27 twice we find open sets $U_0, W_0 \subset X$ such that

$$x \in U_0, \quad X \setminus V \subset W_0 \quad \text{and} \quad \operatorname{cl} U_0 \cap \operatorname{cl} W_0 = \emptyset.$$

We shall define by induction two sequences $U_0, U_1, U_2, \ldots$ and $W_0, W_1, W_2, \ldots$ of open subsets of the space X satisfying, for $i = 0, 1, 2, \ldots$, the conditions

(*) $\operatorname{cl} U_i \cap \operatorname{cl} W_i = \emptyset$,

(**) $U_{i-1} \subset U_i$, $W_{i-1} \subset W_i$ and $F_i \subset U_i \cup W_i$, provided $i > 0$.

The sets U_0, W_0 satisfying (*) and (**) for $i = 0$ have already been defined above. Suppose that sets U_i, W_i satisfying (*) and (**) have been defined for every $i < k$, where $k \geq 1$.

The sets $F_k \cap \operatorname{cl} U_{k-1}$ and $F_k \cap \operatorname{cl} W_{k-1}$ are closed in the subspace $F_k \subset X$ and are disjoint. Since $\operatorname{ind} F_k \leq 0$, by Theorem 6.7.5 there exists an open-and-closed set U' in the space F_k such that

$$F_k \cap \operatorname{cl} U_{k-1} \subset U' \quad \text{and} \quad F_k \cap \operatorname{cl} W_{k-1} \subset F_k \backslash U'.$$

In view of F_k being closed, the sets U' and $F_k \backslash U'$ are closed in the space X. The sets $U' \cup \operatorname{cl} U_{k-1}$ and $(F_k \backslash U') \cup \operatorname{cl} W_{k-1}$ are also closed, and since

$$(U' \cup \operatorname{cl} U_{k-1}) \cap [(F_k \backslash U') \cup \operatorname{cl} W_{k-1}] = (U' \cap \operatorname{cl} W_{k-1}) \cup [(F_k \backslash U') \cap \operatorname{cl} U_{k-1}] = \emptyset,$$

there exist open sets $U_k, W_k \subset X$ such that

$$U' \cup \operatorname{cl} U_{k-1} \subset U_k, \quad (F_k \backslash U') \cup \operatorname{cl} W_{k-1} \subset W_k \quad \text{and} \quad \operatorname{cl} U_k \cap \operatorname{cl} W_k = \emptyset.$$

The sets U_k, W_k just obtained satisfy the conditions (*) and (**) for $i = k$, so the construction of the sequences $U_0, U_1, U_2, \ldots$ and $W_0, W_1, W_2, \ldots$ is complete.

Consider the open sets $U = \bigcup_{i=0}^{\infty} U_i$ and $W = \bigcup_{i=0}^{\infty} W_i$. From (*) and (**) it follows that $U \cap W = \emptyset$ and that $U \cup W = X$, i.e. $U = X \backslash W$, so the set U is open-and-closed and $\operatorname{bd} U = \emptyset$. But $X \backslash V \subset W_0 \subset W$, so $x \in U_0 \subset U = X \backslash W \subset V$, which proves that $\operatorname{ind} X \leq 0$. ■

Two lemmas precede the proof of the sum theorem. The first is a strengthening of the separation theorem for dimension zero; it is called the omission theorem for dimension zero (see Exercise (e)).

6.7.7. LEMMA. *Let X be a separable metric space, and Z a subspace of X satisfying* $\operatorname{ind} Z \leq 0$. *For every pair of disjoint closed sets $A, B \subset X$, there exist disjoint open sets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$ and $Z \cap \operatorname{bd} U = Z \cap \operatorname{bd} V = \emptyset$.*

PROOF. Let $W_1, W_2 \subset X$ be open sets satisfying the conditions

$$A \subset W_1, \quad B \subset W_2 \quad \text{and} \quad \operatorname{cl} W_1 \cap \operatorname{cl} W_2 = \emptyset.$$

By Theorem 6.7.5 there exists an open-and-closed set U_0 of Z such that

$$Z \cap \operatorname{cl} W_1 \subset U_0 \quad \text{and} \quad Z \cap \operatorname{cl} W_2 \subset Z \backslash U_0.$$

Consider the functions $f, g\colon X \to \mathbf{R}$, where

$$f(x) = \rho(x, A \cup U_0) \quad \text{and} \quad g(x) = \rho(x, B \cup (Z \backslash U_0)) \quad \text{for} \quad x \in X.$$

From Theorems 1.4.10 and 1.6.24 it follows that the sets

$$U = \{x \in X : f(x) < g(x)\} \quad \text{and} \quad V = \{x \in X : f(x) > g(x)\}$$

are open in X; evidently $U \cap V = \emptyset$. Now

$$Z \setminus U_0 \subset X \setminus W_1 \subset X \setminus A \quad \text{and} \quad U_0 \subset X \setminus W_2 \subset X \setminus B,$$

so that $x \in A$ implies $f(x) = 0$ and $g(x) > 0$, whereas $x \in B$ implies $g(x) = 0$ and $f(x) > 0$. Hence $A \subset U$ and $B \subset V$. Using the fact that $Z \setminus U_0$ and U_0 are closed in the subspace Z it may readily be checked that for $x \in U_0$ we have $f(x) = 0$ and $g(x) > 0$, whereas for $x \in Z \setminus U_0$ we have $g(x) = 0$ and $f(x) > 0$. Thus $Z = U_0 \cup (Z \setminus U_0) \subset U \cup V$. But $\operatorname{bd} U \cup \operatorname{bd} V \subset X \setminus (U \cup V)$, so $Z \cap \operatorname{bd} U = Z \cap \operatorname{bd} V = \emptyset$. ■

6.7.8. LEMMA. *If the separable metric space X is the union of subspaces Y and Z satisfying* $\operatorname{ind} Y \leq n-1$ *and* $\operatorname{ind} Z \leq 0$, *then* $\operatorname{ind} X \leq n$.

PROOF. We consider an arbitrary point $x \in X$ and its neighbourhood $V \subset X$. Applying Lemma 6.7.7 to the sets $A = \{x\}$ and $B = X \setminus V$ we obtain disjoint open sets $U, W \subset X$ such that $x \in U$, $X \setminus V \subset W$ and $Z \cap \operatorname{bd} U = \emptyset$. The latter equation implies that $\operatorname{bd} U \subseteq Y$, so $\operatorname{ind} \operatorname{bd} U \leq n-1$, by Theorem 6.7.4. Since $U \subset X \setminus W \subset V$, we have $\operatorname{ind} X \leq n$. ■

6.7.9. THEOREM (The Sum Theorem). *If the separable metric space X is the union of a countable sequence $F_1, F_2, \ldots$ of closed sets, with* $\operatorname{ind} F_i \leq n$ *for* $i = 1, 2, \ldots$, *then* $\operatorname{ind} X \leq n$.

PROOF. We proceed by induction on n. The case $n = 0$ is subsumed under Theorem 6.7.6. We suppose the theorem is proved for dimensions less than n where $n > 0$ and we consider a space $X = \bigcup_{i=1}^{\infty} F_i$ where $\operatorname{cl} F_i = F_i$ and $\operatorname{ind} F_i \leq n$ for $i = 1, 2, \ldots$ Theorem 6.7.3 secures the existence of a countable base $\mathcal{B}_i$ of the subspace $F_i \subset X$ with the property that $\operatorname{ind} \operatorname{bd}_{F_i} U \leq n-1$ for every $U \in \mathcal{B}_i$, where the symbol bd_{F_i} denotes the boundary operator for the subspace F_i. Since the sets F_i are closed, it follows that the sets $\operatorname{bd}_{F_i} U$ are closed in X. By the inductive hypothesis the subspace $Y = \bigcup_{i=1}^{\infty} [\bigcup \{\operatorname{bd}_{F_i} U : U \in \mathcal{B}_i\}]$ of X satisfies $\operatorname{ind} Y \leq n-1$, being a countable union of closed sets whose dimension does not exceed $n-1$. Let $Z_i = F_i \setminus Y$ for $i = 1, 2, \ldots$ Since the family $\{Z_i \cap U : U \in \mathcal{B}_i\}$ is a base of the subspace $Z_i \subset X$ and consists of sets which are open-and-closed in Z_i, we have $\operatorname{ind} Z_i \leq 0$. The subspace $Z = X \setminus Y$ of the space X is the union of the closed sets $Z_i = F_i \cap Z$ for $i = 1, 2, \ldots$, so $\operatorname{ind} Z \leq 0$ by Theorem 6.7.6. Applying Lemma 6.7.8 we deduce that $\operatorname{ind} X \leq n$. ■

We pass now to the separation theorem.

6.7.10. LEMMA. *If a separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n$ *where* $n \geq 0$, *then X is a union of two subspaces Y and Z such that* $\operatorname{ind} Y \leq n-1$ *and* $\operatorname{ind} Z \leq 0$.

PROOF. Using Theorem 6.7.3 consider a countable base $\mathcal{B}$ of the space X with the property that $\operatorname{ind} \operatorname{bd} U \leq n-1$ for every $U \in \mathcal{B}$. By the sum theorem (6.7.9) the

subspace $Y = \bigcup\{\operatorname{bd} U : U \in \mathcal{B}\}$ of X satisfies $\operatorname{ind} Y \leq n-1$. Let $Z = X \setminus Y$; now the family $\{Z \cap U : U \in \mathcal{B}\}$ is a base of the subspace $Z \subset X$ and consists of sets which are open-and-closed in Z, so $\operatorname{ind} Z \leq 0$. ■

6.7.11. THEOREM (The Separation Theorem). *Let X be a separable metric space satisfying $\operatorname{ind} X \leq n$, where $n \geq 0$. For every pair of disjoint closed sets $A, B \subset X$, there exist disjoint, open sets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$ and $\operatorname{ind} \operatorname{bd} U \leq n-1$ and $\operatorname{ind} \operatorname{bd} V \leq n-1$.*

PROOF. We use Lemma 6.7.10 to express X in the form $Y \cup Z$ where $\operatorname{ind} Y \leq n-1$ and $\operatorname{ind} Z \leq 0$. By Lemma 6.7.7 there exist disjoint open sets $U, W \subset X$ satisfying $A \subset U$, $B \subset V$ and $Z \cap \operatorname{bd} U = Z \cap \operatorname{bd} V = \emptyset$. The last two equations imply that $\operatorname{bd} U \subseteq Y$ and $\operatorname{bd} V \subseteq Y$ hence $\operatorname{ind} \operatorname{bd} U \leq n-1$ and $\operatorname{ind} \operatorname{bd} V \leq n-1$. ■

It follows from Lemmas 6.7.8 and 6.7.10 that a separable metric space X satisfies the inequality $\operatorname{ind} X \leq n$ for $n \geq 0$ if and only if X is the union of two subspaces Y and Z with $\operatorname{ind} Y \leq n-1$ and $\operatorname{ind} Z \leq 0$. A simple inductive argument now gives the following.

6.7.12. THEOREM (The Decomposition Theorem). *A non-empty separable metric space X satisfies the inequality $\operatorname{ind} X \leq n$ for $n \geq 0$ if and only if it may be expressed as the union of $n+1$ zero-dimensional subspaces.* ■

In the next theorem we give an estimate for the dimension of the product of two metric spaces (see Supplement 6.S.23). We begin with a straightforward lemma (cf. Exercise (h) of Section 1.6).

6.7.13. LEMMA. *Let X_1 and X_2 be arbitrary metric spaces. The following inclusion holds for any subsets $A_1 \subset X_1$ and $A_2 \subset X_2$*

$$\operatorname{bd}(A_1 \times A_2) \subset (\operatorname{bd} A_1 \times X_2) \cup (X_1 \times \operatorname{bd} A_2),$$

where bd *signifies in order of appearance the boundary operator for the spaces $X_1 \times X_2$, X_1 and X_2.*

PROOF. If $(x_1, x_2) \notin (\operatorname{bd} A_1 \times X_2) \cup (X_1 \times \operatorname{bd} A_2)$ then $x_1 \notin \operatorname{bd} A_1$ and $x_2 \notin \operatorname{bd} A_2$. There is thus a neighbourhood $U_1 \subset X_1$ of x_1 and a neighbourhood $U_2 \subset X_2$ of x_2 such that $U_1 \cap A_1 = \emptyset$ or $U_1 \setminus A_1 = \emptyset$ and $U_2 \cap A_2 = \emptyset$ or $U_2 \setminus A_2 = \emptyset$. By Theorem 1.6.6 the set $U_1 \times U_2 \subset X_1 \times X_2$ is a neighbourhood of the point (x_1, x_2); now either $(U_1 \times U_2) \cap (A_1 \times A_2) = (U_1 \cap A_1) \times (U_2 \cap A_2) = \emptyset$ or $(U_1 \times U_2) \setminus (A_1 \times A_2) = [(U_1 \setminus A_1) \times U_2] \cup [U_1 \times (U_2 \setminus A_2)] = \emptyset$, so that $(x_1, x_2) \notin \operatorname{bd}(A_1 \times A_2)$. ■

6.7.14. THEOREM. *For any pair of separable metric spaces X_1, X_2 of which one at least is non-empty, the following inequality holds*

$$\operatorname{ind}(X_1 \times X_2) \leq \operatorname{ind} X_1 + \operatorname{ind} X_2.$$

PROOF. The theorem is obvious when $\operatorname{ind} X_1 = \infty$ or $\operatorname{ind} X_2 = \infty$. So let us assume that $\operatorname{ind} X_i < \infty$ for $i = 1, 2$. We proceed by induction on the value of the

sum $\operatorname{ind} X_1 + \operatorname{ind} X_2$. If $\operatorname{ind} X_1 + \operatorname{ind} X_2 = -1$ then either $X_1 = \emptyset$ or $X_2 = \emptyset$ and the assertion holds. Assume the theorem has been established for any pair of separable metric spaces whose dimensions sum to at most $k-1$ where $k \geq 0$ and consider non-empty spaces X_1 and X_2 such that $\operatorname{ind} X_1 = n_1$, $\operatorname{ind} X_2 = n_2$ and $n_1 + n_2 = k$. Fix any point $(x_1, x_2) \in X_1 \times X_2$ and any neighbourhood of it $V \subset X_1 \times X_2$. From the definition of the metric on the product it follows that there are open sets $V_1 \subset X_1$ and $V_2 \subset X_2$ such that $(x_1, x_2) \in V_1 \times V_2 \subset V$. Consider open sets $U_1 \subset X_1$ and $U_2 \subset X_2$ such that

$$x_i \in U_i \subset V_i \quad \text{and} \quad \operatorname{ind} \operatorname{bd} U_i \leq n_i - 1 \quad \text{for} \quad i = 1, 2.$$

The set $U_1 \times U_2$ is open in $X_1 \times X_2$. Of course, $(x_1, x_2) \in U_1 \times U_2 \subset V_1 \times V_2$; but by Lemma 6.7.13, Theorems 6.7.9 and 6.7.4 and the inductive hypothesis it follows that $\operatorname{ind} \operatorname{bd}(U_1 \times U_2) \leq k-1$, hence $\operatorname{ind}(X_1 \times X_2) \leq k = n_1 + n_2 = \operatorname{ind} X_1 + \operatorname{ind} X_2$. ■

Using Theorem 6.7.14 and the associativity of products (cf. Example 1.3.20) we obtain by induction:

6.7.15. THEOREM. *The metric product of a finite number of zero-dimensional spaces is zero-dimensional.* ■

6.7.16. EXAMPLE. *The spaces N_n^m and L_n^m.* For any pair of integers k, m where $0 \leq k \leq m$ and $m \geq 1$ let Q_k^m be the subspace of $\mathbf{R}^m$ consisting of all points with exactly k coordinates rational. Following Example 6.7.1 and Theorem 6.7.15 we have $\operatorname{ind} Q_m^m = 0$; we shall show that $\operatorname{ind} Q_k^m = 0$ holds also for $k < m$.

For any choice of k distinct natural numbers $i_1, i_2, \ldots, i_k$ not exceeding m and any choice of k rational numbers $r_1, r_2, \ldots, r_k$ the product $\bigtimes_{i=1}^m A_i$ is a closed subspace of the space $\mathbf{R}^m$, where for $j = 1, 2, \ldots, k$ we require $A_{i_j} = \{r_j\}$ and $A_i = \mathbf{R}$ for $i \notin \{i_1, i_2, \ldots, i_k\}$. The set $Q_k^m \cap \bigtimes_{i=1}^m A_i$ is thus closed in the space Q_k^m. Since the space $Q_k^m \cap \bigtimes_{i=1}^m A_i$ is homeomorphic to the subspace of points in $\mathbf{R}^{m-k}$ having all coordinates irrational, we have by Example 6.7.1 and Theorem 6.7.15 that $\operatorname{ind}(Q_k^m \cap \bigtimes_{i=1}^m A_i) = 0$. The sets of the form $Q_k^m \cap \bigtimes_{i=1}^m A_i$ constitute a countable covering of the space Q_k^m, so by Theorem 6.7.6 we have the equation $\operatorname{ind} Q_k^m = 0$.

For any pair of integers m, n with $0 \leq n \leq m$ and $m \geq 1$ consider the subspaces

$$N_n^m = Q_0^m \cup Q_1^m \cup \ldots \cup Q_n^m \quad \text{and} \quad L_n^m = Q_n^m \cup Q_{n+1}^m \cup \ldots \cup Q_m^m$$

of $\mathbf{R}^m$. Thus N_n^m consists of all points of $\mathbf{R}^m$ with at most n coordinates rational, whereas L_n^m consists of all points of $\mathbf{R}^m$ with at least n coordinates rational. It follows from the decomposition theorem (6.7.12) that

$$\operatorname{ind} N_n^m \leq n \quad \text{and} \quad \operatorname{ind} L_n^m \leq m - n \quad \text{for } 0 \leq n \leq m \text{ and } m = 1, 2, \ldots$$

In the next section we shall prove that these inequalities may be replaced by equations (cf. Corollary 6.8.6).

Notice that $\mathbf{R}^m = Q_0^m \cup Q_1^m \cup \ldots \cup Q_m^m$. Using the decomposition theorem (6.7.12) we have a second proof of the inequality $\operatorname{ind} \mathbf{R}^m \leq m$ for $m = 1, 2, \ldots$ (cf. Example 6.7.2). ■

We shall now define for any metric space X the *covering dimension* of X, denoted $\dim X$, which will be an integer greater than or equal to -1, or the symbol ∞. The definition of covering dimension is less intuitive than the definition of the dimension ind; however, the new definition is in a sense more satisfactory. It turns out that although the dimension ind may be defined for all metric spaces by means of the conditions (D1)–(D4), it nevertheless loses, in this wider class of spaces, many of its important properties; for example, it no longer obeys the separation theorem, nor the sum theorem, whereas the dimension dim obeys analogues of the theorems proved so far, in the context of all metric spaces. Moreover, as we show in the next section (see Theorem 6.8.19), for any separable metric space X we have the equation $\dim X = \operatorname{ind} X$, so that dim is the appropriate extension of the notion of dimension to the class of all metric spaces (cf. Supplement 6.S.3). Apart from the dimensions ind and dim a dimension Ind is also studied, but we shall not dwell on it here; in the realm of metric spaces the dimension Ind agrees with the dimension dim (see Supplement 6.S.22).

Before we can give the definition of the covering dimension we need to introduce an auxiliary concept.

Let $\mathcal{U} = \{U_t\}_{t\in T}$ be a covering of the metric space X; the *order* of the covering $\mathcal{U}$ is the largest integer n such that $\mathcal{U}$ contains $n+1$ sets with non-empty intersection; if no such integer exists we say that the covering $\mathcal{U}$ has order equal to ∞. Thus if the order of the covering $\mathcal{U}$ is n, then for any $n+2$ indices $t_1, t_2, \ldots, t_{n+2}$ in T we have $U_{t_1} \cap U_{t_2} \cap \ldots \cap U_{t_{n+2}} = \emptyset$. In particular a covering of order -1 contains only the empty set and a covering of order 0 consists of non-empty pairwise disjoint sets. We denote the order of the covering $\mathcal{U}$ by $\operatorname{ord}\mathcal{U}$.

The covering dimension of a metric space X is determined by the following conditions.

(CD1) $\dim X \leq n$ *for* $n = -1, 0, 1, \ldots$, *provided that for every finite open covering* $\{V_i\}_{i=1}^m$ *of the space* X *there exists an open covering* $\{U_i\}_{i=1}^m$ *of the space such that* $\operatorname{ord}(\{U_i\}_{i=1}^m) \leq n$ *and* $U_i \subset V_i$ *for* $i = 1, 2, \ldots, m$,

(CD2) $\dim X = n$ *if* $\dim X \leq n$ *and* $\dim X \leq n-1$ *does not hold,*

(CD3) $\dim X = \infty$, *if* $\dim X \neq n$ *for* $n = -1, 0, 1, \ldots$

It follows immediately from the definition that the covering dimension is a topological invariant; that is, if two spaces X and Y are homeomorphic, then $\dim X = \dim Y$. Evidently $\dim X = -1$ if and only if $X = \emptyset$.

We close this section by proving that the inequality $\dim X \leq \operatorname{ind} X$ holds for any separable metric space X. This is the easier half of the theorem on the coincidence of the dimensions ind and dim in the class of separable metric spaces. We prove the reverse inequality $\operatorname{ind} X \leq \dim X$ in the next section (cf. Problem 6.P.49).

6.7.17. LEMMA. *Let* X *be a separable metric space and* Z *any subspace of* X. *For any family* $\{F_i\}_{i=1}^m$ *of pairwise disjoint closed subsets in the space* Z *there exists a family* $\{W_i\}_{i=1}^m$ *of pairwise disjoint open subsets of the space* X, *such that* $F_i \subset W_i$ *for* $i = 1, 2, \ldots, m$.

PROOF. The sets

$$W_i = \bigcap_{j \neq i} \{x \in X : \rho(x, F_i) < \rho(x, F_j)\}$$

have the desired properties. ■

6.7.18. THEOREM. *For any separable metric space X the inequality* $\dim X \leq \operatorname{ind} X$ *holds.*

PROOF. We may of course assume that $\operatorname{ind} X < \infty$. If $\operatorname{ind} X = -1$ then $X = \emptyset$ and $\dim X = -1 \leq \operatorname{ind} X$. Let us consider the case $\operatorname{ind} X = 0$. Let $\{V_i\}_{i=1}^m$ be any finite open covering of the space X. By Theorem 6.7.3 the space X has a countable base $\mathcal{B}$ consisting of open-and-closed sets. Arrange those elements of the base $\mathcal{B}$ which are contained in at least one of the sets V_i in a sequence $B_1, B_2, \ldots$ and then for $j = 1, 2, \ldots$ pick an index $i(j) \leq m$ such that $B_j \subset V_{i(j)}$. The sets

$$A_1 = B_1, \quad A_2 = B_2 \setminus A_1, \quad \ldots, \quad A_k = B_k \setminus (A_1 \cup A_2 \cup \ldots \cup A_{k-1}), \quad \ldots$$

are open-and-closed, pairwise disjoint and form a covering of the space X. Setting $U_i = \bigcup \{A_j : i(j) = i\}$ for $i = 1, 2, \ldots, m$ we obtain an open covering $\mathcal{U} = \{U_i\}_{i=1}^m$ of the space X such that $U_i \subset V_i$ for $i = 1, 2, \ldots, m$. Since the elements of this covering are pairwise disjoint, we have $\operatorname{ord} \mathcal{U} = 0$. We conclude that $\dim X \leq 0 = \operatorname{ind} X$.

We now consider the case when $\operatorname{ind} X = n > 0$. Let $\{V_i\}_{i=1}^m$ be any finite open covering of the space X. Applying Theorem 6.7.12 express X as a union

$$X = Z_0 \cup Z_1 \cup \ldots \cup Z_n, \quad \text{where} \quad \operatorname{ind} Z_j = 0 \quad \text{for} \quad j = 0, 1, \ldots, n.$$

For each j we consider the open covering $\{V_i \cap Z_j\}_{i=1}^m$ of the subspace Z_j and, taking advantage of the case considered above, we select a covering $\{U_{i,j}\}_{i=1}^m$ of the space Z_j by pairwise disjoint open subsets of the space Z_j, such that $U_{i,j} \subset V_i \cap Z_j$ for $i = 1, 2, \ldots, m$. Evidently the sets $U_{i,j}$ are also closed in the space Z_j. Applying Lemma 6.7.17 we expand the sets $U_{i,j}$ to pairwise disjoint open subsets $W_{i,j}$ of the space X. The sets

$$U_i = V_i \cap \bigcup_{j=0}^{n} W_{i,j} \quad \text{for } i = 1, 2, \ldots, m,$$

form an open covering of the space X. The order of this covering does not exceed n. Indeed, the intersection of any $n+2$ sets $W_{i_0,j_0}, W_{i_1,j_1}, \ldots, W_{i_{n+1},j_{n+1}}$, where $i_0, i_1, \ldots, i_{n+1}$ are distinct, is empty, since there will be at least two sets among the $n+2$ considered which have second subscripts equal. Since $U_i \subset V_i$ for $i = 1, 2, \ldots, m$ we deduce that $\dim X \leq n = \operatorname{ind} X$. ■

Exercises

a) Give a detailed proof of the topological invariance of the dimension ind; that is, show that if the separable metric spaces X and Y are homeomorphic then $\operatorname{ind} X = \operatorname{ind} Y$. (Hint: Reduce the problem to the case that $\operatorname{ind} X < \infty$ and proceed by induction on $\operatorname{ind} X$).

b) Show that a non-empty subspace A of the real line $\mathbf{R}$ is zero-dimensional if and only if it does not contain any interval.

c) Prove that any separable metric space X for which $\operatorname{ind} X = n$ with $n > 0$ contains closed subspaces $X_0, X_1, \ldots, X_{n-1}$ such that $\operatorname{ind} X_i = i$ for $i = 0, 1, \ldots, n-1$ (cf. Supplement 6.S.25).

d) Show that if a separable metric space X is the union of two sets A_1 and A_2, one at least of which is closed, and $\operatorname{ind} A_i \leq n$ for $i = 1, 2$ then $\operatorname{ind} X \leq n$.

e) Let X be a separable metric space and Z a subspace of X satisfying $\operatorname{ind} Z \leq n$ with $n > 0$. Prove that for any pair of disjoint, closed sets $A, B \subset X$ there exist disjoint, open sets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$ and $\operatorname{ind}(Z \cap \operatorname{bd} U) \leq n-1$ and $\operatorname{ind}(Z \cap \operatorname{bd} V) \leq n-1$. (This is the *omission theorem.*)

f) Show that for any pair of separable subspaces X, Y of an arbitrary metric space the inequality

$$\operatorname{ind}(X \cup Y) \leq \operatorname{ind} X + \operatorname{ind} Y + 1$$

holds. (This is the *addition theorem.*)

g) Prove that the metric and the topological products of an infinite sequence of zero-dimensional spaces are zero-dimensional.

h) Show that a compact metric space X satisfies the inequality $\dim X \leq n$ if and only if for every positive real number ϵ there exists a finite open covering $\mathcal{U}$ of the space X such that $\operatorname{ord} \mathcal{U} \leq n$ and $\operatorname{diam} U < \epsilon$ for every $U \in \mathcal{U}$ (see Supplement 6.S.26).

6.8. Dimension in Euclidean spaces

We begin this section with the fundamental theorem in dimension theory which asserts that $\operatorname{ind} \mathbf{R}^m = m$ for $m = 0, 1, 2, \ldots$

6.8.1. LEMMA. *Suppose given for $i = 1, 2, \ldots, m$ open subsets U_i of the unit cube I^m such that the sets*

$$A_i = \{(x_1, x_2, \ldots, x_m) \in I^m : x_i = 0\} \quad \text{and} \quad B_i = \{(x_1, x_2, \ldots, x_m) \in I^m : x_i = 1\}$$

satisfy the inclusions $A_i \subset U_i$ and $B_i \subset I^m \setminus \operatorname{cl} U_i$, then $\bigcap_{i=1}^m \operatorname{bd} U_i \neq \emptyset$.

PROOF. Let $V_i = I^m \setminus \operatorname{cl} U_i$ and $L_i = \operatorname{bd} U_i$ for $i = 1, 2, \ldots, m$. For each $x \in I^m$ and for $i = 1, 2, \ldots, m$ let us define

$$f_i(x) = \begin{cases} \dfrac{1}{2} \cdot \dfrac{\rho(x, L_i)}{\rho(x, L_i) + \rho(x, A_i)} + \dfrac{1}{2}, & \text{for } x \in I^m \setminus V_i, \\ -\dfrac{1}{2} \cdot \dfrac{\rho(x, L_i)}{\rho(x, L_i) + \rho(x, B_i)} + \dfrac{1}{2}, & \text{for } x \in I^m \setminus U_i. \end{cases}$$

Now $(I^m \setminus V_i) \cup (I^m \setminus U_i) = I^m \setminus (U_i \cap V_i) = I^m$ and $(I^m \setminus V_i) \cap (I^m \setminus U_i) = I^m \setminus (U_i \cup V_i) = L_i$, so by Corollary 1.6.29 the function $f_i : I^m \to I$ is continuous for $i = 1, 2, \ldots, m$. It follows from the definition that

$$f_i(A_i) = \{1\}, \quad f_i(B_i) = \{0\} \quad \text{and} \quad f_i^{-1}(1/2) = L_i.$$

Let us suppose that $\bigcap_{i=1}^m L_i = \emptyset$. Since $f_i(x) = \frac{1}{2}$ only if $x \in L_i$, the continuous map $f: I^m \to I^m$ defined by the formula

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \quad \text{for } x \in I^m$$

does not assume the value $p = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in I^m$. Composing f with the retraction (see Example 3.1.11) of the set $I^m \setminus \{p\}$ onto the surface of the cube I^m, i.e. on the set $B = \bigcup_{i=1}^m (A_i \cup B_i)$, we obtain a continuous map $g: I^m \to I^m$ for which $g(I^m) \subset B$. Now $f_i(A_i) = \{1\}$ and $f_i(B_i) = \{0\}$, so $g(A_i) \subset B_i$ and $g(B_i) \subset A_i$. We conclude that $g(x) \neq x$ for every $x \in I^m$, which contradicts Brouwer's Theorem (3.1.16). This contradiction proves that $\bigcap_{i=1}^m L_i \neq \emptyset$. ■

6.8.2. THEOREM (The Fundamental Theorem of Dimension Theory). *The Euclidean space* $\mathbf{R}^m$ *satisfies the equation* $\operatorname{ind} \mathbf{R}^m = m$ *for* $m = 0, 1, 2, \ldots$

PROOF. We showed in Example 6.7.2 that $\operatorname{ind} \mathbf{R}^0 = 0$, $\operatorname{ind} \mathbf{R}^1 = 1$ and that $\operatorname{ind} \mathbf{R}^m \leq m$ for $m \geq 2$. Let us suppose that $\operatorname{ind} \mathbf{R}^m < m$ for some $m \geq 2$. By Theorem 6.7.12 there therefore exists a decomposition

$$\mathbf{R}^m = Z_1 \cup Z_2 \cup \ldots \cup Z_m \quad \text{where} \quad \operatorname{ind} Z_i = 0 \quad \text{for } i = 1, 2, \ldots, m.$$

Let A_i, B_i for $i = 1, 2, \ldots, m$ be the subsets of the cube I^m defined in Lemma 6.8.1. Using Lemma 6.7.7 we obtain for $i = 1, 2, \ldots, m$ open subsets U_i of the cube I^m such that $A_i \subset U_i$, $B \subset I^m \setminus \operatorname{cl} U_i$ and $Z_i \cap L_i = \emptyset$, where L_i is the boundary of U_i in the space I^m. By Lemma 6.8.1 we have that $\bigcap_{i=1}^m L_i \neq \emptyset$. However, the equations $Z_i \cap L_i = \emptyset$ imply that

$$\bigcap_{i=1}^m L_i = \left(\bigcup_{i=1}^m Z_i \right) \cap \bigcap_{i=1}^m L_i \subset \bigcup_{i=1}^m Z_i \cap L_i = \emptyset,$$

and so the assumption that $\operatorname{ind} \mathbf{R}^m < m$ leads to a contradiction. ■

The following three corollaries are consequences of Theorems 6.8.2 and 6.7.4, Example 6.7.2 and the topological invariance of the dimension ind.

6.8.3. COROLLARY. *The unit cube* I^m *and the unit sphere* S^m *satisfy the equation* $\operatorname{ind} I^m = \operatorname{ind} S^m = m$ *for* $m = 0, 1, 2, \ldots$ ■

More generally:

6.8.4. COROLLARY. *Every* m*-dimensional manifold* X *satisfies the equation* $\operatorname{ind} X = m$. ■

6.8.5. COROLLARY. *The Hilbert cube* $I^{\aleph_0}$ *satisfies the equation* $\operatorname{ind} I^{\aleph_0} = \infty$. ■

The next corollary concerns the spaces N_n^m and L_n^m introduced in Example 6.7.16.

6.8.6. COROLLARY. *The space* N_n^m *consisting of all points of* $\mathbf{R}^m$ *with at most* n *coordinates rational, and the space* L_n^m *consisting of all points of* $\mathbf{R}^m$ *with at least* n *coordinates rational satisfy the equations*

$$\operatorname{ind} N_n^m = n \quad \textit{and} \quad \operatorname{ind} L_n^m = m - n$$

for $0 \leq n \leq m$ *and* $m = 1, 2, \ldots$

PROOF. From Example 6.7.1 and Theorem 6.7.5 we have $\operatorname{ind} N_0^m = \operatorname{ind} L_m^m = 0$; we may therefore assume that $0 < n < m$. In Example 6.7.16 we showed that $\operatorname{ind} N_n^m \leq n$ and $\operatorname{ind} L_n^m \leq m - n$. But $\mathbf{R}^m = N_n^m \cup L_{n+1}^m = N_{n-1}^m \cup L_n^m$, so if the inequality $\operatorname{ind} N_n^m \leq n-1$ or $\operatorname{ind} L_n^m \leq m-n-1$ held we could represent $\mathbf{R}^m$ by Theorem 6.7.12 as a union of m zero-dimensional subspaces. It follows from Theorems 6.8.2 and 6.7.12 that such a representation does not exist. ■

We now turn our attention to a characterization of the m-dimensional subsets of the Euclidean space $\mathbf{R}^m$. It turns out that they are precisely the subsets of $\mathbf{R}^m$ which have non-empty interior. The proof of this uses two theorems on sets without interior points in $\mathbf{R}^m$. We precede the first of these theorems with two easy lemmas. Lemma 6.8.7 is an immediate consequence of the theorem on the invariance of open sets (4.2.12); however, since the result is quite elementary, we give a direct proof in order not to rely on the harder theorem.

6.8.7. LEMMA. *If a subset C of $\mathbf{R}^m$ is dense and has no interior points, then every subset D of $\mathbf{R}^m$ that is homeomorphic to C has no interior points.*

PROOF. Let $f: D \to C$ be a homeomorphism. Let us suppose that $\operatorname{int} D \neq 0$. The set D therefore contains some closed ball $\bar{B}(x;r)$ of $\mathbf{R}^m$. Since the ball $B = B(x;r)$ is an open subset of the space D, its image $A = f(B)$ is an open subset of C. By Theorems 1.10.2 and 1.8.2 the set $\bar{A} = f(\bar{B}(x;r)) \subset C$ is compact. Now A is dense in $\bar{A}$ and by Theorem 1.8.4 it therefore follows that the closure, $\operatorname{cl} A$, of A in the space $\mathbf{R}^m$ coincides with the set $\bar{A}$. But the set C is dense in $\mathbf{R}^m$, so $\mathbf{R}^m = \operatorname{cl} A \cup \operatorname{cl}(C \setminus A)$; moreover $f(x) \notin \operatorname{cl}(C \setminus A)$ hence some ball of $\mathbf{R}^m$ centred at $f(x)$ is contained in the set $\operatorname{cl} A = \bar{A} \subset C$. We have arrived at a contradiction to the hypothesis that C has empty interior. ■

6.8.8. LEMMA. *For any subset D of the space $\mathbf{R}^m$, for any point $x_0 \in \mathbf{R}^m \setminus D$ and any positive real number ϵ there exists a homeomorphism h of D onto $h(D) \subset \mathbf{R}^m$ such that $B(x_0;\epsilon) \cap h(D) = \emptyset$ and*

$$\rho(x, h(x)) \leq \epsilon \quad \text{and} \quad \rho(x,y) \leq \rho(h(x), h(y)) \quad \text{for all} \quad x, y \in D.$$

PROOF. The assignment which takes the point $x \in D$ to the point $h(x)$ on the half-line with endpoint x_0 passing through x and satisfying $\rho(x_0, h(x)) = \rho(x_0, x) + \epsilon$ defines a homeomorphism with the desired properties. ■

6.8.9. THEOREM. *Any subset C of $\mathbf{R}^m$ without interior points is homeomorphic to a subset of a compact set $B \subset \mathbf{R}^m$ without interior points.*

PROOF. We fix a countable base $\{V_i\}_{i=1}^{\infty}$ of the space $\mathbf{R}^m$ consisting of non-empty sets. For every natural number i such that $C \cap V_i = \emptyset$ select an arbitrary point $x_i \in V_i$. Adding the selected points x_i to the set C we obtain a set C' which is dense in $\mathbf{R}^m$. The set C' has no interior points. Indeed if C' contained a non-empty open set U of $\mathbf{R}^m$, then one of the selected points x_i would belong to U, but that is impossible, because then the non-empty open set $U \cap V_i$ of $\mathbf{R}^m$ would be contained in the countable set $C' \setminus C$. Evidently it suffices to show that the set C' is homeomorphic to a subset of some

compact set of $\mathbf{R}^m$ without interior points. Thus, without loss of generality, we may suppose that C is dense and has no interior points.

Let X be a bounded subspace of $\mathbf{R}^m$ which is homeomorphic to C. By Theorem 6.2.4 the space $Z = B(X, \mathbf{R}^m)$ of bounded continuous maps from X into $\mathbf{R}^m$ is complete. Let us consider the subspace $Y \subset Z$ consisting of maps f with the property that $\rho(x,y) \leq \rho(f(x), f(y))$ for all $x, y \in X$. It is readily verified that Y is a closed subspace of Z and so, according to Theorem 1.9.12, the space Y is also complete. Since the inclusion map $i_X: X \to \mathbf{R}^m$ belongs to Y, the space Y is non-empty. Evidently every map $f \in Y$ is a homeomorphism of X onto $f(X) \subset \mathbf{R}^m$.

For $i = 1, 2, \ldots$ let us consider the set

$$B_i = \{f \in Y : V_i \subset \operatorname{cl} f(X)\},$$

where the symbol cl denotes closure in the space $\mathbf{R}^m$. We shall show that the sets B_i are closed and have no interior points in Y.

Let $f \in Y \setminus B_i$. Thus there is a point $x \in \mathbf{R}^m$ and a positive real number ϵ such that $B(x;\epsilon) \subset V_i \setminus \operatorname{cl} f(X)$. For any map $g \in Y$ such that $\tilde{\rho}(f,g) < \epsilon/2$ we have $B(x;\epsilon/2) \subset V_i \setminus \operatorname{cl} g(X)$ and so $g \in Y \setminus B_i$. The set $Y \setminus B_i$ is thus open and the set B_i is closed in the space Y.

We now consider an arbitrary map $f_0 \in Y$ and a positive real number ϵ. The space $D = f_0(X) \subset \mathbf{R}^m$ is homeomorphic to the space X which in turn is homeomorphic to the space C, so by Lemma 6.8.7 the set D is without interior points in $\mathbf{R}^m$ and hence there exists a point $x_0 \in V_i \setminus D$. By Lemma 6.8.8 there exists a homeomorphism $h: D \to h(D) \subset \mathbf{R}^m$ such that $B(x_0; \epsilon/2) \cap h(D) = \emptyset$ and $\rho(x, h(x)) \leq \epsilon/2$ and $\rho(x,y) \leq \rho(h(x), h(y))$ for all $x, y \in D$. The composition $f = hf_0: X \to h(D) \subset \mathbf{R}^m$ is an element of the space Y, since the set $h(D) \subset B(D;\epsilon)$ is bounded and $\rho(x,y) \leq \rho(f_0(x), f_0(y)) \leq \rho(hf_0(x), hf_0(y)) = \rho(f(x), f(y))$ for all $x, y \in X$. Moreover, $x_0 \in V_i \setminus \operatorname{cl} f(X)$, so $f \in Y \setminus B_i$, which in view of the inequality $\tilde{\rho}(f_0, f) < \epsilon$, proves that B_i has no interior points in the space Y.

From Baire's Theorem (6.4.2) it follows that there exists a map f in $Y \setminus \bigcup_{i=1}^{\infty} B_i$. The set $B = \operatorname{cl} f(X)$ satisfies the claim made in the theorem. ■

6.8.10. THEOREM. *Every subset C without interior points in $\mathbf{R}^m$ is homeomorphic to a set $D \subset N_{m-1}^m$.*

PROOF. By Theorem 6.8.9 we may assume that C is a compact set. Arrange in a sequence $x_1, x_2, \ldots$ the points of $\mathbf{R}^m \setminus N_{m-1}^m = L_m^m$, i.e. the points of $\mathbf{R}^m$ all of whose coordinates are rational. For $i = 1, 2, \ldots$ the set $B_i = \{x_i - x : x \in C\}$ is closed in $\mathbf{R}^m$ and has no interior points, so by Baire's Theorem the union $\bigcup_{i=1}^{\infty} B_i$ has no interior points. It is readily verified that by taking $h(x) = x + x_0$, where x_0 is an arbitrary point of the complementary set $\mathbf{R}^m \setminus \bigcup_{i=1}^{\infty} B_i$, we define a homeomorphism h of the set C onto the set $D = h(C) \subset N_{m-1}^m$. ■

6.8.11. THEOREM. *A subset A of the space $\mathbf{R}^m$ has dimension m if and only if* $\operatorname{int} A \neq \emptyset$.

PROOF. Every non-empty open subset of the space $\mathbf{R}^m$ contains a set homeomorphic to $\mathbf{R}^m$, so by the fundamental theorem of dimension theory (6.8.2) if $\operatorname{int} A \neq \emptyset$ then $\operatorname{ind} A = m$. It remains to prove that if $A \subset \mathbf{R}^m$ has no interior points then

$\operatorname{ind} A \leq m-1$. The truth of this implication follows immediately from Theorem 6.8.10 and the inequality $\operatorname{ind} N^m_{m-1} \leq m-1$ (see Example 6.7.16). ■

From Theorem 6.8.11 we shall deduce an important result stating that a closed set whose dimension does not exceed $m-2$ does not separate the space $\mathbf{R}^m$.

6.8.12. LEMMA. *If a non-empty open set $U \subset \mathbf{R}^m$ is not dense in $\mathbf{R}^m$ then* $\operatorname{ind} \operatorname{bd} U = m-1$.

PROOF. We conclude from Theorem 6.8.11 that $\operatorname{ind} \operatorname{bd} U \leq m-1$. When $m=1$ the equation $\operatorname{ind} \operatorname{bd} U = m-1$ follows from the connectedness of the space $\mathbf{R}^1$. We suppose that $m \geq 2$ and that $\operatorname{ind} \operatorname{bd} U \leq m-2$. Let us consider any homeomorphism h of the space $\mathbf{R}^m$ onto the subspace $S^m \setminus \{x_0\}$ of the sphere S^m, where $x_0 \in S^m$ is an arbitrarily selected point. The set $W = h(U)$ is open in S^m and its boundary $\operatorname{bd} W$ in the sphere S^m is contained in the set $h(\operatorname{bd} U) \cup \{x_0\}$. Since the set $\operatorname{bd} U$, and hence also the set $h(\operatorname{bd} U)$, is the union of a countable number of compact sets, it follows from Theorem 6.7.9 that $\operatorname{ind} \operatorname{bd} W \leq m-2$. Now consider any point $x \in W$. Since $\operatorname{cl} W \neq S^m$, for every neighbourhood $V \subset W$ of the point x there exists a homeomorphism $f: S^m \to S^m$ such that $f(x) = x$ and $f(\operatorname{cl} W) \subset V$. The set $f(W)$ is open and satisfies the conditions $x \in f(W) \subset V$ and $\operatorname{ind} \operatorname{bd} f(W) = \operatorname{ind} f(\operatorname{bd} W) = \operatorname{ind} \operatorname{bd} W \leq m-2$. Hence, in view of the topological homogeneity of the sphere, it follows that, contrary to Corollary 6.8.3, $\operatorname{ind} S^m \leq m-1$. The contradiction thus obtained proves that $\operatorname{ind} \operatorname{bd} U = m-1$. ■

6.8.13. THEOREM. *No closed set $L \subset \mathbf{R}^m$ with $\operatorname{ind} L \leq m-2$ separates the space $\mathbf{R}^m$.*

PROOF. Suppose that $\mathbf{R}^m \setminus L = U \cup V$, where $U \cap V = \emptyset$ and the sets U and V are open and non-empty. Evidently $\operatorname{bd} U \subset L$, so $\operatorname{ind} \operatorname{bd} U \leq m-2$ contrary to Lemma 6.8.12. ■

We conclude our study of the dimension of the subspaces of Euclidean spaces with an important theorem characterizing topologically the separable metric space of finite dimension as the subspaces of the Euclidean spaces. To be precise, we shall show that every separable metric space with dimension not exceeding n is homeomorphic to a subspace of the space $\mathbf{R}^{2n+1}$. We shall deduce this from a more general theorem which also implies that in the realm of separable metric spaces the dimensions ind and dim coincide. The required homeomorphism will be obtained by applying Baire's Theorem to an appropriate space of maps (cf. Supplement 6.S.5). The proofs of the promised results, though conceptionally simple, require considerable preparatory calculations.

We begin by introducing the notion of a $\mathcal{U}$-map which plays a fundamental role in the sequel. Let $\mathcal{U}$ be an open covering of the space X and $f: X \to Y$ a continuous map; if every point $y \in Y$ possesses a neighbourhood $V \subset Y$ such that the inverse image $f^{-1}(V)$ is contained in an element of the covering $\mathcal{U}$, then we say that f is a *$\mathcal{U}$-map* (cf. Supplement 6.S.28).

6.8.14. LEMMA. *For every separable metric space X there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of finite open coverings of the space with the property that every continuous map $f: X \to Y$ which is a $\mathcal{U}_i$-map for $i = 1, 2, \ldots$ is a homeomorphism of the space X onto the subspace $f(X)$ of the space Y.*

PROOF. Let $\mathcal{B}$ be a countable base of the space X. Consider all pairs (U,W) of elements of the base $\mathcal{B}$ which satisfy the inclusion $\operatorname{cl} U \subset W$; to every such pair let us assign the two element open covering $\{W, X\setminus \operatorname{cl} U\}$ of the space X and arrange all the coverings so obtained into a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ We shall show that if a continuous map $f: X \to Y$ is a $\mathcal{U}_i$-map for $i = 1,2,\ldots$ then f is a homeomorphism of X onto $f(X) \subset Y$. By Corollary 1.5.7 it suffices to show that for every sequence of points $x_n \in X$, where $n = 0,1,\ldots$ if $\lim_n f(x_n) = f(x_0)$ then $\lim_n x_n = x_0$. Let us suppose that x_0 is not the limit of the sequence $\{x_n\}$. The point x_0 thus has a neighbourhood $W \in \mathcal{B}$ outside of which lie infinitely many terms of the sequence $\{x_n\}$. Consider an arbitrary neighbourhood $U \in \mathcal{B}$ of x_0 satisfying $\operatorname{cl} U \subset W$. Since f is a $\mathcal{U}_i$-map for $\mathcal{U}_i = \{W, X\setminus \operatorname{cl} U\}$, the point $f(x_0)$ has a neighbourhood $V \subset Y$ whose inverse image $f^{-1}(V)$ is contained in some element of $\mathcal{U}_i$; that element must obviously be W. We deduce that infinitely many terms of the sequence $\{f(x_n)\}$ are lying outside of the neighbourhood $V \cap f(X)$ of the point $f(x_0)$ in the space $f(X)$; thus, the point $f(x_0)$ is not the limit of the sequence $\{f(x_n)\}$. ■

6.8.15. LEMMA. *Let X be a non-empty metric space and Y a compact metric space. For every open covering $\mathcal{U}$ of the space X the set of all $\mathcal{U}$-maps is open in the space $C(X,Y)$.*

PROOF. Let $f \in C(X,Y)$ be any $\mathcal{U}$-map. It follows from the definition of a $\mathcal{U}$-map that there exists an open covering $\mathcal{V}$ of Y with the property that for every $V \in \mathcal{V}$ the inverse image $f^{-1}(V)$ is contained in some element of $\mathcal{U}$. Let λ be the Lebesgue number of this covering (see Lemma 1.8.13). To complete the proof it will be sufficient to show that the ball $B(f; \frac{1}{6}\lambda) \subset C(X,Y)$ consists of $\mathcal{U}$-maps. Consider a map $g \in B(f; \frac{1}{6}\lambda)$ and a point $y \in Y$. Let V be an element of $\mathcal{V}$ containing the ball $B(y; \frac{1}{3}\lambda) \subset Y$. If $g(x) \in B(y; \frac{1}{6}\lambda)$, then

$$\rho(y, f(x)) \leq \rho(y, g(x)) + \rho(g(x), f(x)) < \frac{1}{6}\lambda + \frac{1}{6}\lambda = \frac{1}{3}\lambda,$$

so $f(x) \in B(y, \frac{1}{3}\lambda) \subset V$. We conclude that $g^{-1}(B(y; \frac{1}{6}\lambda)) \subset f^{-1}(V)$ which proves that g is a $\mathcal{U}$-map. ■

6.8.16. LEMMA. *Let X be a non-empty metric space and Y a compact metric space. For every closed subset H of the space Y the set $\{f \in C(X,Y) : H \cap \operatorname{cl} f(X) = \emptyset\}$ is open in the space $C(X,Y)$.*

PROOF. Let $f \in C(X,Y)$ be any map for which $H \cap \operatorname{cl} f(X) = \emptyset$. From the compactness of Y, Assertion 1.6.7 and Theorem 1.8.16, there follows the existence of a positive real number r such that $H \cap B(\operatorname{cl} f(X); r) = \emptyset$. To clinch the proof it is enough to notice that every map $g \in B(f; r) \subset C(X,Y)$ satisfies the condition $H \cap \operatorname{cl} g(X) = \emptyset$. ■

6.8.17. LEMMA. *Let X be a non-empty separable metric space, $\mathcal{U}$ a finite open covering of the space X, and H an n-dimensional affine subspace of the space $\mathbf{R}^{2n+1}$. If $\dim X \leq n$, then the $\mathcal{U}$-maps $f: X \to I^{2n+1}$ satisfying the condition $H \cap \operatorname{cl} f(X) = \emptyset$ form a dense set in the space $C(X, I^{2n+1})$.*

PROOF. Consider any map $f_0 \in C(X, I^{2n+1})$ and any positive real number ϵ. Let $\mathcal{W}$ be a finite open covering of the cube I^{2n+1} by sets of diameter less than $\frac{1}{3}\epsilon$. The family $\{V_i\}_{i=1}^m$ consisting of all sets of the form $U \cap f_0^{-1}(W)$ where $U \in \mathcal{U}$ and $W \in \mathcal{W}$ is an open covering of the space X. Since $\dim X \leq n$, there exists an open covering $\{U_i\}_{i=1}^m$ of the space X such that $\operatorname{ord}(\{U_i\}_{i=1}^m) \leq n$ and $U_i \subset V_i$ for $i = 1, 2, \ldots, m$. Without loss of generality we may assume that $U_i \neq \emptyset$ if and only if $i \leq k$. It follows from the construction of the sets V_i that for $i = 1, 2, \ldots, k$ the set U_i is contained in some element of the covering $\mathcal{U}$ and $\operatorname{diam}(f_0(U_i)) < \frac{1}{3}\epsilon$.

Let the affine subspace H be the affine hull of an affinely independent set of points $\{a_0, a_1, \ldots, a_n\}$ in the space $\mathbf{R}^{2n+1}$ and let $x_i \in U_i$ for $i = 1, 2, \ldots, k$. By Theorem 1.6.16 there exist points $b_1, b_2, \ldots, b_k \in \mathbf{R}^{2n+1}$ such that the finite sequence of points $a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_k$ is in general position and $\rho(f_0(x_i), b_i) < \min\{\frac{1}{3}\epsilon - \operatorname{diam}(f_0(U_j)) : j = 1, 2, \ldots, k\}$ for $i = 1, 2, \ldots, k$. We deduce that

$$\operatorname{diam}(\{b_i\} \cup f_0(U_i)) < \frac{1}{3}\epsilon \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Without loss of generality we may further assume that $b_1, b_2, \ldots, b_k \in I^{2n+1}$. Let K be any nerve of the covering $\{U_i\}_{i=1}^k$ of the space X (see Example 2.2.4). Since $\operatorname{ord}(\{U_i\}_{i=1}^m) \leq n$, the dimension of the complex K does not exceed n. From Example 2.5.4 it follows that the family $\mathcal{L}$ consisting of the empty set and all simplices $\Delta(b_{h_0}, b_{h_1}, \ldots, b_{h_p}) \subset I^{2n+1}$ for which $U_{h_0} \cap U_{h_1} \cap \ldots \cap U_{h_p} \neq \emptyset$ is a simplicial complex that is simplicially isomorphic to K. Since the sequence $a_0, a_1, \ldots, a_n, b_1, \ldots, b_k$ is in general position and $\dim \mathcal{L} \leq n$, we have $H \cap \Delta = \emptyset$ for every $\Delta \in \mathcal{L}$.

Let σ denote the metric on the space X. It follows from the equation $X = \bigcup_{i=1}^k U_i$, that for every $x \in X$ the number $r(x) = \sum_{i=1}^k \sigma(x, X \setminus U_i)$ is positive; evidently $r\colon X \to \mathbf{R}$ is a continuous function. Consequently too the map $f\colon X \to \mathbf{R}^{2n+1}$ defined by formula

$$f(x) = \sum_{j=1}^{k} r^j(x) b_j, \quad \text{where} \quad r^j(x) = \sigma(x, X \setminus U_j)/r(x),$$

is continuous. We note that $r^j(x) \geq 0$ for $x \in X$ and $j = 1, 2, \ldots, k$ and $\sum_{j=1}^k r^j(x) = 1$ for $x \in X$. We shall show that f maps the space X into the underlying space of the complex $\mathcal{L}$. Indeed, for every point $x \in X$ we have $r^j(x) > 0$ if $x \in U_j$, while $r^j(x) = 0$ if $x \notin U_j$, and so, denoting by $U_{h_0}, U_{h_1}, \ldots, U_{h_p}$ all the elements of the covering $\{U_i\}_{i=1}^k$ which contain the point x, we have $f(x) \in \Delta(b_{h_0}, b_{h_1}, \ldots, b_{h_p}) \in \mathcal{L}$. In particular it now follows that $H \cap \operatorname{cl} f(X) = \emptyset$. Since $f^{-1}(\operatorname{st} b_j) \subset U_j$ for $j = 1, 2, \ldots, k$ we conclude from Assertions 2.3.12 and 2.3.15 that f is a $\mathcal{U}$-map taking X into the cube I^{2n+1}.

To complete the proof it will be enough to show that $\tilde{\rho}(f_0, f) < \epsilon$. Consider any point $x \in X$ and denote by $U_{h_0}, U_{h_1}, \ldots, U_{h_p}$ all the elements of the covering $\{U_i\}_{i=1}^k$ which contain this point. Now $\rho(f_0(x), b_{h_i}) < \frac{1}{3}\epsilon$ for $i = 0, 1, \ldots, p$, so

$$\begin{aligned} \rho(f_0(x), f(x)) &\leq \operatorname{diam} \operatorname{conv}\{f_0(x), b_{h_0}, b_{h_1}, \ldots, b_{h_p}\} \\ &= \operatorname{diam}\{f_0(x), b_{h_0}, b_{h_1}, \ldots, b_{h_p}\} \leq \frac{2}{3}\epsilon \end{aligned}$$

(see Lemma 1.4.7) and hence $\tilde{\rho}(f_0, f) \leq \frac{2}{3}\epsilon < \epsilon$. ■

6.8.18. THEOREM. *For every separable metric space X satisfying the inequality* $\dim X \leq n$, *where* $n = 0, 1, 2, \ldots$ *there exists a subspace* X' *of the cube* I^{2n+1} *that is homeomorphic to* X *and which is such that* $\operatorname{cl} X' \subset N_n^{2n+1}$.

PROOF. The complement $\mathbf{R}^{2n+1} \setminus N_n^{2n+1}$ is a countable union of sets of the form $\{(x_1, x_2, \ldots, x_{2n+1}) \in \mathbf{R}^{2n+1} : x_{i_j} = r_j \text{ for } j = 1, 2, \ldots, n+1\}$, where $i_1, i_2, \ldots, i_{n+1}$ are distinct natural numbers not exceeding $2n+1$ and $r_1, r_2, \ldots, r_{n+1}$ are arbitrary rational numbers. Each of these sets is an n-dimensional affine subspace of $\mathbf{R}^{2n+1}$, so that $\mathbf{R}^{2n+1} \setminus N_n^{2n+1}$ is the union of n-dimensional affine subspaces $H_1, H_2, \ldots$

Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a sequence of finite open coverings of the space X with the property stated in Lemma 6.8.14. Let us consider the space of maps $C(X, I^{2n+1})$ and denote by G_i for $i = 1, 2, \ldots$ the subset of this space consisting of $\mathcal{U}_i$-maps $f: X \to I^{2n+1}$ satisfying $H_i \cap \operatorname{cl} f(X) = \emptyset$. By Lemmas 6.8.15, 6.8.16 and 6.8.17 the sets G_i are open and dense in $C(X, I^{2n+1})$. It follows from Theorem 6.2.4 and Corollary 6.4.3 that there exists a map $f \in \bigcap_{i=1}^{\infty} G_i$. By Lemma 6.8.14 the map f is a homeomorphism of the space X onto the subspace $X' = f(X)$ of the unit cube I^{2n+1} satisfying the inclusion $\operatorname{cl} X' \subset N_n^{2n+1}$. ■

6.8.19. THEOREM. *For any separable metric space* X *the equation* $\dim X = \operatorname{ind} X$ *holds.*

PROOF. Since $\operatorname{ind} N_n^{2n+1} \leq n$, by Theorem 6.8.18 we have for any separable metric space that $\operatorname{ind} X \leq \dim X$ is true. The reverse inequality has already been demonstrated (Theorem 6.7.18). ■

From Theorems 6.8.18 and 6.8.19 we immediately obtain the following three results.

6.8.20. THEOREM. *Every separable metric space satisfying the inequality* $\operatorname{ind} X \leq n$ *where* $n = 0, 1, \ldots$, *is homeomorphic to a subspace of the space* $\mathbf{R}^{2n+1}$. ■

6.8.21. THEOREM. *The space* N_n^{2n+1} *for* $n = 0, 1, \ldots$ *is* n*-dimensional and contains for every* n*-dimensional separable metric space* X *a subspace homeomorphic to* X. ■

6.8.22. THEOREM. *Every* n*-dimensional separable metric space is homeomorphic to a subspace of an* n*-dimensional compact metric space.* ■

Exercises

a) Show that for every positive real number ϵ there exists a finite open covering $\mathcal{U}$ of the unit cube I^m such that $\operatorname{ord} \mathcal{U} \leq m$ and $\operatorname{diam} \mathcal{U} < \epsilon$ for every $U \in \mathcal{U}$. Hence deduce that $\dim I^m \leq m$ for $m = 0, 1, 2, \ldots$

b) Deduce from Lemma 6.8.1 that the sphere S^{m-1} is not the retract of the ball $\bar{B}^m$ for any m (cf. Theorem 3.1.15).

c) Let X be either the unit cube I^m or the unit sphere S^m. Show that a subset A of the space X has dimension m if and only if $\operatorname{int} A \neq \emptyset$.

d) Let X be either the unit cube I^m or the unit sphere S^m. Show that no closed set $L \subset X$ with $\operatorname{ind} L \leq m - 2$ separates the space X (see Supplement 6.S.27).

e) Let G be an arbitrary region in the space $\mathbf{R}^m$. Prove that no closed set $L \subset G$ with $\operatorname{ind} L \leq m-2$ separates the space G. (Hint: Use the fact that any two points of the set G may be connected by a chain of balls in $\mathbf{R}^m$ that are all contained in G.)

f) Prove that the geometric dimension of a polyhedron X coincides with the dimensions $\operatorname{ind} X$ and $\dim X$.

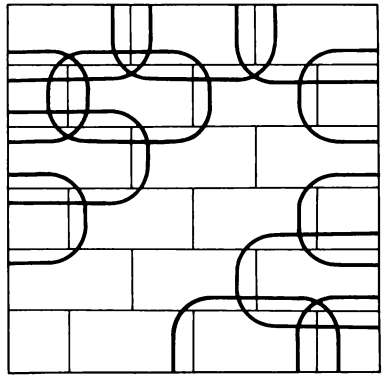


Fig.145. Construction of an open covering of order 2 consisting of sets of arbitrarily small diameters for the case of the square I^2 (see Exercise (a)).

6.S. Supplements

6.S.1. The definition of the topological product of metric spaces (X_i, ρ_i) consists in the specification of a metric for the Cartesian product $X = \bigtimes_{i=1}^{\infty} X_i$ in such a way that Theorem 6.1.9 will be obeyed, that is a metric giving "coordinatewise" convergence. There are a number of such metrics (all of them equivalent, to be sure) and apart from the definition of the topological product adopted here other definitions are in use. In the case when the spaces X_i have a common bound, i.e. when there is a real number a such that $\operatorname{diam} X_i \leq a$ for $i = 1, 2, \ldots$ the metric for X given by the formula

$$\sigma(x,y) = \sum_{i=1}^{\infty} 2^{-i} \rho_i(x_i, y_i) \quad \text{for} \quad x = \{x_1, x_2, \ldots\}, \quad y = \{y_1, y_2, \ldots\} \in X$$

is often used. Two other metrics σ' and σ'' are also in use; these are defined for *any* sequence of metric spaces (X_i, ρ_i), $i = 1, 2, \ldots$, by the equations

$$\sigma'(x,y) = \sum_{i=1}^{\infty} 2^{-i} \min(1, \rho_i(x_i, y_i)) \quad \text{and} \quad \sigma''(x,y) = \sum_{i=1}^{\infty} 2^{-i} \frac{\rho_i(x_i, y_i)}{1 + \rho_i(x_i, y_i)} .$$

The reader will easily convince himself that all these metrics are equivalent to the metric ρ^*.

6.S.2. An interesting operation on metric spaces is the formation of the hyperspace. Let X be a metric space; denote by $\mathcal{H}(X)$ the family of all non-empty, closed and

bounded subsets of the space X. Recall (see Problem 1.P.1) the definition of the Hausdorff metric dist on $\mathcal{H}(X)$

$$\operatorname{dist}(A,B) = \max(\sup\{\rho(x,B) : x \in A\},\ \sup\{\rho(y,A) : y \in B\}) \quad \text{for } A, B \in \mathcal{H}(X).$$

The set $\mathcal{H}(X)$ with the Hausdorff metric is known as the *hyperspace* of X. Problems 6.P.9–6.P.12 are dedicated to a study of the properties of the hyperspace.

6.S.3. Separable metric spaces were introduced by M. Fréchet (1906). The notion of separability may be extended to topological spaces. However, it turns out that in topological spaces more interesting and useful are the notions defined by conditions (2) and (3) of Theorem 6.3.3, that is the notion of a space satisfying the second axiom of countability and the notion of a Lindelöf space (see Section 7.1 and Supplement 7.S.5). This is a typical situation in topology: in extending to general topological spaces a notion that works successfully in the theory of metric spaces, one needs to choose carefully from among the several characterizations of the notion studied one which, when adopted as a definition, leads to the most interesting and useful class of topological spaces. A similar situation arises when the notion of dimension is extended from separable metric spaces to arbitrary metric spaces.

6.S.4. The connection between total boundedness and separability described by Theorem 6.3.15 suggests a method of creating topological notions applicable to metric spaces. Namely, if $\mathcal{P}$ is a metric property (such as boundedness, total boundedness, or completeness), we shall say that the space X has the property $\mathcal{P}$ *topologically* when there exists a space X' with property $\mathcal{P}$ such that X' is homeomorphic to X; in other words, a metric space (X,ρ) has the property $\mathcal{P}$ topologically, if there is a metric ρ' on X equivalent to ρ, such that the space (X,ρ') has property $\mathcal{P}$. It follows from Theorem 6.1.8 that every metric space is topologically bounded. Theorem 6.3.15 asserts that being topologically totally bounded is the same as being separable. Topological completeness was introduced at the end of Section 6.4 under the names of complete metrizability.

6.S.5. Sets whose closure has empty interior are called *nowhere dense*; countable unions of nowhere dense sets are known as sets of the *first category* (sets of the *second category* are those which are not of the first category). Baire's Theorem states that in a complete space every set of the first category has empty interior. It is from this formulation of Baire's Theorem that the term 'category method' derives. It is used to describe proofs which rely on applying the theorem to a space of maps or to a space of sets (see Problem 6.P.9 and 6.P.12) in order to prove that a function or a set satisfying particular conditions exists. Such proofs run along the lines of the example presented in 6.4.4 or, dually, using sets with empty interior (cf. Problem 6.P.15). It is readily noted that though the category method proves only the existence of certain mathematical objects, nevertheless in the course of proof objects are constructed which are their approximates to within arbitrary accuracy. The method may therefore be regarded as effective, in that it allows a description of the construction of the desired object. The category method was introduced and developed at the turn of the thirties by W. Hurewicz, K. Kuratowski and S. Mazurkiewicz. They obtained several new theorems and simplified

many complicated proofs by this method. Somewhat later S. Banach and H. Steinhaus introduced the method into analysis. It is now widely applied by mathematicians world wide. The current book uses the category method in Example 6.4.4, in the proof of Theorem 6.8.18 and in Problems 6.P.15 and 6.P.46. Example 6.4.4 is due to Banach himself; we have replaced a section of Banach's original proof by a direct calculation (given in J. R. Munkres' *Topology, a first course*, Englewood Cliffs, N.Y. 1975). Of the numerous, interesting applications of the category argument let us also mention Mazurkiewicz's theorem stating that in the subspace of the hyperspace $\mathcal{H}(I^2)$ consisting of non-empty continua lying in the square I^2, the hereditarily indecomposable continua (see Supplement 4.S.3) form a dense $\mathcal{G}_\delta$-set (a proof, none too difficult, may be found in the paper by S. Mazurkiewicz, *Sur les continus absolument indécomposables*, Fund. Math. 16 (1930), 151–159).

6.S.6. The notion of the completion of a metric space, especially in the rendering of F. Hausdorff as described in Problem 6.P.20, draws on the Cantor-Mérey theory of real numbers. The latter theory defines irrational numbers as equivalence classes consisting of Cauchy sequences of rational numbers; so, the set of real numbers is defined as the completion of the set of rational numbers. (Recall that in Dedekind's theory real numbers are defined as cuts in the set of rational numbers). Earlier, in the context of projective spaces and the Möbius space (see Supplement 1.S.18), we met enlargement of spaces through the addition of 'missing points'. Such processes often lead to very important and interesting objects. We shall meet them again in Section 7.5 when we study compactifications of a topological space (cf. Supplement 7.S.23).

The completion can also be defined for pseudometric spaces (see Supplement 1.S.1). Extending to pseudometric spaces the notion of a Cauchy sequence and repeating the construction of Problem 6.P.20, we obtain for any pseudometric space X a complete pseudometric space $\tilde{X}$ containing a dense subset isometric to X. (The notions of completeness, density and isometry carry over in the natural way to pseudometric spaces). Clearly the pseudometric space $\tilde{X}$ is not determined uniquely. It is easily checked that the metric space $\hat{\tilde{X}}$ determined by $\tilde{X}$ (see Problem 1.P.2) is isometric to the completion $\tilde{\hat{X}}$ of the metric space $\hat{X}$ determined by the space X.

6.S.7. Sets of type $\mathcal{G}_\delta, \mathcal{G}_{\delta\sigma}, \ldots$ and $\mathcal{F}_\sigma, \mathcal{F}_{\sigma\delta}, \ldots$ are examples of Borel sets. This is the name given to the subsets of a metric space which can be obtained from the open sets, or dually from the closed sets, by forming countable unions and intersections and taking complements. More precisely, the family of *Borel sets* of a metric space X is the smallest family $\mathcal{S}$ satisfying the following conditions:

(BS1) *the family $\mathcal{S}$ contains all the open sets of X,*

(BS2) *if $A \in \mathcal{S}$, then $X \setminus A \in \mathcal{S}$,*

(BS3) *if $A_n \in \mathcal{S}$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$.*

The existence of a smallest family $\mathcal{S}$ satisfying conditions (BS1)–(BS3) follows from the simple observation, that the family of all subsets of X satisfies these conditions and that the intersection of any collection of families satisfying (BS1)–(BS3) also satisfies these conditions. It is easily checked that condition (BS1) of the definition of Borel sets

may be replaced by the condition

(BS1′) *the family $\mathcal{S}$ contains all the closed sets of X,*

and the condition (SB3) may be replaced by the condition

(BS3′) *if $A_n \in \mathcal{S}$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{S}$.*

The theory of Borel sets has been extensively developed and is an interesting branch of metric topology; an exhaustive treatment of the theory may be found in [9]. It turns out that for the real line $\mathbf{R}$ (more generally: in spaces containing "sufficiently many" non-isolated points) the hierarchy of open sets, $\mathcal{G}_\delta$-sets, $\mathcal{G}_{\delta\sigma}$-sets etc. is strictly increasing, any countable union of families in this hierarchy does not exhaust the family of Borel sets and there do exist subsets which are not Borel sets.

6.S.8. The notion of local connectedness arises from localization of connectedness. In similar fashion local properties can be manufactured from various topological properties. Generally we may define a space to have the *property $\mathcal{P}$ locally at the point x*, if for every neighbourhood U of the point x there is a set $A \subset U$ such that $x \in \operatorname{int} A$ and the subspace A of the space X has property $\mathcal{P}$. If a space has property $\mathcal{P}$ locally at every point, then we say that the space has *property $\mathcal{P}$ locally*. It is easily checked that localization of topological properties as here defined yields again topological properties. In certain particular cases the definition of a local property $\mathcal{P}$ can be simplified, for instance, if $\mathcal{P}$ is a hereditary property with respect to open sets (i.e. carries over from a space to subspaces which are open subsets), then it suffices to require the existence of a neighbourhood having property $\mathcal{P}$ (cf. the definition of local separability given in Problem 6.P.8) and if $\mathcal{P}$ is a hereditary property with respect to closed sets, then it suffices to require in the definition the existence of a neighbourhood whose closure has property $\mathcal{P}$ (cf. the definition of local compactness given in Section 7.5). Notions obtained by the above described process of localization applied to various topological properties in general yields new and interesting topological properties. However, it can happen that localization offers nothing new; for example, local complete metrizability is equivalent to complete metrizability (see Problem 7.P.59). Observe also that, although some theorems about local properties follow merely from the way in which the definition is framed (see Problems 6.P.29 and 6.P.30), all the deeper facts relating to various local properties are not embodied in the framework of the definition alone and require separate proof.

The general scheme for localization of topological properties described above does not always provide the best local analogue of a property under study. For example, to localize the notion of pathwise connectedness (see Section 3.4) and, more generally, of connectedness in dimension m (see Supplement 3.S.3) and also of contractibility (see Section 6.6), we employ a different strategy, which in these instances leads to more successful local properties. The concept of local pathwise connectedness, that is of local connectedness in dimension 0, is also a localization of the concept of pathwise connectedness in the sense discussed here (see Exercise (h) of Section 6.5). For local connectedness in dimension m, with m greater than zero, this is no longer the case; a description of an appropriate example is left to the reader.

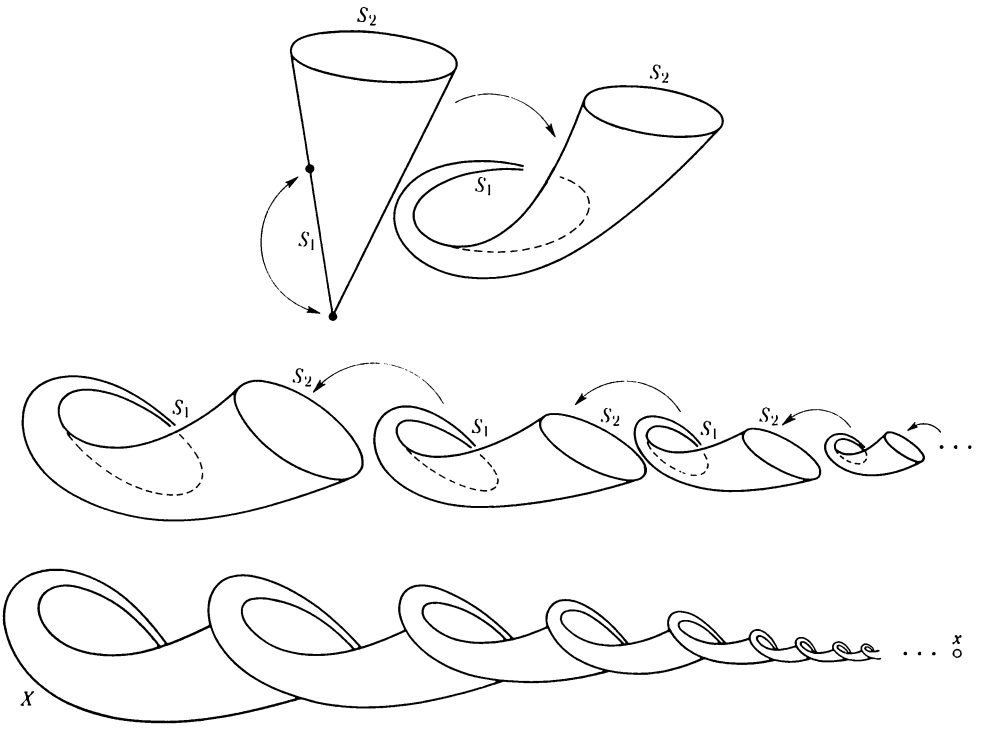


Fig.146. The space X is locally contractible, but the point x does not belong to the interior of any small connected set of dimension 1. Above, a basic element of the space X and an indication of the method of pasting the elements together.

6.S.9. The name "Sierpiński Theorem" is often applied to a particular case of Theorem 6.5.13, namely the case when X is a continuum, which is how Sierpiński (1920) formulated the theorem. Since every compact, locally connected space, and similarly any space covered by a finite number of continua, has finitely many components, Sierpiński's Theorem as formulated here easily follows from the particular case discussed.

6.S.10. The name "Mazurkiewicz-Moore Theorem" is often applied to a particular case of Theorem 6.5.17, viz. the case when X is compact; in that form the theorem was proved independently by S. Mazurkiewicz (1913) and R. L. Moore (1916). The generalization to complete spaces, requiring in fact only minor changes in the proof, was given later by K. Menger and R. L. Moore.

6.S.11. Theorem 6.5.24 was obtained independently by S. Mazurkiewicz (1913) and H. Hahn (1914); associated with it is the term *Peano continuum* used synonymously with the term "locally connected continuum". G. Peano (1890) gave the first example of a continuous map from the unit interval I onto the square I^2 (see Example 4.4.5).

6.S.12. Apart from the notion of separating a space, introduced in Section 4.2, also studied is the related notion of cutting. We say that the set A lying in a metric space X *cuts* the space if the complement $X \setminus A$ contains two points x, y with the property that every continuum $C \subset X$ containing x and y meets A. Evidently every set which separates a space X cuts the space, but not conversely (see Problem 6.P.31). It follows from Theorem 6.5.17 that the two concepts of separating and of cutting by a closed set are equivalent for locally connected complete spaces. An interesting generalization of Theorem 6.8.13 (cf. Exercise (e) of Section 6.8) is the *theorem of Mazurkiewicz* asserting that for an arbitrary region G in the space $\mathbf{R}^m$ no set $M \subset G$ with $\dim M \leq m-2$ cuts the space G (see [3], p. 80).

6.S.13. Among the more important results on continua is the *theorem of Moore* stating that every non-trivial continuum contains at least two points which do not separate it (see [10], p. 177). Associated with this theorem is an interesting characterization of the arc obtained for the case of plane arcs by N. J. Lennes (1911) and in the general case independently by W. Sierpiński (1916), S. Straszewicz (1916) and R. L. Moore (1920). It is that every continuum containing exactly two non-separating points is an arc (see [10], p. 179). Moore also gave a similar characterization of the closed curve: every continuum that is separated by every two-point set is a closed curve (see [10], p. 180).

6.S.14. The notions of absolute retract and absolute neighbourhood retract were introduced by K. Borsuk (1931). The researches of K. Borsuk and his students and collaborators led to the creation of the theory of retracts, a distinct branch of geometric topology. A full account of the theory may be found in the monographs [2] and [7]. We should add that apart from the terms "absolute retract" and "absolute neighbourhood retract" the abbreviations AR and ANR are in frequent use.

The theory of retracts may be carried over from compact to arbitrary metric spaces. The generalized theory is based on the following modified definitions: a metric space X is called an *absolute retract* (*absolute neighbourhood retract*) *in the class of metric spaces*, if every homeomorphic image A of X which is a closed subspace of an arbitrary metric space Y is a retract (neighbourhood retract) of Y. An important result in the theory is the *theorem of Hanner* (see [7], p. 98) stating that a metric space X is an absolute neighbourhood retract in the class of metric spaces if and only if every point $x \in X$ has a neighbourhood which is an absolute neighbourhood retract in the class of metric spaces. In particular it follows from this theorem that manifolds are absolute neighbourhood retracts.

6.S.15. A space Y is called an *absolute extensor (absolute neighbourhood extensor) in the class of spaces* K, if every continuous map $f: A \to Y$ defined on the closed subset A of a space $X \in K$ has a continuous extension $f^*: X \to Y$ (has a continuous extension $f^*: U \to Y$, where U is an open set of X containing A). Theorems 6.6.1–6.6.4 imply that a compact metric space Y is an absolute extensor (absolute neighbourhood extensor) in the class of all metric spaces, or, equivalently, in the class of all compact metric spaces, if and only if, the space Y is an absolute retract (absolute neighbourhood retract).

Similarly, a metric space Y is an absolute extensor (absolute neighbourhood extensor) in the class of metric spaces, if and only if, the space Y is an absolute retract (absolute neighbourhood retract) in the class of metric spaces. The proof depends on the *theorem of Wojdysławski* (see [2], p. 79) which asserts that every metric space is isometric to a closed subspace of a convex set of a normed space (in fact, a closed subspace of the smallest convex set – in the appropriate function space – containing the image of the given metric space under the isometry described in Theorem 6.2.5) and also on the theorem of Dugundji on the extension of maps (see Supplement 7.S.10). The theory of extensors for various classes of spaces is embraced by the theory of retracts.

6.S.16. We say that the space X has *property* C^n (or that X is a C^n*-space*) if X is connected in dimension m for $m = 0, 1, 2, \ldots, n$ (see Supplement 3.S.3). Similarly, we say that X has *property* LC^n (or that X is an LC^n*-space*) if X is locally connected in dimension m for $m = 0, 1, 2, \ldots, n$ (see Supplement 3.S.3). The properties C^n and LC^n for $n = 0, 1, 2, \ldots$ constitute two sequences of topological properties bridging the gap, as it were, between pathwise connectedness and contractibility and respectively the gap between local pathwise connectedness and local contractibility (see Problem 6.P.33).

We should add that a metric space X satisfying the inequality $\dim X \leq n$ is locally contractible (contractible and locally contractible) if and only if it is an LC^n-space (an LC^n- and a C^n-space); proofs may be found in [7], p. 168 and 175 and, under the additional hypothesis that X is compact, in [2], p. 122.

6.S.17. As stated in the commentary to Theorem 6.6.18, within the realm of finite dimensional compact spaces the class of absolute retracts coincides with the class of spaces which are both contractible and locally contractible and the class of absolute neighbourhood retracts coincides with the class of locally contractible spaces – the proof may be found in [2], p. 122. Similar characterizations hold for the spaces which are absolute retracts or absolute neighbourhood retracts in the class of metric spaces – again within the realm of metric spaces with finite dimension dim (see [7], p. 168 and 175). It thus follows from Supplement 6.S.16 that a metric space satisfying the inequality $\dim X \leq n$ is an absolute retract (absolute neighbourhood retract) in the class of metric spaces if and only if it is both an LC^n- and a C^n-space (an LC^n-space).

We may add that there exists an (infinite dimensional) compact metric space which is contractible and locally contractible, but which is not an absolute neighbourhood retract (see [2], p. 126).

6.S.18. Theorem 6.6.20 can in fact be strengthened. As was shown recently, for every absolute neighbourhood retract X there exists a polyhedron which is homotopically equivalent to X. The proof is very difficult (see J. E. West, *Compact ANR's have finite type*, Bull. Amer. Math. Soc. 81 (1975), 163–165).

6.S.19. We quote two further results concerning operations on retracts. It transpires that if X is a non-empty compact metric space, Y an absolute retract (absolute

neighbourhood retract) in the class of metric spaces (see Supplement 6.S.14), then the function space $C(X,Y)$ is also an absolute retract (absolute neighbourhood retract) in the class of metric spaces (see [2], p. 89 or [7], p. 186 and 187). As was proved by M. Wojdysławski for every locally connected continuum X the hyperspace $\mathcal{H}(X)$ (see Supplement 6.S.2) is an absolute retract (see M. Wojdysławski, *Rétractes absolus et hyperespaces des continus*, Fund. Math. 32 (1939), 184–192). One of the most interesting problems in the theory of retracts, open for 40 years, was Wojdysławski's hypothesis asserting that for every non-trivial locally connected continuum X the hyperspace $\mathcal{H}(X)$ is homeomorphic to the Hilbert cube. The hypothesis was proved by D. W. Curtis and R. M. Schori (*Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math. 101 (1978), 19–38), and a simpler proof was given by H. Toruńczyk (*On CE-images of the Hilbert cube and characterization of Q-manifolds*, Fund. Math. 106 (1980), 31–40).

6.S.20. Although absolute neighbourhood retracts share many of the properties of polyhedra and are considered to have regular structure, there nevertheless exist even among them various peculiar spaces. For instance there exists a two-dimensional absolute neighbourhood retract which cannot be expressed as a finite (nor countable) union of absolute retracts. Also there exists an absolute neighbourhood retract lying in $\mathbf{R}^3$ which is the common boundary of three regions. Examples of peculiar absolute neighbourhood retracts, among them the two mentioned above, may be found in Chapter 6 of [2].

6.S.21. The dimension ind was introduced by P. S. Urysohn (1922) and K. Menger (1923). Both authors, working independently, laid the foundations for the theory of dimension. Earlier, but less precisely and less systematically, H. Poincaré, L. E. J. Brouwer and H. Lebesgue concerned themselves with the concept of dimension; the third of these in principle discovered the notion of covering dimension dim. The theory of the dimension ind, initially developed for compact metric spaces, was then extended to separable metric spaces. Many of the results in dimension theory can be generalized to arbitrary metric spaces and to certain classes of topological spaces. In the generalized theory use is made of the covering dimension dim or the dimension Ind (see Supplement 6.S.22). A full account of the theory of dimension may be found in [3]; the earlier [8] contains a very fine exposition of the theory for separable metric spaces.

6.S.22. Apart from the dimension function ind, known as the small inductive dimension, an alternative function is also studied, namely the *large inductive dimension* Ind, which is defined for all metric spaces. The dimension Ind is also defined recursively:

(LD1) $\operatorname{Ind} X = -1$ *if and only if* $X = \emptyset$,

(LD2) $\operatorname{Ind} X \leq n$ *where* $n = 0,1,2,\ldots$, *if for every closed* $A \subset X$ *and every open* $V \subset X$ *containing* A, *there exists an open* $U \subset X$ *such that* $A \subset U \subset V$ *and* $\operatorname{Ind} \operatorname{bd} U \leq n-1$,

(LD3) $\operatorname{Ind} X = n$, *if* $\operatorname{Ind} X \leq n$ *and* $\operatorname{Ind} X \leq n-1$ *does not hold*,

(LD4) $\operatorname{Ind} X = \infty$ *if* $\operatorname{Ind} X \neq n$ *for* $n = -1,0,1,\ldots$

The dimension Ind is a topological invariant. It follows from Theorem 6.7.11 by a straightforward inductive argument that for every separable X the equation $\operatorname{ind} X = \operatorname{Ind} X$ holds. This equation fails to hold in the class of all metric spaces (assuming, that is, that the dimension ind is regarded as defined for arbitrary metric spaces by the conditions (D1)–(D4)); however, it is true for all metric spaces X that $\operatorname{Ind} X = \dim X$ (see [3], p. 254).

6.S.23. In Theorem 6.7.14 the inequality sign cannot be replaced by an equality. For instance the space X considered in Problems 6.P.43 and 6.P.44 has $\operatorname{ind}(X \times X) < \operatorname{ind} X + \operatorname{ind} X$. Also there exist compact metric spaces X_1 and X_2 such that $\operatorname{ind} X_1 = \operatorname{ind} X_2 = 2$, but $\operatorname{ind}(X_1 \times X_2) = 3$ (see L. S. Pontryagin, *Sur une hypothèse fondamentale de la théorie de la dimension*, C. R. Acad. Sci. Paris 190 (1930), 1105–1107).

6.S.24. The dimension of a space may be characterized by the extension properties of maps into spheres. It may be proved that a separable metric space X satisfies the inequality $\operatorname{ind} X \leq n$, where $n \geq 0$, if and only if every continuous map $f: A \to S^n$ defined on a closed subset A of X has a continuous extension $f^*: X \to S^n$ (see [3], p. 90).

6.S.25. In connection with Exercise (c) of Section 6.7, we remark that the continuum hypothesis implies the existence of a separable metric space of dimension ∞ all of whose finite-dimensional subspaces are countable (see [3], p. 82; we recall that the continuum hypothesis, that is the assertion that the cardinal number $\mathfrak{c} = 2^{\aleph_0}$ is the immediate successor of the cardinal number $\aleph_0$, is independent of the usual axioms of set theory). There also exists a compact metric space of dimension ∞, all of whose finite-dimensional subspaces are zero-dimensional (see J. J. Walsh, *Infinite dimensional compacta containing no n-dimensional* ($n \geq 1$) *subsets*, Topology 18 (1979), 91–95; a simpler example of such a space was described by R. Pol, *Countable dimensional universal sets*, Trans. Amer. Math. Soc. 297 (1986), 255–268).

6.S.26. The characterization of the dimension dim given in Exercise (h) of Section 6.7 does not extend to arbitrary separable metric spaces. One can define (see [3], p. 113) a two-dimensional subspace X of the unit cube I^3 with the property that for every positive real number ϵ there exists a finite open covering $\mathcal{U}$ such that $\operatorname{ord} \mathcal{U} \leq 1$ and $\operatorname{diam} U < \epsilon$ for every $U \in \mathcal{U}$.

6.S.27. A compact metric space X with $\operatorname{ind} X = n$, where $n \geq 1$, is called an *n-dimensional Cantor manifold* if no closed set $L \subset X$ with $\operatorname{ind} L \leq n - 2$ separates the space X. The unit cube I^n and the unit sphere S^n are n-dimensional Cantor manifolds (see Exercise (d) of Section 6.8). It may be proved (see [3], p. 93) that every compact metric space X such that $\operatorname{ind} X = n$, where $n \geq 1$, contains an n-dimensional Cantor manifold.

6.S.28. Associated with the notion of a $\mathcal{U}$-map is the notion of an ϵ-map. Let ϵ be a positive real number and $f: X \to Y$ a continuous map; if $\operatorname{diam} f^{-1}(y) < \epsilon$ for each $y \in Y$, then we say that f is an *ϵ-map*. In the realm of compact metric spaces it is

usual to work with ϵ-maps. The passage to arbitrary separable metric spaces requires a replacement of this class of maps by the $\mathcal{U}$-maps. The situation is well illustrated by Problem 6.P.51; the characterization of the dimension dim by ϵ-maps into polyhedra does not carry over from compact spaces to separable spaces and must be replaced by a characterization via $\mathcal{U}$-maps. In fact the two-dimensional subspace X of the unit cube I^3 mentioned in Supplement 6.S.26 has the property that for every $\epsilon > 0$ there exists an ϵ-map $f: X \to Z$ of the space X into a polyhedron Z whose geometric dimension equals 1.

The $\mathcal{U}$-map f constructed in the course of Lemma 6.8.17 is called a *κ-map* corresponding to the covering $\{U_i\}_{i=1}^k$ and the points $b_1, b_2, \ldots, b_k$. The notion of κ-map introduced by K. Kuratowski (1933) played a major role in the development of topology in conjunction with the notion of the nerve of a covering. Application of these notions brought together the two fundamental investigative methods of the time in topology – the set theoretic and the combinatorial.

6.S.29. In connection with theorem 6.8.20 we remark that for every natural number n there exist n-dimensional polyhedra which are not homeomorphic to any subspace of the space $\mathbf{R}^{2n}$. This property is possessed, for example, by the union of all the faces of the $(2n+2)$-dimensional simplex which have dimension not exceeding n. The proof for the case $n = 1$ is elementary (see Problem 6.P.54); the proof for general n is more complicated (see [3], p. 132).

6.S.30. By generalizing the construction of the Cantor set, the Sierpiński curve and the Menger curve (see Section 4.4) one can define for every pair of integers m, n with $0 \leq n \leq m$, and $m \geq 1$ a certain compact space M_n^m of dimension n lying in the unit cube I^m. The space M_0^1 is the Cantor set, the space M_1^2 the Sierpiński curve, the space M_1^3 – the Menger curve. It transpires (see [3], p. 126) that every separable metric space X satisfying $\operatorname{ind} X \leq n$ is homeomorphic to a subspace of M_n^{2n+1}. The space M_n^{2n+1} is known under the name *Menger's n-dimensional universal space.* The space N_n^{2n+1} is known under the name *Nöbeling's n-dimensional universal space* (cf. Theorem 6.8.21).

6.S.31. Using the notion of dimension a precise definition can be given for the notion of a curve. A *curve* is just a one-dimensional continuum. Before the introduction of the notion of dimension only the plane curves were defined correctly; these were defined as plane continua with empty interior in $\mathbf{R}^2$ and were called *curves in the sense of Cantor.* From Theorem 6.8.11 it follows that curves in the sense of Cantor are identical with curves lying in $\mathbf{R}^2$. Of course the Sierpiński curve and the Menger curve are curves in the sense of the definition given above. It may be proved (see Problem 6.P.55) that every curve $X \subset \mathbf{R}^2$ is homeomorphic to a subspace of the Sierpiński curve which is therefore also known as Sierpiński's *universal* curve. As noted in Supplement 6.S.30 the Menger curve has a similar property in relation to all curves and is therefore also called Menger's *universal* curve.

6.P. Problems

6.P.1. Let a sequence of non-empty metric spaces (X_i, ρ_i) for $i = 1, 2, \ldots$ be given. Prove that, if the Cartesian product $\bigtimes_{i=1}^{\infty} X_i$ has a metric ρ for which the conclusion of Theorem 6.1.4 holds and if for some choice of points $a_i \in X_i$ for $i = 1, 2, \ldots$ it is true that for each natural number m the map from the metric product $\bigtimes_{i=1}^{m} X_i$ onto a subset of the space $(\bigtimes_{i=1}^{\infty} X_i, \rho)$ as defined in Exercise (d) of Section 6.1 is an isometry, then the series $\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^2$ is convergent. Show that the assumption that the conclusion of Theorem 6.1.4 holds cannot be omitted.

6.P.2. For $i = 1, 2, \ldots$ let X_i be the discrete space of cardinality $\mathfrak{m} \geq \aleph_0$ equipped with the zero-one metric. Prove that the formula

$$\sigma(x, y) = \begin{cases} 1/m, & \text{if } x_i = y_i \text{ for } i < m \text{ and } x_m \neq y_m, \\ 0, & \text{if } x_i = y_i \text{ for all } i, \end{cases}$$

where $x = \{x_1, x_2, \ldots\}$, $y = \{y_1, y_2, \ldots\} \in \bigtimes_{i=1}^{\infty} X_i$, defines a metric on the product $\bigtimes_{i=1}^{\infty} X_i$. Check that σ is equivalent to the metric of the topological product $\bigtimes_{i=1}^{\infty} X_i$. The space $\bigtimes_{i=1}^{\infty} X_i$ equipped with the metric σ is called the *Baire space of weight* $\mathfrak{m}$ and is denoted $B(\mathfrak{m})$ (cf. Problem 6.P.23).

6.P.3. Show that it is not possible to equip the set $C(I, I)$ with a metric yielding convergence which is identical with pointwise convergence as defined for sequences of functions. (Hint: Assume that ρ is such a metric; for $n = 1, 2, \ldots$ consider the function $d_n \in C(I, I)$ defined by the formula $d_n(x) = \sup\{f(x) : f \in C(I, I) \text{ and } \rho(f, f_0) < 1/n\}$ where $x \in I$ and $f_0 \in C(I, I)$ vanishes at all points of I; check that $\lim_n d_n(x) = 0$ for each $x \in I$. Observe that for some n there exists a sequence $x_1, x_2, \ldots$ of points of I and a sequence $U_1, U_2, \ldots$ of open sets of I such that $d_n(x_i) < 1$, $x_i \in U_i$ for $i = 1, 2, \ldots$ and $U_i \cap U_j = \emptyset$ for $i \neq j$. Consider the functions $f_i \in C(I, I)$ satisfying $f_i(I \setminus U_i) = \{0\}$ and $f_i(x_i) = 1$ for $i = 1, 2, \ldots$)

6.P.4. Let X, Y be arbitrary metric spaces; suppose given a sequence of maps $f_n \colon X \to Y$ for $n = 1, 2, \ldots$ and a map $f_0 \colon X \to Y$. We say that the sequence $\{f_n\}$ is *continuously convergent* to the map f_0 (which is its *limit*) if for every sequence of points $x_0, x_1, x_2, \ldots$ of the space X the condition $\lim_n x_n = x_0$ implies the condition $\lim_n f_n(x_n) = f_0(x_0)$.

Check that the limit of a continuously convergent sequence consisting of arbitrary maps is a continuous map. Prove that if X is a non-empty compact space then continuous convergence is identical with uniform convergence in $C(X, Y)$. Deduce that if X is a non-empty compact metric space then for any equivalent metrics ρ and ρ' on the set Y the metrics $\tilde{\rho}$ and $\tilde{\rho'}$ on the sets $C(X, Y)$ are equivalent (cf. Exercise (a) of Section 6.2 and Problem 7.P.45).

6.P.5. Let X be a proper compact subset of the sphere S^m. Prove that the set X separates the sphere S^m if and only if the space of maps $C(X, S^{m-1})$ is disconnected. (This is *Borsuk's Theorem.*) (Hint: Use Example 3.2.1, Exercise (a) of Section 3.2, Exercise (i) of Section 6.5 and Theorem 4.2.1.)

6.P.6. Show that if X is a non-empty compact space, Y an arbitrary metric space then the set of continuous maps $f: X \to Y$ that are homeomorphisms from X onto $f(X)$ consistutes a $\mathcal{G}_\delta$-set in the space $C(X, Y)$.

6.P.7. Show that a metric space X is separable if and only if every family of pairwise disjoint, non-empty, open subsets of X is countable.

6.P.8. Say that the space X is *locally separable at a point* x if the point x has a neighbourhood U which, when regarded as a subspace of X, is separable; if the space is locally separable at each point say that X is *locally separable* (see Supplement 6.S.8). Check that $J(\mathbf{c})$ is not locally separable. Give an example of a metric space which is not locally separable at any point. Prove that every locally separable connected metric space is separable (cf. Problem 7.P.60).

6.P.9. Give an example of two equivalent totally bounded metrics on the countable discrete space X such that the corresponding Hausdorff metrics on the set $\mathcal{H}(X)$ are not equivalent (see Supplement 6.S.2). Prove that if a metric space X is complete (totally bounded, compact) then the hyperspace $\mathcal{H}(X)$ is also complete (totally bounded, compact). Give an example of a separable, connected metric space X for which the space $\mathcal{H}(X)$ is neither separable, nor connected.

6.P.10. Let X be a non-empty metric space and $\mathcal{H}(X)$ its hyperspace (see Supplement 6.S.2). Observe that the map which assigns to each point $x \in X$ the singleton set $\{x\} \in \mathcal{H}(X)$ is an isometry of X onto a subspace of its hyperspace. Prove that the map which assigns to each set $A \in \mathcal{H}(X)$ the function $f_A: X \to \mathbf{R}$ defined by the formula

$$f_A(x) = \rho(x, A) - \rho(x, x_0),$$

where $x_0 \in X$ is fixed, is an isometry of the hyperspace $\mathcal{H}(X)$ onto a subspace of the space of maps $BC(X, \mathbf{R})$.

6.P.11. Let X be a non-empty compact metric space and Y an arbitrary metric space. Show that the correspondence assigning each map $f \in C(X, Y)$ its graph, that is, the set $\{(x, f(x)) : x \in X\} \subset X \times Y$, defines a homeomorphism of the space $C(X, Y)$ onto a subspace of the hyperspace $\mathcal{H}(X \times Y)$. Hence deduce Theorem 6.3.8. (Hint: Notice that if Y is a separable space, then the space $C(X, Y)$ is homeomorphic to a subspace of the space $C(X, I^{\aleph_0})$ and apply Problem 6.P.9.)

6.P.12. Let X be a metric space; denote by $\mathcal{Z}(X)$ the subspace of the hyperspace $\mathcal{H}(X)$ consisting of all non-empty compact subspaces of the space X. Observe that the finite subsets of X form a dense subspace of the space $\mathcal{Z}(X)$. Show that if metrics ρ and ρ' on the set X are equivalent then the restrictions of the corresponding Hausdorff metrics to $\mathcal{Z}(X)$ are also equivalent. Prove that if the space X is complete (totally bounded, compact, separable, connected) then the space $\mathcal{Z}(X)$ equipped with the Hausdorff metric is also complete (totally bounded, compact, separable, connected).

6.P.13. Prove that a metric space is complete if and only if every family $\{F_t\}_{t \in T}$ of closed subsets of the space X containing, for each positive real number ϵ, a set of diameter less than ϵ and satisfying the condition that $\bigcap_{i=1}^{m} F_{t_i} \neq \emptyset$ for every finite

sequence of indices $t_1, t_2, \ldots, t_m \in T$, itself has non-empty intersection (cf. the notion of a family with the finite intersection property introduced in Section 7.5 and Theorem 7.5.2).

6.P.14. For each point $x = (x^1, 0) \in \mathbf{R}^2$ with x^1 irrational pick a real number $r_x > 0$ and let $B_x = B(y_x; r_x)$, where $y_x = (x^1, r_x)$. Using Baire's Theorem prove that the union of the sets B_x contains the interior of a rectangle contiguous to the axis of abscissae.

6.P.15. Check that the formula

$$\rho(f,g) = \sigma(f,g) + \sum_{i=1}^{\infty} 2^{-i} \min(1, \sigma(f^{(i)}, g^{(i)})),$$

where $\sigma(f,g) = \sup\{|f(r) - g(r)| : r \in I\}$ defines a metric on the set $C^\infty(I) \subset C(I, \mathbf{R})$ consisting of functions f all of whose derivatives $f^{(i)}$ are continuous, and that $C^\infty(I)$ is complete under this metric. Use the category method to show that there exist functions in $C^\infty(I)$ which are not analytic at any point (recall that a function $f \in C^\infty(I)$ is analytic at the point $a \in I$ if, for every real number r in some neighbourhood of a in the interval I, the Taylor series $\sum_{i=0}^{\infty} f^{(i)}(a)(r-a)^i/i!$ converges to $f(r)$). Observe that functions which have the desired property form a dense set in $C^\infty(I)$. (Hint: Observe that if a function f is analytic at the point a then $\sup\{\sqrt[i]{f^{(i)}(a)/i!} : i = 1, 2, \ldots\} < \infty$. Deduce that the set of functions which are analytic at some point is contained in the union $\bigcup_{a \in Q \cap I} \bigcup_{n=1}^{\infty} B_{a,n}$ where $B_{a,n} = \{f \in C^\infty(I) : |f^{(i)}(a)| \leq i! n^i \text{ for } i = 1, 2, \ldots\}$ and Q denotes the set of rational numbers. Show that the sets $B_{a,n}$ are closed and have empty interior. To show that the set $B_{a,n}$ has empty interior consider a function $f \in B_{a,n}$ and the ball $B(f; 2\epsilon)$, choose a natural number m such that $2^{1-m} < \epsilon$ and a number $b > 2$ satisfying the inequality $\epsilon b^m > (2m)! n^{2m}$ and prove that the function $g \in C^\infty(I)$ defined by the formula $g(r) = f(r) + \epsilon b^{-m} \cos b(r-a)$ for $r \in I$ belongs to $B(f, 2\epsilon) \backslash B_{a,n}$.)

6.P.16. Let $\Phi \in C(K, \mathbf{R})$ be a continuous function defined on a square K centered on $(x_0^1, x_0^2) \in \mathbf{R}^2$, satisfying the Lipschitz condition with respect to the second variable. Show by applying Banach's fixed point theorem that there exist a neighbourhood U of the point x_0^1 and an interval $B = [x_0^2 - a, x_0^2 + a]$, for some $a > 0$, with the property that the differential equation $x^{2\prime} = \Phi(x^1, x^2)$ has exactly one solution φ satisfying the initial condition $\varphi(x_0^1) = x_0^2$ with graph lying in the set $U \times B$; that is, there exists exactly one differentiable function $\varphi: U \to B$ such that $\varphi'(x^1) = \Phi(x^1, \varphi(x^1))$ for $x^1 \in U$ and $\varphi(x_0^1) = x_0^2$. (Hint: Let $K = A \times B$ where $A = [x_0^1 - a, x_0^1 + a]$ and $B = [x_0^2 - a, x_0^2 + a]$ with $a > 0$ and suppose $|\Phi(x^1, x_1^2) - \Phi(x^1, x_2^2)| \leq c|x_1^2 - x_2^2|$ for $x^1 \in A$ and $x_1^1, x_2^2 \in B$. For some positive real number $\delta < \frac{1}{c}$ such that $\delta \sup\{\Phi(x^1, x^2) : (x^1, x^2) \in K\} \leq a$ take $U = (x_0^1 - \delta, x_0^1 + \delta)$ and consider the function space $C(U, B)$ with metric σ defined by the formula $\sigma(f,g) = \sup\{|f(r) - g(r)| : r \in U\}$. Consider the map $F: C(U,B) \to C(U,B)$ defined by

$$[F(f)](x) = x_0^2 + \int_{x_0^1}^{x} \Phi(x^1, f(x^1)) dx^1$$

for $x \in U$.)

6.P.17. Show that if a subspace A of a totally bounded space X is the image of X under an isometry $f: X \to A$, then A is a dense subspace of X. Deduce that a compact metric space cannot be mapped isometrically onto a proper subspace of itself. Observe that there exist isometries of totally bounded spaces onto proper subspaces. (Hint: Check that for each $x \in X$ the sequence $x, f(x), ff(x), \ldots$ contains a subsequence converging to x.)

6.P.18. Prove that if X is a totally bounded space then every non-contractive map f of the space X onto a subspace A, that is a map $f: X \to A$ satisfying $\rho(x,y) \leq \rho(f(x), f(y))$ for $x, y \in X$, is an isometry. (Hint: Check that for non-contractive maps the claim in the hint to the previous problem holds good and apply it to the Cartesian product $f \times f: X \times X \to A \times A$.)

6.P.19. Prove that if X is a totally bounded space then every non-expansive map f of the space X onto a dense subspace A, that is a map $f: X \to A$ satisfying $\rho(f(x), f(y)) \leq \rho(x,y)$ for $x, y \in X$, is an isometry. (Hint: Extend f to a map $\tilde{f}: \tilde{X} \to \tilde{A}$ where $\tilde{X}$ and $\tilde{A}$ denote the completions of the spaces X and A and use the previous problem.)

6.P.20. Let X be any metric space. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences let $\{x_n\} R \{y_n\}$ hold if and only if $\lim_n \rho(x_n, y_n) = 0$. Check that R is an equivalence relation on the set of Cauchy sequences in X and show that the formula $\rho([\{x_n\}], [\{y_n\}]) = \lim_n \rho(x_n, y_n)$ defines a metric on the set $\tilde{X}$ of equivalence classes of R. Observe that the map which takes a point $x \in X$ to the point $[\{x_n\}]$ where $x_n = x$ for $n = 1, 2, \ldots$ defines an isometry of the space X onto a dense subspace of $\tilde{X}$ and show that $\tilde{X}$ is the completion of the space X.

6.P.21. Show that every $\mathcal{G}_\delta$-set A in a metric space X is homeomorphic to a closed subspace of the topological product $X \times \mathbf{R}^{\aleph_0}$. Observe that this provides an immediate solution to Exercise (f) of Section 6.4. Show that a separable metric space X is completely metrizable if and only if X is homeomorphic to a closed subspace of the $\aleph_0$-power of the real line $\mathbf{R}^{\aleph_0}$. (Hint: Express the complement $X \backslash A$ as a union of closed sets $F_1, F_2, \ldots$ and consider the map taking a point $x \in A$ to the point $\{x, \rho(x, F_1)^{-1}, \rho(x, F_2)^{-1}, \ldots\} \in X \times \mathbf{R}^{\aleph_0}$.)

6.P.22. Prove that a metric space X is completely metrizable if and only if for every homeomorphism f of the space X onto a subspace $X' = f(X)$ of a metric space Y the subspace X' is a $\mathcal{G}_\delta$-set in Y. (Hint: Use Lavrentiev's Theorem (6.4.14) and Exercise (f) of Section 6.4).

6.P.23. Prove that the space of irrational numbers P is homeomorphic to the $\aleph_0$-power of the discrete space of cardinality $\aleph_0$ (cf. Problem 6.P.2). (Hint: Show that for every metric ρ on the space P equivalent to its usual metric, and for every positive real number ϵ and every non-empty open set $U \subset P$, there exists an infinite sequence $A_1, A_2, \ldots$ of non-empty pairwise disjoint, open-and-closed sets with $\operatorname{diam} A_n < \epsilon$ for $n = 1, 2, \ldots$, such that $U = \bigcup_{n=1}^{\infty} A_n$. Next fix a complete metric ρ on P and for

every finite sequence $n_1, n_2, \ldots, n_k$ of natural numbers define a non-empty open-and-closed set $F_{n_1,n_2,\ldots,n_k} \subset P$ with diameter less than $1/k$ such that $P = \bigcup_{n=1}^{\infty} F_n$ and $F_{n_1 n_2 \ldots n_k} = \bigcup_{n=1}^{\infty} F_{n_1 n_2 \ldots n_k n}$, and also $F_{n_1 n_2 \ldots n_k} \cap F_{m_1 m_2 \ldots m_k} = \emptyset$ for distinct sequences of indices.)

6.P.24. Prove that every completely metrizable space X without any isolated points contains a subspace homeomorphic to the Cantor set (see Problem 6.P.37). (Hint: Fix a complete metric ρ on X and for every finite sequence $n_1, n_2, \ldots, n_k$ consisting of noughts and ones define a non-empty open set $V_{n_1 n_2 \ldots n_k} \subset X$ of diameter less than $1/k$ in such a way that $\operatorname{cl} V_{n_1 n_2 \ldots n_k n} \subset V_{n_1 n_2 \ldots n_k}$ for $n = 0, 1$, and $\operatorname{cl} V_{n_1 n_2 \ldots n_k 0} \cap \operatorname{cl} V_{n_1 n_2 \ldots n_k, 1} = \emptyset$.)

6.P.25. Show that every separable completely metrizable space is either countable or has cardinality $\mathfrak{c}$ (cf. Problem 7.P.5).

6.P.26. Prove that the quasi-components of the topological product $\bigtimes_{i=1}^{\infty} X_i$ take the form $\bigtimes_{i=1}^{\infty} K_i$ where for $i = 1, 2, \ldots$ each K_i is a quasi-component of X_i (cf. Exercise (g) of Section 6.1).

6.P.27. Suppose given a separable metric space X and a sequence $f_1, f_2, \ldots$ of bounded continuous functions defined on X and taking values in the real line $\mathbf{R}$. Prove that there exists a compact metric space Z containing X as a dense subspace such that for $i = 1, 2, \ldots$ the function f_i has a continuous extension $f_i^*: Z \to \mathbf{R}$. (Hint: Assume that σ is a totally bounded metric on the space X and $|f_i(x)| \leq 1$ for $x \in X$ and $i = 1, 2, \ldots$; consider the metric ρ defined by the formula $\rho(x, y) = \sigma(x, y) + \sum_{i=1}^{\infty} 2^{-i} |f_i(x) - f_i(y)|$ for $x, y \in X$ and take Z to be the completion of the space (X, ρ).)

6.P.28. Give an example of a connected subspace of the plane which can be expressed as a countable union of disjoint non-empty closed subsets.

6.P.29. Let $\mathcal{P}$ be any topological property. Show that if a space X has property $\mathcal{P}$ locally (see Supplement 6.S.8) and A is an open subset of X then the subspace A has property $\mathcal{P}$ locally. Observe that this need not be the case for a closed subset A.

6.P.30. Let $\mathcal{P}$ be a topological property such that for every sequence $Y_1, Y_2, \ldots$ of spaces having property $\mathcal{P}$ the topological product $\bigtimes_{i=1}^{\infty} Y_i$ also has property $\mathcal{P}$. Prove that if all the spaces of the sequence $X_1, X_2, \ldots$ have property $\mathcal{P}$ locally (see Supplement 6.S.8) and there exists a natural number m such that the spaces X_i for $i > m$ have property $\mathcal{P}$ then the topological product $\bigtimes_{i=1}^{\infty} X_i$ has property $\mathcal{P}$ locally. Observe that in general the assumption that the spaces X_i for $i > m$ have property $\mathcal{P}$ cannot be omitted.

6.P.31. Give an example of a set which cuts the plane but does not separate it (see Supplement 6.S.12).

6.P.32. Prove that if a space X and its subspace A are absolute retracts then the set A is a strong deformation retract of the space X (see Supplement 3.S.2).

6.P.33. Observe that if X is an LC^n-space (see Supplement 6.S.16) and A is an open subset of the space X then the subspace A is also an LC^n-space.

Show that every retract (neighbourhood retract) of a C^n-space (an LC^n-space) is also C^n (also LC^n).

Verify that every contractible (locally contractible) space is a C^n- (an LC^n-) space for $n = 0, 1, 2, \ldots$

Give an example of a compact metric space which is an LC^n-space with $n = 0, 1, 2, \ldots$ but is not locally contractible. (Hint: Consider the subspace $X = \bigcup_{n=1}^{\infty} X_n$ of the Hilbert cube, where $X_n = \{x = \{x_1, x_2, \ldots\} \in I^{\aleph_0} : \rho(x, a_n) = 1/[2n(n+1)]$ and $x_i = 0$ for $i > n\}$ and $a_n = \{(2n+1)/[2n(n+1)], 0, 0, \ldots\}$ for $n = 1, 2, \ldots$)

6.P.34. Prove that if X is an absolute retract lying in the Euclidean space $\mathbf{R}^m$ then no component of the complement $\mathbf{R}^m \setminus X$ is bounded. Deduce that no absolute retract in the space $\mathbf{R}^m$ for $m \geq 2$ can separate the space $\mathbf{R}^m$. (Hint: Show that if A is a compact subspace of the space $\mathbf{R}^m$, then for every continuous map $f: A \to \mathbf{R}^m$ with $f(x) = x$ for $x \in \operatorname{bd} A$ we have $A \subset f(A)$.)

6.P.35. Prove that if X is an absolute neighbourhood retract lying in the Euclidean space $\mathbf{R}^m$, then the complement $\mathbf{R}^m \setminus X$ has only finitely many components. (Hint: Consider an open set $U \subset \mathbf{R}^m$ containing X such that X is a retract of U and show by using the fact noted in the hint to the last problem, that every component of the complement $\mathbf{R}^m \setminus X$ meets $\mathbf{R}^m \setminus U$.)

6.P.36. Show that every zero-dimensional separable metric space is homeomorphic to a subspace of the Cantor set. Deduce using Problem 6.P.23 that every zero-dimensional separable metric space is homeomorphic to a subspace of the space of irrational numbers. (Hint: Consider a countable base $\{U_i\}_{i=1}^{\infty}$ of the space X consisting of open-and-closed sets and the map $f: X \to D^{\aleph_0}$ where $f(x) = \{f_1(x), f_2(x), \ldots\}$ for $x \in X$ and $f_i: X \to D$ is the characteristic function of the set U_i (see Problem 1.P.21).)

6.P.37. Show that every zero-dimensional compact metric space without any isolated points is homeomorphic to the Cantor set. (Hint: Modify the construction described in the hint to Problem 6.P.23 so that the sets $F_{n_1 n_2 \ldots n_k}$ are defined only for $n_1 \leq m_1, n_2 \leq m_2, \ldots, n_k \leq m_k, \ldots$, where the sequence of numbers $m_1, m_2, \ldots$ consists of powers of 2.)

6.P.38. Prove that every zero-dimensional compact subset of the plane $\mathbf{R}^2$ is contained in some arc of the plane. (This is the *Denjoy-Riesz Theorem.*) (Hint: Make use of Problems 6.P.37 and 4.P.17.)

6.P.39. Prove that every non-empty closed subset A of a zero-dimensional separable metric space X is a retract of X. Using this fact and Corollary 6.3.12 show that every non-empty compact metric space is the continuous image of the Cantor set. (Hint: We may assume that the space X is totally bounded. Define a sequence $F_1, F_2, \ldots$ of pairwise disjoint, open-and-closed subsets of the space X such that $X \setminus A = \bigcup_{i=1}^{\infty} F_i$ and $\lim_i \operatorname{diam} F_i = 0$. Choose from the set A points $a_1, a_2, \ldots$ with the property that $\rho(a_i, F_i) \leq \rho(x, F_i) + 1/i$ for each $x \in A$ and map F_i into a_i.)

6.P.40. Show that if a non-empty separable metric space X has the property that every non-empty closed subset A of the space X is a retract of X then X is zero-dimensional.

6.P.41. Prove that every separable metric space (X,ρ) is isometric to a subset of the space $(C(C,\mathbf{R}),\sigma)$ where C is the Cantor set and $\sigma(f,g)=\sup\{|f(x)-g(x)|:x\in C\}$ for $f,g\in C(C,\mathbf{R})$. (Hint: Fix a point $x_0\in X$, arrange into a sequence $a_1,a_2,\ldots$ the points of a dense subset of X and consider a sequence of functions $f_1,f_2,\ldots$ where the function $f_i\colon X\to\mathbf{R}$ is defined by $f_i(x)=\rho(x,a_i)-\rho(x,x_0)$ for $x\in X$. Using Problem 6.P.27 and Theorems 6.2.5 and 6.4.12 find a compact metric space Z such that the space (X,ρ) is isometric to a subspace of the space $C(Z,\mathbf{R})$. Deduce from Problem 6.P.39 that the space $C(Z,\mathbf{R})$ is isometric to a subspace of the space $C(C,\mathbf{R})$.)

6.P.42. Prove that every separable metric space (X,ρ) is isometric to a subset of the space $(C(I,\mathbf{R}),\sigma)$ where $\sigma(f,g)=\sup\{|f(x)-g(x)|:x\in I\}$ for $f,g\in C(I,\mathbf{R})$. (Hint: Show that the space $C(C,\mathbf{R})$ is isometric to a subspace of the space $C(I,\mathbf{R})$ and use the previous problem.)

6.P.43. A metric space is said to be *totally disconnected*, if for every pair of distinct points x,y of the space X there exist disjoint, open-and-closed sets $U,V\subset X$ such that $x\in U$, $y\in V$.

Show that a compact metric space X is totally disconnected if and only if $\operatorname{ind}X\leq 0$. (Hint: Use Theorem 6.5.5).

Prove that the subspace X of the Hilbert space $\mathbf{R}^\omega$ (see Example 1.1.8) consisting of sequences all of whose terms are rational is totally disconnected but not zero-dimensional. (Hint: Let $x_0\in X$ be the sequence all of whose terms are zero. Show that if a neighbourhood U of the point x_0 is contained in $B(x_0;1)$ then $\operatorname{bd}U\neq\emptyset$. For this purpose construct inductively a sequence of rational numbers $a_1,a_2,\ldots$ such that

$$x_k=\{a_1,a_2,\ldots,a_k,0,0,\ldots\}\in U \quad\text{and}\quad \rho(x_k,X\backslash U)\leq 1/k \quad\text{for}\quad k=1,2,\ldots$$

and verify that the point $a=\{a_1,a_2,\ldots\}$ belongs to $\operatorname{bd}U$.)

6.P.44. Prove that the space X of the last problem is one-dimensional. Observe that the spaces X and X^m for $m=1,2,\ldots$ are homeomorphic. (Hint: Show that for each natural number n the point $x_0\in X$ has a neighbourhood $U_n\subset X$ such that $\operatorname{diam}U_n\leq 1/n$ and $\operatorname{ind}\operatorname{bd}U_n=0$. Take advantage of the fact that $\{x\in X:\rho(x_0,x)=1/n\}$ is homeomorphic to a subset of the Hilbert cube consisting of points all of whose coordinates are rational.)

6.P.45. Prove that for every subspace A of a separable metric space X there exists a subspace $A^*\subset X$ such that A^* is a $\mathcal{G}_\delta$-set in X, $A\subset A^*$ and $\operatorname{ind}A=\operatorname{ind}A^*$. (This is the *enlargement theorem.*) (Hint: Consider first the case when $\operatorname{ind}A=0$ and then use the Decomposition Theorem (6.7.12). Use may also be made of Theorems 6.8.22 and 6.4.14.)

6.P.46. Prove that if a compact metric space X has $\dim X \leq n$ with $n \geq 0$ and contains no isolated points then there exists a continuous map $f: C \to X$ of the Cantor set onto the space X such that $\operatorname{card} f^{-1}(x) \leq n+1$ for each $x \in X$ (cf. Problem 6.P.37). (Hint: Let $Y = \{f \in C(C,X) : f(C) = X\}$. By Problem 6.P.39 the subspace $Y \subset C(C,X)$ is non-empty and by Theorems 6.2.4 and 1.9.12 is complete. Let $B_k \subset Y$ consist of functions $f \in C(C,X)$ for which there exist $n+2$ points $x_0, x_1, \ldots, x_{n+1} \in C$ with $|x_i - x_j| \geq 1/k$ for $i \neq j$ and $f(x_0) = f(x_1) = \ldots = f(x_{n+1})$. Prove that the sets $B_1, B_2, \ldots$ have empty interior and are closed in Y and use Baire's Theorem.)

6.P.47. Show that if a compact space X satisfies $\dim X \leq n$ where $n \geq 0$ then there exists a continuous map $f: A \to X$ of a closed subset of the Cantor set onto the space X such that $\operatorname{card} f^{-1}(x) \leq n+1$ for each $x \in X$. (Hint: Check that $\dim(X \times C) \leq n$.)

6.P.48. Prove that if a compact metric space X is the image of a closed subset A of the Cantor set under a continuous map $f: A \to X$ such that $\operatorname{card} f^{-1}(x) \leq n+1$ for each $x \in X$ then $\operatorname{ind} X \leq n$. (Hint: Proceed by induction on n.)

6.P.49. Deduce from the last two problems and Theorem 6.7.18 that for every compact metric space X the equation $\dim X = \operatorname{ind} X$ holds (cf. Theorem 6.8.19).

6.P.50. Prove that the Hilbert cube $I^{\aleph_0}$ cannot be expressed as a union of countably many zero-dimensional subspaces. (Hint: Use Lemmas 6.7.7 and 6.8.1.)

6.P.51. Prove that a separable metric space X satisfies $\dim X \leq n$ where $n \geq 0$ if and only if for every finite open covering $\mathcal{U}$ of the space there exists a $\mathcal{U}$-map $f: X \to Z$ of the space X into a polyhedron Z whose geometric dimension does not exceed n.

Prove that a compact metric space X satisfies $\dim X \leq n$ where $n \geq 0$ if and only if for every positive real number ϵ there exists an ϵ-map $F: X \to Z$ of the space X into a polyhedron Z whose geometric dimension does not exceed n (see Problem 6.P.52 and Supplement 6.S.28). (Hint: Analyse the proof of Lemma 6.8.17.)

6.P.52. Prove that for every subset A of a polyhedron $|K|$ there exists a continuous map $w: A \to |K|$ and a subcomplex $K_0 \subset K$ such that $w(A) = |K_0|$ and $w(A \cap S) \subset S$ for each $S \in K$. (This is the *sweeping out theorem.*) Observe that the proof of the sweeping out theorem may be considerably simplified if A is a closed subset of $|K|$. Show, using the sweeping out theorem, that in Problem 6.P.51 the words "into a polyhedron" may be replaced by "onto a polyhedron".

6.P.53. Observe that Theorem 6.8.11 follows from the property of the space $\mathbf{R}^m$ formulated in Problem 4.P.8.

6.P.54. Let X be the union of all those faces of a four-dimensional simplex whose dimensions do not exceed 1. Using Jordan's Theorem (4.2.5) prove that the space X is not homeomorphic to any subspace of the plane $\mathbf{R}^2$ (cf. Theorem 6.8.20 and Supplement

6.S.29). (Hint: Deduce from Jordan's Theorem that every θ-curve in the plane $\mathbf{R}^2$, i.e., every set that can be represented as the union $L_0 \cup L_1 \cup L_2$ of three arcs that have only the end-points in common, separates $\mathbf{R}^2$ into three components D_0, D_1, D_2 in such a way that $\operatorname{bd} D_0 = L_0 \cup L_1$, $\operatorname{bd} D_1 = L_1 \cup L_2$ and $\operatorname{bd} D_2 = L_0 \cup L_2$.)

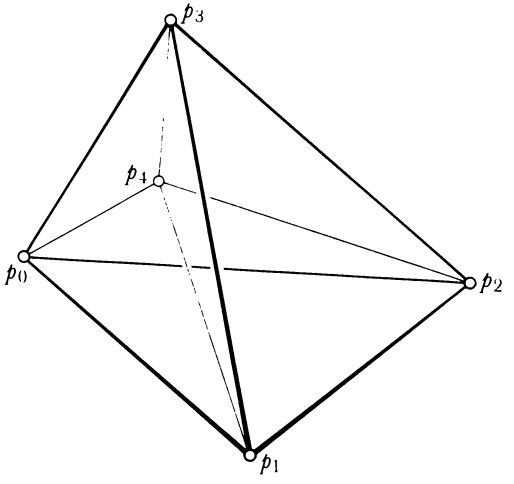


Fig.147. The union of all those faces of the four-dimensional simplex whose dimensions do not exceed 1 is not homeomorphic to any subspace of the plane $\mathbf{R}^2$ (see Problem 6.P.54).

6.P.55. Prove that every curve $X \subset \mathbf{R}^2$ is homeomorphic to a subspace of the Sierpiński universal curve (see Supplement 6.S.31). (Hint: Construct a set homeomorphic to Sierpiński's universal curve so that it contains X. For this purpose remove rectangles from an arbitrary square containing X, much as the subsquares of I^2 are removed in the construction of the Sierpiński curve.)

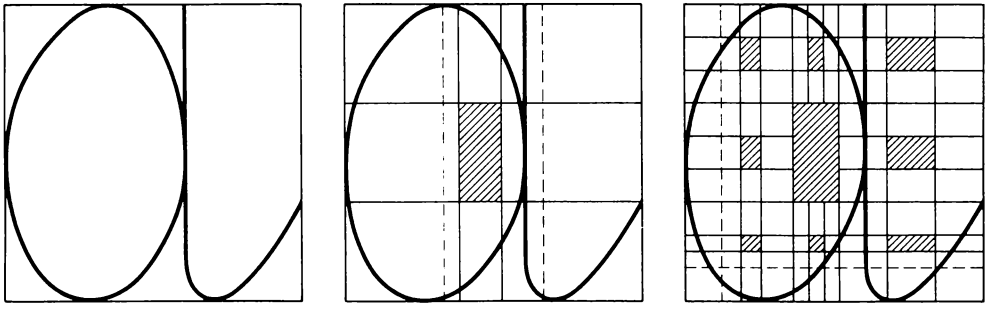


Fig.148. Construction of a set homeomorphic to Sierpiński's universal curve and containing a given plane curve (see Problem 6.P.55).

Chapter 7
Topological spaces

Topological research was initially concerned only with the class of metric spaces. Through its own internal impetus and the ever widening applications in other branches of mathematics, topology quickly broadened its scope of inquiry to more general spaces. The current chapter presents the more important concepts and methods of General Topology.

Section 7.1 introduces the concept of a topological space and extends to such spaces the basic topological notions defined initially in the context of metric spaces in Chapters 1 and 6.

In Section 7.2 we introduce the notion of a continuous map, the significance of which ranks equally in topology with that of a topological space; we then carefully analyse the notion. We also define homeomorphisms, introduce the notion of a topological property, and explain briefly what topology is concerned with.

The next section is devoted to separation axioms; these are restrictions imposed on topological spaces concerning the mutual separation of points and closed sets. We prove in this section Urysohn's Lemma, one of the more important theorems of General Topology.

Operations on topological spaces are the subject of Section 7.4. We will be concerned with the operations of subspace, topological product and quotient space.

Section 7.5, the longest and most important in the chapter, is devoted to compact spaces. It turns out that compactness may be carried across from metric spaces to topological spaces with the preservation of all the essential features. Most important of these is the fact that the class of compact spaces is closed under topological products (Tikhonov's Theorem). The topic of the latter part of the section is the compactification of topological spaces, an idea which finds frequent application in several branches of mathematics. In particular we define and study the Stone-Čech compactification and the Alexandrov compactification. We close the section with two important results which, though they do not strictly speaking belong to topology, succinctly illustrate the importance of compactness in mathematics: these are the Stone-Weierstrass Theorem on the approximation of continuous functions with compact domain, and Stone's Theorem on the representation of Boolean algebras.

The final section is devoted to the metrization problem for topological spaces and to a study of the class of paracompact spaces; the latter embraces both the compact spaces and the metrizable spaces, and finds wide use in contemporary mathematical research.

7.1. The concept of a topological space

By a *topological space* we mean an arbitrary set X together with a family $\mathcal{O}$ of subsets of X satisfying the following axioms:

(T1) *the set X and the empty set $\emptyset$ belong to $\mathcal{O}$,*
(T2) *if $U_1 \in \mathcal{O}$ and $U_2 \in \mathcal{O}$ then $U_1 \cap U_2 \in \mathcal{O}$,*
(T3) *if $U_s \in \mathcal{O}$ for each $s \in S$, then $\bigcup_{s \in S} U_s \in \mathcal{O}$.*

The elements of X are conventionally referred to as *points* of the space, and the members of $\mathcal{O}$ as the *open* sets of X while the family $\mathcal{O}$ is called its *topology* (see Supplement 7.S.1). We note that formally a topological space is a pair $(X, \mathcal{O})$. The same set X may in general have defined on it several families $\mathcal{O}$ of subsets of X satisfying the axioms (T1)–(T3); when we endow a set X with a topology we mean that one such family is being picked out. If a topology $\mathcal{O}$ on a set X is fixed, or the context makes quite clear how it is defined, then the space $(X, \mathcal{O})$ will be denoted more simply by the single symbol X (see Supplement 7.S.2).

Axiom (T1) asserts that the empty set and the whole space are open, while axioms (T2) and (T3) assert that the intersection of two open sets is an open set and, likewise, the union of an arbitrary family of open sets is an open set. An immediate consequence of axiom (T2) is the following.

7.1.1. ASSERTION. *The intersection of a finite number of open sets is an open set.* ■

Suppose given an arbitrary metric space (X, ρ). As was shown in Section 1.6 (see Theorems 1.6.3 and 1.6.4) the family $\mathcal{O}$ consisting of the open sets of the metric space X satisfies the axioms (T1)–(T3). The metric space (X, ρ) thus determines a topological space $(X, \mathcal{O})$. We say that the topology $\mathcal{O}$ is *induced* by the metric ρ. A topological space whose topology can be induced by some metric is called a *metrizable* space.

7.1.2. EXAMPLE. Let X be any set. The topology on X induced by the discrete metric on X is known as the *discrete* topology and the set X with this topology is called a *discrete topological space*, or briefly a *discrete space*. Evidently every subset of a discrete space is open; that is, the topology of a discrete space coincides with the family of all subsets of X. ■

7.1.3. EXAMPLE. Let X be an arbitrary set. The family consisting only of the empty set $\emptyset$ and the set X itself, is a topology on the set X; it is known as the *coarse* topology or the *antidiscrete* topology. The set X together with this topology is called an *antidiscrete space*. An antidiscrete space with two or more points is not metrizable, since every metric space with two or more points contains two non-empty disjoint open sets. ■

More interesting examples of non-metrizable topological spaces are given later in this section.

The discrete and the antidiscrete topologies on a set X with $\operatorname{card} X \geq 2$ are like two opposite poles for the family of all topologies on the set X. The discrete topology is the richest and the antidiscrete is the poorest while between them lie very many other topologies on the set X (cf. Supplement 7.S.3).

We should also remark that different metrics on a set X may well induce identical topologies. For example the discrete topology on the set $\mathbf{N}$ of natural numbers is induced both by the discrete metric and the usual subspace metric of the real line, as well as by any metric which is a positive scalar multiple of either. Exercises (a) and (b) of Section 1.2 describe two metrics on the Cartesian product $X = \bigtimes_{i=1}^{m} X_i$ of metric spaces X_i, which are both distinct from the metric of the metric product of the spaces, but both yield the same topology (cf. Supplement 6.S.1). It follows from Theorem 1.6.24 that the metrics ρ and ρ' on a set X induce identical topologies if and only if they are equivalent in the sense defined in Supplement 1.S.16; that is, when the identity map $\mathrm{id}_X\colon (X,\rho) \to (X,\rho')$ is a homeomorphism. Equivalence of metrics thus amounts to the same as coincidence of their induced topologies. Thus, from the topological point of view, the substitution of a metric ρ on a set X for a metric ρ' equivalent to it does not alter matters, since the topological properties of the space considered remain unchanged.

We now show how to define, in the context of topological spaces, the notions of closed set, dense set, boundary set and the operations of closure and interior.

Let X be any topological space. A *closed* set of the space X is any set $C \subset X$ whose complement $X\backslash C$ is open in X. It follows from Corollary 1.6.12 that if the topology of a space X is induced by a metric ρ, the closed sets of the topological space X coincide with the closed sets of the metric space (X,ρ).

7.1.4. THEOREM. *The family $\mathcal{C}$ of closed subsets of a topological space X has the following properties:*

(C1) *the set X and the empty set $\emptyset$ belong to $\mathcal{C}$,*

(C2) *if $C_1 \in \mathcal{C}$ and $C_2 \in \mathcal{C}$, then $C_1 \cup C_2 \in \mathcal{C}$,*

(C3) *if $C_s \in \mathcal{C}$ for every $s \in S$, then $\bigcap_{s\in S} C_s \in \mathcal{C}$.*

PROOF. Property (C1) follows from axiom (T1) since X is the complement of the empty set, while the empty set is the complement of X.

If $C_1 \in \mathcal{C}$ and $C_2 \in \mathcal{C}$, the complements $U_1 = X\backslash C_1$ and $U_2 = X\backslash C_2$ are open sets of X. By De Morgan's Laws and axiom (T2) it follows that the set

$$X\backslash(C_1 \cup C_2) = (X\backslash C_1) \cap (X\backslash C_2) = U_1 \cap U_2$$

is open in X and so $C_1 \cup C_2 \in \mathcal{C}$.

If $C_s \in \mathcal{C}$ for every $s \in S$, the complement $U_s = X\backslash C_s$, where $s \in S$, is open in X. By De Morgan's Laws and axiom (T3) it follows that the set

$$X\backslash \bigcap_{s\in S} C_s = \bigcup_{s\in S} (X\backslash C_s) = \bigcup_{s\in S} U_s$$

is open in X and so $\bigcap_{s\in S} C_s \in \mathcal{C}$. ■

Sets which are simultaneously closed and open in the topological space X are called *open-and-closed* sets of X. The empty set and the whole space X are open-and-closed. Finite unions and finite intersections of sets which are open-and-closed are again open-and-closed.

Consider now an arbitrary space X and an arbitrary set $A \subset X$. By property (C1) the family $\mathcal{C}_A$ of those closed sets of X which contain A is non-empty and by property (C3) the intersection of the family $\mathcal{C}_A$ is a closed set. We call this intersection the *closure* of the set A in the space X and denote it by $\operatorname{cl} A$. Evidently a set A is closed if and only if it coincides with its closure, that is when $A = \operatorname{cl} A$.

7.1.5. ASSERTION. *The inclusion* $\operatorname{cl} A \subset C$ *holds for every closed set* C *containing* A. ■

It is easy to see that if the topology of the space X is induced by a metric ρ, then the closure of a set A in the topological space X is identical with its closure in the metric space (X, ρ).

7.1.6. ASSERTION. *If* $A \subset B$, *then* $\operatorname{cl} A \subset \operatorname{cl} B$. ■

7.1.7. THEOREM. *The closure operation has the following properties:*
(CO1) $\operatorname{cl} \emptyset = \emptyset$,
(CO2) $A \subset \operatorname{cl} A$,
(CO3) $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$,
(CO4) $\operatorname{cl} \operatorname{cl} A = \operatorname{cl} A$.

PROOF. Properties (CO1) and (CO2) are obvious. It follows from Assertion 7.1.6 that $\operatorname{cl} A \subset \operatorname{cl}(A \cup B)$ and that $\operatorname{cl} B \subset \operatorname{cl}(A \cup B)$, so $\operatorname{cl} A \cup \operatorname{cl} B \subset \operatorname{cl}(A \cup B)$. By (CO2) we have $A \subset \operatorname{cl} A$ and $B \subset \operatorname{cl} B$ so $A \cup B \subset \operatorname{cl}(A \cup B)$. Since the set $\operatorname{cl} A \cup \operatorname{cl} B$ is closed we have by Assertion 7.1.5 that $\operatorname{cl}(A \cup B) \subset \operatorname{cl} A \cup \operatorname{cl} B$ hence the closure operation has property (CO3).

Property (CO4) holds since $\operatorname{cl} A$ is closed. ■

For any set A in a topological space X the union of all open sets which are contained in A is an open set. We call this union the *interior* of the set A and denote it by $\operatorname{int} A$. Evidently a set A is open if and only if it coincides with its interior, that is when $A = \operatorname{int} A$.

The theorem below shows that the interior operator is closely connected with the closure operator.

7.1.8. THEOREM. *For every set* $A \subset X$ *the following equation holds:*

$$\operatorname{int} A = X \setminus \operatorname{cl}(X \setminus A).$$

PROOF. It follows from property (CO2) that $X \setminus A \subset \operatorname{cl}(X \setminus A)$ and so $X \setminus \operatorname{cl}(X \setminus A) \subset X \setminus (X \setminus A) = A$. Since the set $X \setminus \operatorname{cl}(X \setminus A)$ is open

$$X \setminus \operatorname{cl}(X \setminus A) \subset \operatorname{int} A.$$

For every open set $U \subset A$ we have $X \setminus A \subset X \setminus U = \operatorname{cl}(X \setminus U)$. Appealing to Assertion 7.1.6 we conclude that $\operatorname{cl}(X \setminus A) \subset \operatorname{cl}(X \setminus U) = X \setminus U$, and so $U \subset X \setminus \operatorname{cl}(X \setminus A)$. The open set $\operatorname{int} A$ is contained in A so in particular

$$\operatorname{int} A \subset X \setminus \operatorname{cl}(X \setminus A)$$

and this, together with the reverse inclusion proved above, gives the required equation. ■

We note that from Theorem 7.1.8 and Corollary 1.6.11 it follows that, if the topology of a space X is induced by a metric ρ, the interior of a set A in the topological space X coincides with its interior in the metric space (X, ρ).

The following is a consequence of Theorems 7.1.7 and 7.1.8 and De Morgan's Laws.

7.1.9. THEOREM. *The interior operation has the following properties:*

(IO1) $\operatorname{int} X = X$,

(IO2) $\operatorname{int} A \subset A$,

(IO3) $\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$,

(IO4) $\operatorname{int} \operatorname{int} A = \operatorname{int} A$. ■

It is not possible to introduce the notion of a ball in the context of topological spaces; its place is taken by the notion of a neighbourhood. Just as in a metric space, any open set of a topological space X is called a *neighbourhood* of the point $x \in X$ when it contains the point.

7.1.10. ASSERTION. *A set $A \subset X$ is open if and only if for each point $x \in A$ there is a neighbourhood of the point x contained in the set A.* ■

7.1.11. ASSERTION. *A set $A \subset X$ is closed if and only if for every point $x \in X \setminus A$ there is a neighbourhood of the point x disjoint from the set A.* ■

7.1.12. ASSERTION. *A point x belongs to the set* $\operatorname{int} A$ *if and only if there is a neighbourhood of the point x which is contained in the set A.* ■

By Theorem 7.1.8 we have $X \setminus \operatorname{cl} A = \operatorname{int}(X \setminus A)$ so by Assertion 7.1.12 we obtain the following.

7.1.13. ASSERTION. *A point x belongs to the set* $\operatorname{cl} A$ *if and only if every neighbourhood of the point x meets the set A.* ■

7.1.14. COROLLARY. *If U is an open set and $U \cap A = \emptyset$ then also $U \cap \operatorname{cl} A = \emptyset$.*

In particular *if the sets U and V are open and disjoint then $U \cap \operatorname{cl} V = V \cap \operatorname{cl} U = \emptyset$.* ■

Let X be any topological space and A a subset of the space. We say that the set A is *dense* in the space X if it satisfies the equation $\operatorname{cl} A = X$. We say that the set A is *boundary* in the space X if it satisfies the equation $\operatorname{int} A = \emptyset$. It is easy to see that if the topology of the space X is induced by a metric then the notions just introduced coincide with the notions considered in Section 1.6.

The following is an immediate consequence of Theorem 7.1.8.

7.1.15. ASSERTION. *A set A is a boundary set in the space X if and only if its complement $X \setminus A$ is a dense set in X.* ■

The next two assertions follow from Assertions 7.1.13 and 7.1.12.

7.1.16. ASSERTION. *A set $A \subset X$ is dense if and only if it has non-empty intersection with every non-empty open set.* ■

7.1.17. ASSERTION. *A set $A \subset X$ is boundary if and only if it contains no non-empty open set.* ■

We now prove a theorem which will be of use to us in Section 7.5.

7.1.18. THEOREM. *If a set A is dense in a space X, then for every open set $U \subset X$ we have the equation*

$$\operatorname{cl} U = \operatorname{cl}(U \cap A).$$

PROOF. For every point $x \in \operatorname{cl} U$ and every neighbourhood W of the point, the set $W \cap U$ is non-empty and open in X. By Assertion 7.1.16 we thus have $W \cap U \cap A \neq \emptyset$ whence we conclude that $x \in \operatorname{cl}(U \cap A)$. Thus the inclusion $\operatorname{cl} U \subset \operatorname{cl}(U \cap A)$ holds; the reverse inclusion is obvious. ■

A family $\mathcal{B}$ consisting of open sets of a topological space X is called a *base of the space X* if every non-empty open set of the space may be expressed as a union of some collection of members of $\mathcal{B}$. If the topology of the space X is induced by a metric, this notion coincides with the notion of base studied in Section 6.3.

A family $\mathcal{P}$ consisting of open sets of a topological space X is called a *subbase of the space X* if the family of all finite intersections of members of the family $\mathcal{P}$, that is the sets of form $U_1 \cap U_2 \cap \ldots \cap U_m$ where $U_i \in \mathcal{P}$ for $i = 1, 2, \ldots, m$ and $m = 1, 2, \ldots$, form a base of the space X.

7.1.19. ASSERTION. *A family $\mathcal{B}$ of subsets of a space X is a base of the space if and only if $\mathcal{B}$ consists of open sets of X and for every point $x \in X$ and for every neighbourhood U of x there exists a set $V \in \mathcal{B}$ such that $x \in V \subset U$.* ■

Evidently a topological space X may have several bases, one such is the family of all open sets of X.

7.1.20. THEOREM. *Every base $\mathcal{B}$ of a space X has the following properties:*

(B1) *if $V_1 \in \mathcal{B}$ and $V_2 \in \mathcal{B}$ then for every point $x \in V_1 \cap V_2$ there is a set $V \in \mathcal{B}$ such that $x \in V \subset V_1 \cap V_2$,*

(B2) *for every point $x \in X$ there is a set $V \in \mathcal{B}$ such that $x \in V$.*

PROOF. Property (B1) follows because $V_1 \cap V_2$ is open and property (B2) because the set X is open. ■

Since every set of cardinal numbers is well ordered by the relation $\leq$, it follows that for any fixed topological space X there is a smallest cardinal among those of the form $\operatorname{card} \mathcal{B}$, where $\mathcal{B}$ is a base of the space X. This smallest number is called the *weight of the space X* and is denoted by $w(X)$.

A family $\mathcal{B}_x$ consisting of neighbourhoods of a point x of a topological space X is called a *local base of the space X at the point x* if for every neighbourhood U of the point x there is a set $V \in \mathcal{B}_x$ such that $V \subset U$. If $\mathcal{B}$ is a base of the space X, then for every point $x \in X$ the family $\mathcal{B}_x = \{V \in \mathcal{B} : x \in V\}$ is a local base of the space X at the point x. Conversely, if for each point $x \in X$ a local base $\mathcal{B}_x$ of the space X at the point x is given, the union $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a base of the space X.

The collection $\{\mathcal{B}_x\}_{x \in X}$ where, for each x, the family $\mathcal{B}_x$ is a local base of the space X at the point is called a *neighbourhood system of the space X*.

7.1.21. THEOREM. *Every neighbourhood system $\{\mathcal{B}_x\}_{x\in X}$ of a space X has the following properties:*

(BP1) *for each point $x \in X$ the family $\mathcal{B}_x$ is non-empty and each of its members contains the point x,*

(BP2) *if $x \in V \in \mathcal{B}_y$, there is a set $V' \in \mathcal{B}_x$ such that $V' \subset V$,*

(BP3) *if $V_1 \in \mathcal{B}_x$ and $V_2 \in \mathcal{B}_x$, there is a set $V \in \mathcal{B}_x$ such that $V \subset V_1 \cap V_2$.*

PROOF. Property (BP1) follows from the definition of a local base. Properties (BP2) and (BP3) follow because V and $V_1 \cap V_2$ are open sets. ■

For a fixed point x of a topological space there is a smallest cardinal number of the form $\operatorname{card}\mathcal{B}_x$, where $\mathcal{B}_x$ is a local base of the space X at the point x. This smallest number is called the *character of the point x in the space X* and is denoted $\chi(x, X)$. The *character of the topological space X* denoted $\chi(X)$ is the smallest cardinal number $\mathbf{m}$ such that $\mathbf{m} \geq \chi(x, X)$ for every point $x \in X$.

It could happen that the character of a space is greater than the character of each point of that space, but it evidently does not exceed the weight of the space.

If $\chi(X) \leq \aleph_0$ we say that the topological space X satisfies the *first axiom of countability*. If $w(X) \leq \aleph_0$ we say that the topological space X satisfies the *second axiom of countability*. Evidently every space satisfying the second axiom of countability satisfies the first axiom of countability.

7.1.22. THEOREM. *Every metrizable space satisfies the first axiom of countability.*

PROOF. Suppose that the topology of X is induced by the metric ρ. For each point $x \in X$ the family $\mathcal{B}_x = \{B(x; 1/n) : n = 1, 2, \ldots\}$, where $B(x; 1/n)$ denotes a ball in the metric space (X, ρ), is a local base of the space X at the point x. Since $\operatorname{card}\mathcal{B}_x \leq \aleph_0$ for each $x \in X$, we have $\chi(X) \leq \aleph_0$. ■

It is easy to check that the weight of a discrete space of cardinality $\mathbf{m}$ is also $\mathbf{m}$ and so there exist metrizable spaces of arbitrary large weight.

A topological space X is said to be *separable* if X contains a countable dense set. If the topology of the space X is induced by a metric the notion just introduced coincides with the notion of separability studied in Section 6.3.

The following is an immediate consequence of Theorem 6.3.3.

7.1.23. THEOREM. *A metrizable space satisfies the second axiom of countability if and only if it is separable.* ■

In Example 7.1.28 we describe a separable space which does not satisfy the second axiom of countability. However, the following does hold.

7.1.24. THEOREM. *Every space satisfying the second axiom of countability is separable.*

PROOF. Let X be a space satisfying the second axiom of countability and let $\mathcal{B}$ be a countable base consisting of non-empty sets. Consider a set A obtained by arbitrarily selecting one point from each member of the base $\mathcal{B}$. Obviously $\operatorname{card} A \leq \operatorname{card}\mathcal{B} \leq \aleph_0$. For every non-empty open set $U \subset X$ there is a set $V \in \mathcal{B}$ with $V \subset U$ and so $\emptyset \neq A \cap V \subset A \cap U$, hence by Assertion 7.1.16 it follows that the set A is dense in X. Thus the space X is separable. ■

We now describe a method for generating a topology which is frequently employed. It relies on selecting a family of sets $\mathcal{B}$, which is taken as a base of the space being constructed, and then defining the topology as the family of those sets which may be expressed as unions of some collections of members of $\mathcal{B}$.

7.1.25. THEOREM. *If a family $\mathcal{B}$ of subsets of a set X has the properties* (B1) *and* (B2) *of Theorem* 7.1.20, *then the family $\mathcal{O}$ consisting of those sets which are the unions of subfamilies of $\mathcal{B}$, is a topology on the set X. Moreover, the family $\mathcal{B}$ is a base of the space obtained by endowing X with the topology $\mathcal{O}$.*

PROOF. The family $\mathcal{O}$ satisfies axiom (T1) since the empty set is the union of the empty subfamily of $\mathcal{B}$ while the set X is, by property (B2), the union of the whole family $\mathcal{B}$.

Consider two sets U_1 and U_2 belonging to the family $\mathcal{O}$; we thus have

$$U_1 = \bigcup \mathcal{B}_1 \quad \text{and} \quad U_2 = \bigcup \mathcal{B}_2 \quad \text{where} \quad \mathcal{B}_1 \subset \mathcal{B} \quad \text{and} \quad \mathcal{B}_2 \subset \mathcal{B}.$$

But,

$$U_1 \cap U_2 = \bigcup \{V_1 \cap V_2 : \ V_1 \in \mathcal{B}_1 \quad \text{and} \quad V_2 \in \mathcal{B}\},$$

so to show that the family $\mathcal{O}$ satisfies axiom (T2) it suffices to check that if $V_1 \in \mathcal{B}$ and $V_2 \in \mathcal{B}$ then $V_1 \cap V_2 \in \mathcal{O}$. It follows from property (B1) that for each point $x \in V_1 \cap V_2$ there is a set $V_x \in \mathcal{B}$ such that $x \in V_x \subset V_1 \cap V_2$. The union of the subfamily $\{V_x : x \in V_1 \cap V_2\}$ of $\mathcal{B}$ is of course the set $V_1 \cap V_2$ and so $V_1 \cap V_2 \in \mathcal{O}$.

It is immediate from the definition of the family $\mathcal{O}$ that it satisfies axiom (T3).

The last part of the theorem is obvious. ■

The topology $\mathcal{O}$ on the set X described by the last theorem is called *the topology generated by the base $\mathcal{B}$*.

The next theorem describes another method of generating a topology which relies on distinguishing a neighbourhood system. We leave the proof of the theorem to the reader.

7.1.26. THEOREM. *If a collection $\{\mathcal{B}_x\}_{x \in X}$ of families of subsets of a set X has the properties* (BP1)–(BP3) *of Theorem* 7.1.21, *then the family $\mathcal{O}$ consisting of those sets which are the unions of subfamilies of the family $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a topology on the set X. Moreover the collection $\{\mathcal{B}_x\}_{x \in X}$ is a neighbourhood system of the space obtained by endowing X with the topology $\mathcal{O}$.* ■

The topology $\mathcal{O}$ on the set X described by the last theorem is called *the topology generated by the neighbourhood system $\{\mathcal{B}_x\}_{x \in X}$*.

A topology may be generated by distinguishing the family of closed sets, or by defining a closure operation, or an interior operation (see Problems 7.P.1, 7.P.2 and 7.P.3). These methods, though used rather rarely, have theoretical significance since they lead to alternative axiomatic descriptions of the concept of a topological space (see Supplement 7.S.2).

7.1.27. EXAMPLE. Let X be any infinite set. Choose any point $x_0 \in X$ and consider the family $\mathcal{B}$ of subsets of the set X consisting of all singletons contained in $X \setminus \{x_0\}$ and of

all sets of form $X\setminus F$ where F is a finite set contained in $X\setminus\{x_0\}$. It is easily checked that the family $\mathcal{B}$ has properties (B1) and (B2). We investigate the topology generated by the base $\mathcal{B}$.

In the space X every singleton set $\{x\}$, except $\{x_0\}$, is open-and-closed; the set $\{x_0\}$ is closed, but is not open. For every subset $A \subset X$ we have

$$\operatorname{cl} A = \begin{cases} A, & \text{if } A \text{ is finite,} \\ A \cup \{x_0\}, & \text{otherwise,} \end{cases}$$

and

$$\operatorname{int} A = \begin{cases} A, & \text{if } X\setminus A \text{ is finite,} \\ A\setminus\{x_0\}, & \text{otherwise.} \end{cases}$$

The form of the closure implies in particular that the space X does not contain a pair of disjoint, infinite closed sets.

Every base of the space X contains all the singleton sets except $\{x_0\}$, so $w(X) \geq \operatorname{card}(X\setminus\{x_0\}) = \operatorname{card} X$; but $\operatorname{card}\mathcal{B} = \operatorname{card} X$, so $w(X) = \operatorname{card} X$. Every neighbourhood of the point x_0 has the form $X\setminus F$ where F is a finite set contained in $X\setminus\{x_0\}$. Consider an arbitrary local base $\{X\setminus F_s\}_{s\in S}$ of the space X at the point x_0. For each point $x \in X\setminus\{x_0\}$ there is an index $s \in S$ such that $x \in F_s$, otherwise the neighbourhood $X\setminus\{x\}$ of the point x_0 would not contain any member of the local base, which cannot be. We thus have $\bigcup_{s\in S} F_s = X\setminus\{x_0\}$. From the finiteness of the sets F_s it follows now that $\operatorname{card} S \geq \operatorname{card} X$ and so $\chi(x_0, X) = \operatorname{card} X$. For every $x \neq x_0$ we have of course that $\chi(x, X) = 1$ and so $\chi(X) = \operatorname{card} X$.

From the last equation and Theorem 7.1.22 it follows that if the set X is uncountable, then X is not metrizable. The reader may readily check that if $\operatorname{card} X = \aleph_0$ the space X is metrizable and, for any metric ρ on the set X inducing the topology of the space, the metric space (X, ρ) is homeomorphic to the subspace $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ of the real line. ■

Fig.149. Sets of the form $[x, w)$ form a base of the Sorgenfrey line.

7.1.28. EXAMPLE. *The Sorgenfrey line.* Let K be the set of real numbers. We investigate the family $\mathcal{B}$ of subsets of the set K consisting of the half-open intervals $[x, w)$ where $x, w \in K$ and w is rational. It is easy to check that the family $\mathcal{B}$ has properties (B1) and (B2). Consider the topology on K generated by the base $\mathcal{B}$. Note that all the members of the base $\mathcal{B}$ and, more generally, all the intervals $[x, y)$ for $x, y \in K$ are open-and-closed sets of the space K.

For every family $\mathcal{R}$ of open sets of K of cardinality less than $\operatorname{card} K = \mathfrak{c}$ there is a point $x_0 \in K$ which is not the greatest lower bound of any set belonging to $\mathcal{R}$. The open set $[x_0, x_0 + 1)$ cannot therefore be expressed as a union of sets belonging to the family $\mathcal{R}$, so no family of cardinality less than $\mathfrak{c}$ consisting of open sets can be a base of the space K. Since $\operatorname{card}\mathcal{B} = \mathfrak{c}$ we have $w(K) = \mathfrak{c}$. The reader may easily check

that the space K satisfies the first axiom of countability. The set of rational numbers is dense in the space K, so K is a separable space which does not satisfy the second axiom of countability. It follows in particular that K is not a metrizable space (cf. Theorem 7.1.23). The space just described is known variously as the *Sorgenfrey line* or the *arrow space* (on account of the shape of the basic open sets). ■

7.1.29. EXAMPLE. *The Niemytzki plane.* Let L be the upper half-plane of $\mathbf{R}^2$, i.e. the subset of $\mathbf{R}^2$ defined by the condition $x^2 \geq 0$. Denote by L_1 the line with equation $x^2 = 0$ and by L_2 the set $L \backslash L_1$. For each point $x \in L_1$ and every positive real number r let $U(x;r)$ consists of the points of L_2 lying inside the disc of radius r of L which is tangential to L_1 at the point x and let $U_n(x) = U(x;\frac{1}{n}) \cup \{x\}$ for $n = 1, 2, \ldots$ For each point $x \in L_2$ and every positive real number r let $U(x;r)$ consist of the points of L lying inside the disc of radius r centred at x and let $U_n(x) = U(x;\frac{1}{n})$ for $n = 1, 2, \ldots$ It is readily verified that the collection $\{\mathcal{B}_x\}_{x \in L}$ where $\mathcal{B}_x = \{U_n(x) : n = 1, 2, \ldots\}$ has properties (BP1)–(BP3). We investigate the topology on L generated by the neighbourhood system $\{\mathcal{B}_x\}_{x \in L}$.

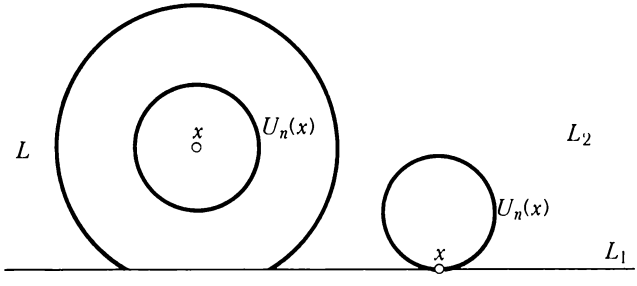


Fig.150. The sets $U_n(x)$ form a local base for the Niemytzki plane at the point x.

The set L_2 is open in L and its complement L_1 is closed in L; it is also the case that all subsets of L_1 are closed. It is easily checked that $w(L) = \mathfrak{c}$ and that the space L satisfies the first axiom of countability. The set of points $(x^1, x^2) \in L$ with both coordinates rational is countable and dense in L so that L is a separable space which does not satisfy the second axiom of countability. It follows in particular that L is not metrizable. The space is known as the *Niemytzki plane* or the *tangent-disc* space. ■

Let A be a subset of a topological space X. We say that the point $x \in X$ is an *accumulation point* of the set A if $x \in \mathrm{cl}(A \backslash \{x\})$; in other words the point x is an accumulation point of A if every neighbourhood of the point x meets the set $A \backslash \{x\}$. The set of accumulation points of a set A is called the *derivative* of the set A. A point of a set A which is not an accumulation point of A is called an *isolated point* of the set. Evidently a point x is an isolated point of the space X if and only if the singleton $\{x\}$ is open in the space.

The intersection of the closure of a set A with the closure of its complement $X \backslash A$ is called the *boundary* of the set A in the space X and is denoted $\mathrm{bd}\, A$.

7.1.30. ASSERTION. *A set A is open-and-closed in the space X if and only if* $\mathrm{bd}\, A = \emptyset$. ■

It follows from the way the definitions have been formulated that if the topology of a space X is induced by a metric then the notions of accumulation point and isolated point and the notion of boundary coincide with their counterparts in Section 1.6.

As may be seen from the considerations above, many of the fundamental topological notions previously introduced in the study of metric spaces carry over to the context of topological spaces. In the ensuing sections we shall see that in similar fashion many of the theorems and constructions of Chapters 1 and 6 may be generalized. The notion of convergence of a sequence of points also carries over to topological spaces. Specifically, we say that a sequence of points $\{x_n\}$ of a topological space X *converges* to a point x if for every neighbourhood U of the point x there exists an index k such that $x_n \in U$ for all $n \geq k$. Any point to which the sequence $\{x_n\}$ converges is called a *limit* of the sequence; in a topological space a sequence may in general have several limits (cf. Exercise (c) of Section 7.3). It is easy to check that if the topology of a space X is induced by a metric, then the notions introduced here coincide with the notions of convergence and limit introduced in Section 1.5. It turns out, however, that in the context of topological spaces the notion of convergence of a sequence looses its fundamental properties. In particular, if a point x belongs to the closure $\operatorname{cl} A$ this does not imply the existence of a sequence of points of the set A converging to the point x (see Example 7.1.31). Accordingly, in topological spaces the notion of convergence of a sequence looses its usefulness. The basic investigative tool for these spaces is the notion of neighbourhood. Nevertheless we remark that in certain classes of topological spaces, for example in the class of spaces satisfying the first axiom of countability, the notion of convergence of a sequence retains its most important properties (see Exercise (f); cf. Supplement 7.S.7). We might add that in topological spaces it is possible instead to consider a notion of convergence for objects other than ordinary sequence, namely for what are known as nets. The latter notion leads to a theory of convergence quite similar to the theory of convergence of sequences in metric spaces (see Supplement 7.S.6).

7.1.31. EXAMPLE. Let X be the subset of the plane $\mathbf{R}^2$ consisting of the point $(0,0)$ and all the points (m,n) where m,n are positive integers. For each positive integer m the set of points (m,n) for $n = 1,2,\ldots$, will be called a column. For the point $x = (0,0)$ we define $\mathcal{B}_x$ to be the family of all subsets of X which may be obtained by removing from X a finite number of columns and a finite number of points from each column not so removed. For the point $x = (m,n)$ where m and n are positive integers we define $\mathcal{B}_x$ to be the family consisting of the one singleton $\{(m,n)\}$. It is easily checked that the collection $\{\mathcal{B}_x\}_{x\in X}$ has properties (BP1)–(BP3). We consider the topology on X generated by the neighbourhood system $\{\mathcal{B}_x\}_{x\in X}$.

The reader will have no difficulty in checking that the point $x = (0,0)$ lies in the closure of the set $A = X\backslash\{x\}$ in the space X, but no sequence of points from the set A converges to x. ■

We make the final remark that the notions of interior point, boundary point and limit point of a set $A \subset X$ carry over from metric spaces to topological spaces with the preservation of their basic properties, but they then play a far less significant role than in metric spaces.

Exercises

a) Check that for every pair of subsets A, B of a topological space the following inclusions hold:

$$\operatorname{cl}(A \cap B) \subset \operatorname{cl} A \cap \operatorname{cl} B \quad \text{and} \quad \operatorname{cl} A \setminus \operatorname{cl} B \subset \operatorname{cl}(A \setminus B).$$

Observe that the inclusion relation may not, in general, be replaced by equality.

b) Show that for every sequence $A_1, A_2, \ldots$ of subsets of a topological space the following equation holds:

$$\operatorname{cl}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} \operatorname{cl} A_i \cup \bigcap_{i=1}^{\infty} \operatorname{cl}\left(\bigcup_{j=0}^{\infty} A_{i+j}\right).$$

Give an example to show that the closure of a countable union of sets may be different from the union of their closures.

c) Check that the union of two boundary sets of which one at least is closed is a boundary set.

d) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two families of subsets of a set X both satisfying properties (B1) and (B2). Show that for the topologies generated by the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ to coincide, it is necessary and sufficient that for any chosen point $x \in X$ and any $V_1 \in \mathcal{B}_1$ containing x there exist a set $V'_2 \in \mathcal{B}_2$ such that $x \in V'_2 \subset V_1$, and also for any $V_2 \in \mathcal{B}_2$ containing x there exist a set $V'_1 \in \mathcal{B}_1$ such that $x \in V'_1 \subset V_2$.

State and prove an analogous theorem for the topology generated by a neighbourhood system.

e) Let $\mathcal{P}$ be any family of subsets of a set X whose union is the whole of X and let $\mathcal{B}$ be the family of finite intersections of members of $\mathcal{P}$. Show that the family $\mathcal{O}$ consisting of sets which are unions of subfamilies of $\mathcal{B}$ is a topology on the set X.

f) Show that if a space X satisfies the first axiom of countability, then for every set $A \subset X$ and every point $x \in \operatorname{cl} A$ there is a sequence $\{x_n\}$ of points of A converging to the point x (see Supplement 7.S.7).

g) Verify that the boundary operation has the following properties:

$$\operatorname{int} A = A \setminus \operatorname{bd} A, \quad \operatorname{cl} A = A \cup \operatorname{bd} A, \quad \operatorname{bd}(X \setminus A) = \operatorname{bd} A,$$

$$\operatorname{bd}(A \cup B) \subset \operatorname{bd} A \cup \operatorname{bd} B, \quad \operatorname{bd}(A \cap B) \subset \operatorname{bd}(A) \cup \operatorname{bd}(B).$$

Observe that if the sets A and B satisfy $A \cap \operatorname{cl} B = \emptyset = B \cap \operatorname{cl} A$ then $\operatorname{bd}(A \cup B) = \operatorname{bd} A \cup \operatorname{bd} B$.

7.2. Maps on topological spaces

A map $f: X \to Y$ from a topological space X into a topological space Y is called *continuous* if for every open set V in the space Y the inverse image $f^{-1}(V)$ is open in the space X.

From Theorem 1.6.24 it follows that if the topologies of the spaces X and Y are both induced by metrics, the notion of continuity introduced here coincides with that introduced in Section 1.3.

7.2.1. ASSERTION. *For a map $f: X \to Y$ of a topological space X into a topological space Y to be continuous it is necessary and sufficient that for every closed set B of the space Y the inverse image $f^{-1}(B)$ is closed in the space X.* ■

The equation $(gf)^{-1}(W) = f^{-1}g^{-1}(W)$ implies the following.

7.2.2. ASSERTION. *If the maps $f: X \to Y$ and $g: Y \to Z$ are continuous then the composition $gf: X \to Z$ is a continuous map.* ■

We now give three frequently used tests for continuity, formulated respectively in terms of bases, neighbourhood systems and the closure operation.

7.2.3. THEOREM. *For any map $f: X \to Y$ of a topological space X into a topological space Y the following conditions are equivalent:*

(1) *the map f is continuous,*

(2) *there is a base $\mathcal{D}$ of the space Y such that for every set $V \in \mathcal{D}$ the inverse image $f^{-1}(V)$ is open in the space X,*

(3) *there are neighbourhood systems $\{\mathcal{B}_x\}_{x\in X}$, $\{\mathcal{D}_y\}_{y\in Y}$ of the respective spaces X and Y such that for every point $x \in X$ and every set $V \in \mathcal{D}_{f(x)}$ there is a set $U \in \mathcal{B}_x$ with $f(U) \subset V$,*

(4) *for every set $A \subset X$ the inclusion $f(\operatorname{cl} A) \subset \operatorname{cl} f(A)$ holds.*

PROOF. (1) $\Rightarrow$ (2). The implication is obvious since for $\mathcal{D}$ we may take the topology of Y.

(2) $\Rightarrow$ (3). Suppose the base $\mathcal{D}$ of the space Y has the property described in (2) and for $y \in Y$ put $\mathcal{D}_y = \{V \in \mathcal{D} : y \in V\}$. Consider any neighbourhood system $\{\mathcal{B}_x\}_{x\in X}$ for the space X. For every point $x \in X$ and every set $V \in \mathcal{D}_{f(x)}$ the set $f^{-1}(V)$ is a neighbourhood of the point x, so there exists a set $U \in \mathcal{B}_x$ such that $U \subset f^{-1}(V)$. Since $f(U) \subset V$, we see that (2) implies (3).

(3) $\Rightarrow$ (4). Suppose that the neighbourhood systems $\{\mathcal{B}_x\}_{x\in X}$ and $\{\mathcal{D}_y\}_{y\in Y}$ have the properties described in (3) and consider any set $A \subset X$ and any point $x \in \operatorname{cl} A$. For every $V \in \mathcal{D}_{f(x)}$ there is a set $V \in \mathcal{B}_x$ such that $f(U) \subset V$. From Assertion 7.1.13 we know $U \cap A \neq \emptyset$, so $\emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap f(A)$, hence $f(x) \in \operatorname{cl} f(A)$. We have thus shown that $f(\operatorname{cl} A) \subset \operatorname{cl} f(A)$, that is condition (4) holds.

(4) $\Rightarrow$ (1). It suffices to show that if a map f satisfies condition (4) then for every closed set B of Y the inverse image $f^{-1}(B)$ is closed in X. Using (4) with $A = f^{-1}(B)$ we conclude that $f(\operatorname{cl} f^{-1}(B)) \subset \operatorname{cl} f f^{-1}(B) \subset \operatorname{cl} B = B$, from which it follows that $\operatorname{cl} f^{-1}(B) \subset f^{-1}(B)$, that is $f^{-1}(B)$ is closed in X. ■

The characterization of continuity given in condition (3) of the last theorem leads to a definition of continuity at a point. We shall say that a map $f: X \to Y$ of a topological space X into a topological space Y is *continuous at the point $x \in X$* if for every neighbourhood $V \subset Y$ of the point $f(x)$ there is a neighbourhood $U \subset X$ of the point x such that $f(U) \subset V$. It is easily seen that a map $f: X \to Y$ is continuous if and

only if it is continuous at every point of the space X. Clearly the notion of continuity at a point as introduced here generalizes the notion studied in Section 1.3.

Observe that the map $f: X \to Y$ of the space X into the space Y where the topology of Y is induced by a metric ρ, is continuous, if and only if, for each point $x \in X$ and every positive real number ϵ there is a neighbourhood U of the point x such that $\rho(f(x), f(x')) < \epsilon$ whenever $x' \in U$.

The notion of uniform convergence introduced in Section 1.5 for maps from a metric space X into a metric space Y may be extended, without altering the definition, to maps from a topological space X into a metrizable space Y with a fixed metric ρ inducing the topology of Y. We will say that a sequence $\{f_n\}$ of maps from a topological space X into a metrizable space Y with metric ρ is *uniformly convergent* to the map $f_0: X \to Y$, known as the *limit* of the sequence, if for every positive real number ϵ there is an index k such that $\rho(f_n(x), f_0(x)) < \epsilon$ for $n \geq k$ and for any $x \in X$.

Repeating the proof of Theorem 1.5.15 we obtain the following.

7.2.4. THEOREM. *The limit of a uniformly convergent sequence of continuous maps from a topological space X into a metrizable space Y, with a fixed metric ρ, is a continuous map of the space X into the space Y.* ■

We shall use the last theorem in the case where Y is the real line with the usual metric. We will in due course discover that continuous maps from a topological space into the real line $\mathbf{R}$ and into the unit interval I play a particularly important role in topology. As indicated in Chapter 0 maps from topological spaces into the real line and into its subsets will be called functions. Of course a function f defined on a topological space X is continuous if and only if for each point $x \in X$ and every positive real number ϵ there is a neighbourhood $U \subset X$ of the point x such that $|f(x) - f(x')| < \epsilon$ whenever $x' \in U$.

Arithmetic operations on continuous functions lead to continuous functions. More precisely, if $f: X \to \mathbf{R}$ and $g: X \to \mathbf{R}$ are continuous functions then the functions $f + g, f - g, f \cdot g$ as well as $\min(f, g)$ and $\max(f, g)$, where

$$(f \pm g)(x) = f(x) \pm g(x), \quad (f \cdot g)(x) = f(x) \cdot g(x), \quad \text{for } x \in X$$

and

$$[\min(f, g)](x) = \min[f(x), g(x)] \quad \text{and} \quad [\max(f, g)] = \max[f(x), g(x)], \quad \text{for } x \in X,$$

are continuous functions defined on X taking values in the real line $\mathbf{R}$. Likewise, if $f: X \to \mathbf{R}$ is a continuous function, then the function $|f|$ where $|f|(x) = |f(x)|$ for $x \in X$ is continuous. The same applies to functions taking values in the unit interval I, assuming of course that the appropriate operations are executable in I.

We now give two examples of continuous functions which are used in the next section.

7.2.5. EXAMPLE. Consider the Sorgenfrey line K defined in Example 7.1.28 and the base $\mathcal{B}$ studied there. Let $[x, w)$ be any member of the base $\mathcal{B}$. Since the set $[x, w)$ is open-and-closed in K, the function $f: K \to I$ defined by the formula

$$f(y) = \begin{cases} 0, & \text{where } x \leq y < w, \\ 1, & \text{otherwise} \end{cases}$$

is continuous. ■

7.2.6. EXAMPLE. Consider the Niemytzki plane L defined in Example 7.1.29 and the neighbourhood system $\{\mathcal{B}_x\}_{x\in L}$ studied there. Let $x \in L$ and let $U_n(x)$ be any member of the local base $\mathcal{B}_x$ at the point. For each $y \in U_n(x)\backslash\{x\}$ denote by $\bar{y}$ the point other than x, on the intersection of the circumference of the disc $U_n(x)$ with the half-line with endpoint x passing through y.

It is easy to check (cf. Exercise (b) of Section 7.4) that the function $f: L \to I$ defined by

$$f(x) = \begin{cases} 0, & \text{if } y = x, \\ 1, & \text{if } y \in L\backslash U_n(x), \\ \dfrac{\|x-y\|}{\|x-\bar{y}\|}, & \text{if } y \in U_n(x)\backslash\{x\}, \end{cases}$$

is continuous. ■

Continuous maps do not preserve weight or character. For instance the space X of Example 7.1.31 is a continuous image of the discrete space of cardinality $\aleph_0$, but as may easily be checked (see Exercise (f) of Section 7.1), it does not satisfy the first, hence neither the second, axiom of countability.

However the following does hold.

7.2.7. THEOREM. *If f is a continuous map of a separable space X onto a space Y, then the space Y is separable.*

PROOF. Let A be a countable dense set in the space X. The set $f(A) \subset Y$ is countable and, via the equivalence of conditions (1) and (4) of Theorem 7.2.3, dense in the space Y. Thus the space Y is separable. ■

A continuous map $f: X \to Y$ is *open (closed)* if for every open (closed) set A of the space X the image $f(A)$ is open (closed) in the space Y. Evidently the composition of open (closed) maps is an open (closed) map.

7.2.8. ASSERTION. *A continuous map $f: X \to Y$ is open if and only if there is a base $\mathcal{B}$ of the space X such that for every set $U \in \mathcal{B}$ the image $f(U)$ is open in the space Y.* ■

7.2.9. THEOREM. *A continuous map $f: X \to Y$ is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ containing the set $f^{-1}(y)$ there is a neighbourhood $V \subset Y$ of the point y such that $f^{-1}(V) \subset U$.*

PROOF. Suppose the map $f: X \to Y$ is closed. Then, for every point $y \in Y$ and every open set $U \subset X$ containing $f^{-1}(y)$ the set $V = Y\backslash f(X\backslash U)$ is a neighbourhood of the point y in the space Y and

$$f^{-1}(V) = f^{-1}(Y\backslash f(X\backslash U)) = X\backslash f^{-1}f(X\backslash U) \subset X\backslash(X\backslash U) = U.$$

Next suppose that the map $f: X \to Y$ satisfies the condition of the theorem and consider a closed set $A \subset X$. If a point $y \in Y$ does not lie in $f(A)$ then the open set $U = X\backslash A \subset X$ contains $f^{-1}(y)$ hence there is a neighbourhood $V \subset Y$ of the point y such that $f^{-1}(V) \cap A = \emptyset$. Clearly $V \cap f(A) = \emptyset$ so $y \notin \operatorname{cl} f(A)$. We have thus shown

that $Y \setminus f(A) \subset Y \setminus \operatorname{cl} f(A)$, that is $\operatorname{cl} f(A) \subset f(A)$, which means that the set $f(A)$ is closed. ■

7.2.10. EXAMPLE. The map $f: \mathbf{R}^2 \to \mathbf{R}^2$ sending each point $(x^1, x^2) \in \mathbf{R}^2$ to its projection x^1 along the axis of abscissae is open but not closed, since the projection of the closed set $\{(x^1, x^2) : x^1 x^2 = 1\}$ is the set $\mathbf{R}^2 \setminus \{0\}$ which is not closed.

The map $g: \mathbf{R} \to I$ defined by

$$g(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geq 1, \end{cases}$$

is closed but not open.

The composition $gf: \mathbf{R}^2 \to I$ is continuous but is neither open nor closed. ■

A continuous map $f: X \to Y$ between topological spaces which is bijective and is such that the inverse map $f^{-1}: Y \to X$ is continuous is called a *homeomorphism*. Two topological spaces X and Y for which there is a homeomorphism taking X onto Y are said to be *homeomorphic*. Both these definitions are consistent with the definitions of the corresponding notions studied in the case of metric spaces in Section 1.3.

For every space X the identity map $\mathrm{id}_X: X \to X$ is a homeomorphism. It is easy to check that the inverse map of a homeomorphism and also the composition of two homeomorphisms are both homeomorphisms. It follows that the relation "the space X is homeomorphic to the space Y" is an equivalence.

7.2.11. ASSERTION. *For any bijective map of a space X onto a space Y the following conditions are equivalent:*

(1) *the map f is a homeomorphism,*

(2) *the map f is open,*

(3) *the map f is closed,*

(4) *the set $f(A)$ is open (closed) in Y if and only if the set A is open (closed) in X.* ■

A *homeomorphism invariant* or *topological property* is any property of topological spaces which holds of a space X if and only if it holds of all spaces homeomorphic to Y. Since a homeomorphism $f: X \to Y$ establishes a bijective correspondence between points of X and Y and between the open sets of X and Y, every property defined exclusively in terms of the concept of open set and the concepts of set theory is a topological property. The study of topological properties is the subject of topology. When we investigate a given space we try to ascertain what are its topological properties. In developing the general theory we ordinarily study a topological property and its relationship to other properties and seek to ascertain which operations on spaces preserve the property. Evidently, from the topological point of view, two homeomorphic spaces may be regarded as the same object.

Every property $\mathcal{P}$ determines a class of spaces, namely those possessing property $\mathcal{P}$. If $\mathcal{P}$ is a topological property, then the class of spaces determined by $\mathcal{P}$ is *topologically invariant*, that is, if it contains a space X it contains all spaces homeomorphic to X. Conversely, every topologically invariant class determines a topological property,

namely, membership of the class. In the last section we defined four topological properties as follows metrizability, separability and satisfaction of the first and second axiom of countability. We shall subsequently learn several other topological properties, that is topologically invariant classes of topological spaces. Whenever we introduce a new class of spaces we shall not remark on its topological invariance, since we shall never consider any classes other than topologically invariant ones. Invariance usually follows from the form of the definition when the only concepts to occur are set-theoretic, or are reducible to the concept of an open set, since these are preserved under homeomorphisms. The definitions of separable spaces and spaces satisfying the first or second axiom of countability are just such examples. The proof of the invariance of the class of metrizable spaces hinges on the remark that if $h: X \to Y$ is a homeomorphism and the topology of the space X is induced by the metric ρ, then the topology of the space Y may be induced by the metric σ where $\sigma(x, y) = \rho(h^{-1}(x), h^{-1}(y))$ for $x, y \in Y$ (see Supplement 7.S.8).

One of the more important topological properties is connectedness. The analysis of this notion undertaken in the first chapter, in the context of metric spaces, carries over almost without change to the context of topological spaces. We therefore limit ourselves to merely stating, for topological spaces, the definitions and theorems corresponding to those in Section 1.7. The proofs given in Section 1.7 continue to hold for the more general case considered here. For the time being we give only the analogues of Theorems 1.7.1 and 1.7.3; further results concerning the relationship of connectedness to the operations of subspace and topological product will be given in Section 7.4.

A topological space which cannot be expressed as the union of two non-empty, disjoint closed subsets is said to be *connected.*

7.2.12. THEOREM. *For a topological space to be disconnected it is necessary and sufficient that there exist a continuous map of the space onto the two-point discrete space.* ■

7.2.13. THEOREM. *If f is a continuous map of a connected space X onto a space Y, then Y is also connected.* ■

7.2.14. EXAMPLE. *The spaces $D(\mathbf{m})$ and $A(\mathbf{m})$.* Let X and Y be discrete spaces of the same cardinality. It is not difficult to check that any bijective map of the set X onto the set Y is a homeomorphism of the space X onto the space Y. A discrete space X does not therefore depend, up to homeomorphism, on the nature of the points of the set X but only on its cardinality. We shall continue to denote the discrete space of cardinality $\mathbf{m}$ by $D(\mathbf{m})$.

The situation is similar when two infinite sets X and Y have equal cardinalities and are both endowed with the topology described in Example 7.1.27, except that in this case we need to consider bijective maps of the set X onto the set Y carrying the distinguished point $x_0 \in X$ to the distinguished point $y_0 \in Y$. We shall continue to denote the space X obtained by the process described in Example 7.1.27 from a set X of cardinality $\mathbf{m} \geq \aleph_0$ by $A(\mathbf{m})$. ■

Exercises

a) Check that a map $f: X \to Y$ is continuous if and only if there is a subbase $\mathcal{P}$ of the space Y with the property that for every set $V \in \mathcal{P}$ the inverse image $f^{-1}(V)$ is open in X.

b) Show that if a map $f: X \to Y$ is continuous at the point $x \in X$ and the map $g: Y \to Z$ is continuous at the point $y = f(x)$, then the composition $gf: X \to Z$ is continuous at the point x.

c) Define a continuous map of the Sorgenfrey line K onto the discrete space $D(\aleph_0)$ and show that there does not exist a continuous map of the Sorgenfrey line K onto the discrete space $D(\mathfrak{c})$.

d) Show that the Sorgenfrey line and the Niemytzki plane are not homeomorphic.

e) Define topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on a set X of cardinality $\aleph_0$ such that the space $(X, \mathcal{O}_1)$ is metrizable, the space $(X, \mathcal{O}_2)$ is not metrizable, but a point $x \in X$ is the limit of a sequence $\{x_n\}$ in the topology $\mathcal{O}_1$ if and only if it is the limit of the sequence in the topology $\mathcal{O}_2$. (Hint: See Example 7.1.31).

f) Prove that if f is an open map of a space X satisfying the first (second) axiom of countability onto a space Y, then the space Y also satisfies the first (second) axiom of countability (cf. Exercise (m) of Section 7.4).

g) Show that if f is a continuous real-valued function defined on a connected space X then for any pair of points $a, b \in X$ and any real number $r \in \mathbf{R}$ with $f(a) \leq r \leq f(b)$ there is a point $c \in X$ such that $f(c) = r$.

7.3. Separation Axioms

The class of all topological spaces is very broad and the number of interesting theorems pertinent to all topological spaces is small. By imposing additional conditions on spaces we obtain narrower classes of spaces about which more interesting theorems can be proved. Restrictions placed on topological spaces are various. In Section 7.1 we considered the axioms of countability which stipulate locally or globally the existence of small bases. In this section we deal with conditions, which are known as axioms of separation. They concern the mutual separation of points and closed sets (see Supplement 7.S.9).

A topological space is said to be *T_1-space* if for every pair of distinct points $x, y \in X$ there is an open set $U \subset X$ such that $x \in U$ but $y \notin U$. Note that there then also exists an open set $V \subset X$ such that $y \in V$ but $x \notin V$; this follows from an application of the definition of a T_1-space to the pair of points y, x.

7.3.1. THEOREM. *A topological space is a T_1-space if and only if for each point $x \in X$ the singleton set $\{x\}$ is closed in X.*

PROOF. If X is a T_1-space, then for every point $x \in X$ we have the equation

$$\{x\} = \bigcap \{X \setminus U : x \notin U \in \mathcal{O}\},$$

where $\mathcal{O}$ is the topology of the space X, so by property (C3) of Theorem 7.1.4 the set

$\{x\}$ is closed. On the other hand if for each point $x \in X$ the set $\{x\}$ is closed then X is a T_1-space since for any pair of distinct points $x, y \in X$ the open set $U = X \setminus \{y\}$ contains the point x and omits the point y. ■

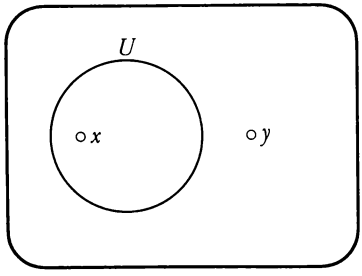


Fig.151. A T_1-space.

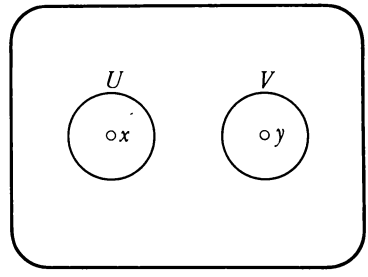


Fig.152. A T_2-space.

Anti-discrete spaces containing at least two points are not T_1-spaces. Metrizable spaces and the spaces described in Examples 7.1.27–7.1.29 and 7.1.31 are T_1-spaces.

A topological space X is said to be a *T_2-space* or a *Hausdorff space* if for every pair of distinct points $x, y \in X$ there are open sets $U, V \subset X$ such that

$$x \in U, \quad y \in V \quad \text{and} \quad U \cap V = \emptyset.$$

Every T_2-space is evidently a T_1-space. Metrizable spaces and the spaces of Examples 7.1.27–7.1.29 and 7.1.31 are T_2-spaces. We give an example of a T_1-space which is not a T_2-space.

7.3.2. EXAMPLE. Let X be any infinite set. The family $\mathcal{O}$ consisting of the empty set and all subsets of X which have finite complement is a topology on X. The set X endowed with this topology is obviously a T_1-space, but since any two non-empty open sets of X have non-empty intersection, the space X is not a T_2-space. ■

7.3.3. ASSERTION. *If a collection $\{\mathcal{B}_x\}_{x \in X}$ of families of subsets of a set X has the properties* (BP1)–(BP3) *of Theorem* 7.1.21 *and the following property:*

(BP4) *for every pair of distinct points $x, y \in X$ there are sets $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ with $U \cap V = \emptyset$,*

then the space X with the topology generated by the neighbourhood system $\{\mathcal{B}_x\}_{x \in X}$ is a Hausdorff space. ■

In the subsequent sections of this chapter we shall make use of an important property of Hausdorff spaces embodied in the following theorem.

7.3.4. THEOREM. *For any pair of continuous maps from a topological space X into a Hausdorff space Y, the set $\{x \in X : f(x) = g(x)\}$ is closed in the space X.*

PROOF. It suffices to show that the set $A = \{x \in X : f(x) \neq g(x)\}$ is open. For each $x \in A$ there are in Y open sets U and V such that $f(x) \in U$, $g(x) \in V$ and $U \cap V = \emptyset$. The set $f^{-1}(U) \cap f^{-1}(V)$ is a neighbourhood of the point x and is contained in A, so A is open. ■

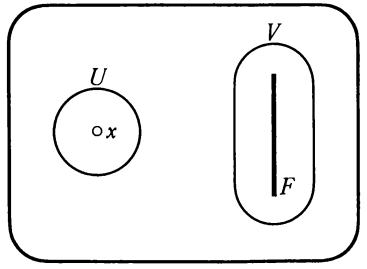


Fig.153. A T_3-space.

A topological space X is said to be a T_3-*space* or a *regular* space if X is a T_1-space and for every point $x \in X$ and every closed set $F \subset X$ with $x \notin F$ there are open sets $U, V \subset X$ such that

$$x \in U, \quad F \subset V \quad \text{and} \quad U \cap V = \emptyset.$$

7.3.5. THEOREM. *A topological space X is a T_3-space if and only if X is a T_1-space and for every point $x \in X$ and every neighbourhood W of the point x taken from a fixed base $\mathcal{B}$ of the space there is a neighbourhood U of the point such that* $\operatorname{cl} U \subset W$.

PROOF. Suppose that X is a T_3-space and consider a point $x \in X$ and a neighbourhood $W \in \mathcal{B}$ of the point. It follows from the definition of regularity that there are open sets $U, V \subset X$ satisfying

$$x \in U, \quad F = X \setminus W \subset V \quad \text{and} \quad U \cap V = \emptyset.$$

By Corollary 7.1.14 we have $V \cap \operatorname{cl} U = \emptyset$ and so $\operatorname{cl} U \subset X \setminus V \subset W$.

Assume now that a T_1-space X satisfies the condition of the theorem, and consider a point $x \in X$ and a closed set $F \subset X$ such that $x \notin F$. Since $\mathcal{B}$ is a base of the space X there exists a set $W \in \mathcal{B}$ satisfying $x \in W \subset X \setminus F$. Let U be a neighbourhod of x such that $\operatorname{cl} U \subset W$. The open set $V = X \setminus \operatorname{cl} U$ satisfies $F \subset X \setminus W \subset X \setminus \operatorname{cl} U = V$ and $U \cap V = \emptyset$, hence X is a T_3-space. ■

Every regular space is a Hausdorff space. It was precisely in order to guarantee this conclusion that we assumed not only the separation of points from closed sets, but also that singletons are closed; an antidiscrete space with at least two points has the separation property under discussion but is not regular since it is not a T_1-space. Metrizable spaces and the spaces of Examples 7.1.27–7.1.29 and 7.1.31 are T_3-spaces. We give an example of a T_2-space which is not a T_3-space.

7.3.6. EXAMPLE. Let X be the set of real numbers; denote by F the subset of X consisting of the reciprocals of the non-zero integers. For each $x \in X$ and for $i = 1, 2, \ldots$ put $U_i(x) = (x - (1/i), x + (1/i))$ and

$$\mathcal{B}_x = \begin{cases} \{U_i(x)\}_{i=1}^{\infty}, & \text{if } x \neq 0, \\ \{U_i(x) \setminus F\}_{i=1}^{\infty}, & \text{if } x = 0. \end{cases}$$

It is easy to check that the collection $\{\mathcal{B}_x\}_{x\in X}$ has the properties (BP1)–(BP4) and so by Assertion 7.3.3 the set X with topology generated by the neighbourhood system $\{\mathcal{B}_x\}_{x\in X}$ is a Hausdorff space. The set F is closed in X and does not contain 0, nevertheless for arbitrary open sets $U, V \subset X$, such that $0 \in U$ and $F \subset V$, we have $U \cap V \neq \emptyset$ and so the space X is not regular. ■

A topological space X is said to be a $T_{3\frac{1}{2}}$-*space*, or a *Tikhonov* space, or a *completely regular* space if X is a T_1-space and for every point $x \in X$ and every closed set $F \subset X$ with $x \notin F$ there is a continuous function $f: X \to I$ such that

$$f(x) = 0 \quad \text{and} \quad f(F) \subset \{1\}.$$

Unlike the definitions of the T_i-spaces for $i \leq 3$, the definition of the class of $T_{3\frac{1}{2}}$-spaces refers not only to the concepts of set theory and the concept of closed set and of open set, but also to the concept of a continuous real-valued function. Thus, while the topological invariance of the T_i-classes for $i \leq 3$ follows immediately from the form of the definitions, in the case of $T_{3\frac{1}{2}}$-spaces a proof is called for. The proof however rests on the obvious remark that the composition fh of a homeomorphism h and a continuous function into the unit interval is also a continuous function (cf. Supplement 7.S.8 and Problem 7.P.9).

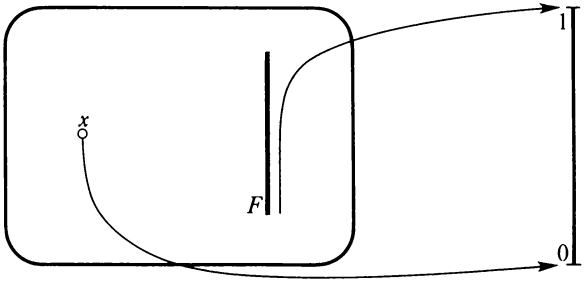


Fig.154. A $T_{3\frac{1}{2}}$-space.

The reader will find no difficulty in proving the following.

7.3.7. THEOREM. *A topological space X is a $T_{3\frac{1}{2}}$-space if and only if X is a T_1-space and for every point $x \in X$ and every neighbourhood W of the point x taken from a fixed base $\mathcal{B}$ of the space, there is a continuous function $f: X \to I$ such that $f(x) = 0$ and $f(X \setminus W) \subset \{1\}$.* ■

Every completely regular space X is regular. To see this note that if the continuous function $f: X \to I$ satisfies the conditions $f(x) = 0$ and $f(F) \subset \{1\}$, then the sets $U = f^{-1}([0, 1/2))$ and $V = f^{-1}((1/2, 1])$ are open in X and satisfy $x \in U$, $F \subset V$ and $U \cap V = \emptyset$. Appealing to the continuity of the distance of a point from a fixed set (see Theorem 1.4.10) it is easy to check that every metrizable space is completely regular. We showed in Examples 7.2.5 and 7.2.6 that the Sorgenfrey line and the Niemytzki plane are completely regular. The spaces of Examples 7.1.27 and 7.1.31 are also completely regular: they are regular and have a base consisting of open-and-closed sets, and each

space with these properties is necessarily completely regular. Examples of regular spaces which are not completely regular are more difficult than the examples considered above; one such space is described in Problem 7.P.8.

A topological space X is called a *T_4-space* or a *normal* space if X is a T_1-space and for every pair of disjoint closed sets $A, B \subset X$ there are open sets $U, V \subset X$ such that

$$A \subset U, \quad B \subset V \quad \text{and} \quad U \cap V = \emptyset.$$

Evidently every normal space is regular. It follows from Theorem 7.3.9, to be proved below, that normal spaces are also completely regular. Every metrizable space is by Theorem 1.6.27 a normal space. The reader will find no difficulty in checking that the space of Example 7.1.27 is normal.

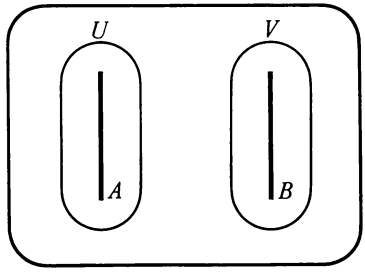


Fig.155. A T_4-space.

The Sorgenfrey line is also normal: if a pair of disjoint closed sets $A, B \subset K$ is given, choosing for each $a \in A$ an interval $[a, x(a))$ disjoint from B and for each $b \in B$ an interval $[b, x(b))$ disjoint from A and putting

$$U = \bigcup_{a \in A} [a, x(a)), \quad V = \bigcup_{b \in B} [b, x(b)),$$

we obtain open sets such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The last equation is a consequence of the fact that $[a, x(a)) \cap [b, x(b)) = \emptyset$ for $a \in A$ and $b \in B$ otherwise we should have either $b \in [a, x(a))$ or $a \in [b, x(b))$ depending on whether $a < b$ or $b < a$.

The normality of the space of Example 7.1.31 will follow from Theorem 7.3.12. On the other hand the Niemytzki plane turns out to be an example of a completely regular space which is not normal (see Example 7.4.36 and Problem 7.P.10).

7.3.8. EXAMPLE. We show that the Niemytzki plane L is not normal. We have already remarked that (see Example 7.1.29) every subset A of L_1 is closed in L and that the set $C \subset L$ consisting of points with both coordinates rational is countable and dense in L. These two facts will form the basis of our proof (see Exercise (g)).

Suppose that L is a normal space. For every subset $A \subset L_1$ there are, accordingly, open sets U_A, $V_A \subset L$ such that

$$A \subset U_A, \quad L_1 \setminus A \subset V_A \quad \text{and } U_A \cap V_A = \emptyset.$$

Let $C_A = C \cap U_A \subset C$. We show that if $A \neq B$ where $A, B \subset L_1$ then $C_A \neq C_B$. A contradiction will thus arise, since L_1 contains $2^{\mathfrak{c}}$ different subsets while C has only $\mathfrak{c}$ different subsets.

Suppose then that $A \neq B$; by the symmetry of the hypothesis we may suppose that $A \setminus B \neq \emptyset$. Since $A \setminus B \subset U_A \cap V_B$ the open set $U_A \cap V_B$ is non-empty, and thus we have $\emptyset \neq C \cap U_A \cap V_B \subset C_A \setminus U_B \subset C_A \setminus C_B$, so that $C_A \neq C_B$. ■

We give other specimens of completely regular spaces which are not normal in Examples 7.4.36 and 7.4.38.

We now prove a fundamental result about normal spaces know for historic reasons as *Urysohn's Lemma.*

7.3.9. THEOREM (Urysohn). *If X is a normal space, then for every pair of disjoint closed sets $A, B \subset X$ there is a continuous function $f: X \to I$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.*

PROOF. For every rational number w of the interval $[0,1]$ we define an open set $U_w \subset X$ so that

$$(*) \qquad A \subset U_0, \quad U_1 \subset X \setminus B$$

and

$$(**) \qquad \operatorname{cl} U_w \subset U_{w'}, \quad \text{if} \quad w < w'.$$

Put $w_1 = 0$ and $w_2 = 1$ and arrange the rational numbers of the interval $(0,1)$ into a sequence $w_3, w_4, \ldots$ We define inductively the sets U_{w_i} for $i = 1, 2, \ldots$ From the normality of the space X follows the existence of open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The open sets $U_0 = U$ and $U_1 = X \setminus B$ satisfy $(*)$. Since $V \cap \operatorname{cl} U = \emptyset$ we have $\operatorname{cl} U \subset X \setminus V \subset X \setminus B$, that is $\operatorname{cl} U_0 \subset U_1$. Assume that the sets U_{w_i} have already been defined for $i = 1, 2, \ldots, m-1$ where $m \geq 3$ and that they satisfy

$$\operatorname{cl} U_{w_i} \subset U_{w_j} \quad \text{if} \quad i, j < m \quad \text{and} \quad w_i < w_j.$$

Denote by w_- and w_+ those of the numbers $w_1, w_2, \ldots, w_{m-1}$ which are closest to w_m respectively on the left and right. Of course $w_- < w_+$ and so $\operatorname{cl} U_{w_-} \subset U_{w_+}$. The sets $\operatorname{cl} U_{w_-}$ and $X \setminus U_{w_+}$ are disjoint and closed in X so there exist open sets $U, V \subset X$ such that

$$\operatorname{cl} U_{w_-} \subset U, \quad X \setminus U_{w_+} \subset V \quad \text{and} \quad U \cap V = \emptyset.$$

Now $V \cap \operatorname{cl} U = \emptyset$, so $\operatorname{cl} U \subset X \setminus V \subset U_{w_+}$. Taking $U_{w_m} = U$ we have of course

$$\operatorname{cl} U_{w_i} \subset U_{w_j} \quad \text{if} \quad i, j \leq m \quad \text{and} \quad w_i < w_j.$$

The open sets U_w for w a rational number of the interval $[0,1]$ just obtained satisfy $(*)$ and $(**)$.

Now put for $x \in X$

$$f(x) = \begin{cases} \inf\{w : x \in U_w\}, & \text{if} \quad x \in U_1, \\ 1, & \text{if} \quad x \in X \setminus U_1. \end{cases}$$

By $(*)$ we have $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$ so to complete the proof it remains to show that $f: X \to I$ is a continuous function. Since $f^{-1}((a,b)) = f^{-1}((a,1]) \cap f^{-1}([0,b))$, it will suffice to prove that sets of the form $f^{-1}((a,1])$ and of the form $f^{-1}([0,b))$, where $a < 1$ and $b > 0$, are open in X.

The inequality $f(x) > a$ signifies that there is a rational number $w' > a$ such that $x \notin U_{w'}$ which by $(**)$ is equivalent to the existence of a rational number $w > a$ such that $x \notin \operatorname{cl} U_w$, hence the set

$$f^{-1}((a,1]) = \bigcup_{w>a} (X \setminus \operatorname{cl} U_w) = X \setminus \bigcap_{w>a} \operatorname{cl} U_w$$

is open. The inequality $f(x) < b$ is equivalent to the existence of a rational number $w < b$ such that $x \in U_w$ hence the set

$$f^{-1}([0,b)) = \bigcup_{w<b} U_w$$

is also open. ■

The reader will find it easy to check that every T_1-space X satisfying the conclusion of Urysohn's Lemma is necessarily normal.

7.3.10. LEMMA. *If X is a T_1-space and if for every closed set $F \subset X$ and every open set $W \subset X$ containing F there is a sequence $W_1, W_2, \ldots$ of open sets of X such that $F \subset \bigcup_{n=1}^{\infty} W_n$ and $\operatorname{cl} W_n \subset W$ for $n = 1, 2, \ldots$, then X is a normal space.*

PROOF. Consider a disjoint pair of closed sets $A, B \subset X$. Taking $F = A$ and $W = X \setminus B$ we obtain a sequence $W_1, W_2, \ldots$ of open sets of X satisfying

$$A \subset \bigcup_{n=1}^{\infty} W_n \quad \text{and} \quad B \cap \operatorname{cl} W_n = \emptyset \quad \text{for} \quad n = 1, 2, \ldots.$$

Then taking $F = B$ and $W = X \setminus A$ we obtain a sequence $V_1, V_2, \ldots$ of open sets of X satisfying

$$B \subset \bigcup_{n=1}^{\infty} V_n \quad \text{and} \quad A \cap \operatorname{cl} V_n = \emptyset \quad \text{for} \quad n = 1, 2, \ldots.$$

For $n = 1, 2, \ldots$ consider the open sets

$$G_n = W_n \setminus \bigcup_{i \le n} \operatorname{cl} V_i \quad \text{and} \quad H_n = V_n \setminus \bigcup_{i \le n} \operatorname{cl} W_i.$$

It follows from the properties of the sets W_n and V_n that

$$A \subset U = \bigcup_{n=1}^{\infty} G_n \quad \text{and} \quad B \subset V = \bigcup_{n=1}^{\infty} H_n.$$

To complete the proof it is enough therefore to show that $U \cap V = \emptyset$. Since $G_i \cap V_j = \emptyset$ for $j \le i$, we also have $G_i \cap H_j = \emptyset$ for $j \le i$. Similarly $G_i \cap H_j = \emptyset$ for $j \ge i$. We thus have $G_i \cap H_j = \emptyset$ for $i, j = 1, 2, \ldots$ whence it follows that $U \cap V = \emptyset$. ■

It may readily be noted that the condition stated in the lemma is not only sufficient but also necessary for the normality of a T_1-space X.

7.3.11. THEOREM. *Every regular space satisfying the second axiom of countability is normal.*

PROOF. Consider a regular space X with a countable base $\{V_n\}_{n=1}^{\infty}$, a closed set $F \subset X$ and an open set $W \subset X$ containing F. For each point $x \in F$ there is a natural number $n(x)$ such that $x \in V_{n(x)} \subset \operatorname{cl} V_{n(x)} \subset W$. Arranging the members of the family $\{V_{n(x)} : x \in F\}$ into a sequence $W_1, W_2, \ldots$ we obtain open sets satisfying the hypothesis of Lemma 7.3.10. ■

7.3.12. THEOREM. *Every countable regular space is normal.*

PROOF. Consider a countable regular space X and a closed set $F \subset X$ together with an open set W containing F. Say $F = \{x_1, x_2, \ldots\}$. For $n = 1, 2, \ldots$ there is an open set $W_n \subset X_n$ such that $x_n \in W_n \subset \operatorname{cl} W_n \subset W$. The open sets $W_1, W_2, \ldots$ satisfy the hypothesis of Lemma 7.3.10. ■

A common generalization of Theorems 7.3.11 and 7.3.12 will be stated in Problem 7.P.15.

We conclude the section by showing that closed maps preserve normality.

7.3.13. LEMMA. *A topological space X is normal if and only if it is a T_1-space and for every pair of open sets satisfying $U \cup V = X$ there is a pair of closed sets $A, B \subset X$ such that $A \subset U$, $B \subset V$ and $A \cup B = X$.* ■

7.3.14. THEOREM. *If f is a closed map of a normal space X onto a space Y then the space Y is also normal.*

PROOF. It follows from Theorem 7.3.1 that Y is a T_1-space. Consider an arbitrary pair of open sets $U, V \subset Y$ satisfying $U \cup V = Y$. The sets $U' = f^{-1}(U)$ and $V' = f^{-1}(V)$ are open in X and $U' \cup V' = X$, so by Lemma 7.3.13 there is a pair of closed sets $A', B' \subset X$ such that $A' \subset U'$, $B' \subset V'$ and $A' \cup B' = X$. The sets $A = f(A')$ and $B = f(B')$ are closed in Y and satisfy

$$A \subset ff^{-1}(U) = U, \quad B \subset ff^{-1}(V) = V \quad \text{and} \quad A \cup B = f(X) = Y,$$

so the space is normal by Lemma 7.3.13. ■

Exercises

a) Show that in a T_1-space X the derived set of any set is closed and that finite subsets of X have empty derived set. Check that every finite T_1-space is discrete.

b) Prove that for each cardinal number $\mathbf{m} \geq \aleph_0$ there is a separable T_1-space of weight $\mathbf{m}$ (cf. Lemma 7.5.33).

c) Show that if X is a Hausdorff space then every sequence of points of the space X has at most one limit. Give an example of a T_1-space which is not Hausdorff in which every sequence has at most one limit.

d) Show that in Theorems 7.3.5 and 7.3.7 it is enough to consider neighbourhoods W belonging to a fixed subbase $\mathcal{P}$ of the space X.

e) Prove that for every finite family $\{A_i\}_{i=1}^m$ of pairwise disjoint, closed subsets of a normal space X there is a family $\{U_i\}_{i=1}^m$ of pairwise disjoint, open sets of the space X such that $A_i \subset U_i$ for $i = 1, 2, \ldots, m$. Verify that if the sets A_i are singletons then it is enough to suppose that X is a Hausdorff space.

f) Prove that if a family $\{A_i\}_{i=1}^\infty$ of pairwise disjoint, finite subsets of a regular space X has the property that the derived set of the union $\bigcup_{i=1}^\infty A_i$ is an empty set then there is a family $\{U_i\}_{i=1}^\infty$ of open subsets of the space X such that $A_i \subset U_i$ for $i = 1, 2, \ldots$ and $\operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset$ for $i \neq j$ (cf. Problem 7.P.11).

g) Prove that if a separable space X has a subset D of cardinality $\mathfrak{c}$ such that every subset A of D is closed in X then X is not normal.

h) Give an example of an open map of the real line onto the two-point antidiscrete space (cf. Exercise (l) of Section 7.4).

i) Show that every connected $T_{3\frac{1}{2}}$-space containing at least two points has cardinality at least $\mathfrak{c}$.

j) Let X be the set of points $(w^1, w^2) \in \mathbf{R}^2$ where w^1 and w^2 are rational and $w^2 \geq 0$. For each point $x = (w^1, w^2) \in X$ and each $n = 1, 2, \ldots$ put

$$U_n(x) = \{x\} \cup \{(w,0) \in X : |w - (w^1 - w^2/\sqrt{3})| < 1/n\} \cup \\ \cup \{(w,0) \in X : |w - (w^1 + w^2/\sqrt{3})| < 1/n\}.$$

Check that the collection $\{\mathcal{B}_x\}_{x \in X}$ where $\mathcal{B}_x = \{U_n(x) : n = 1, 2, \ldots\}$ has properties (BP1)–(BP4) and show that the space X with topology generated by the neighbourhood system $\{\mathcal{B}_x\}_{x \in X}$ is connected.

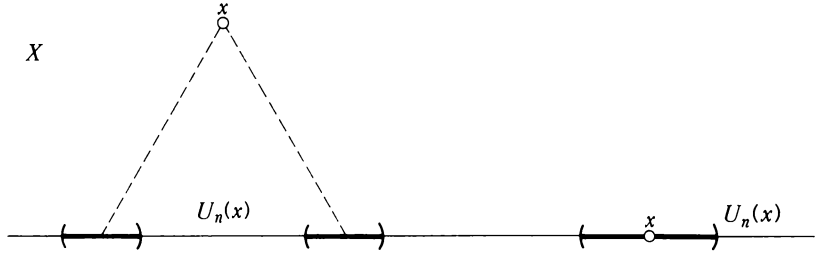


Fig.156. The sets $U_n(x)$ form a local base at the point x in the space X (see Exercise (j)).

7.4. Operations on topological spaces

We begin with the subspace operation. Suppose given a space X and a set $A \subset X$. It is easy to see that the family $\mathcal{O}$ of sets of the form $A \cap U$ where U is an open set of X satisfies the axioms (T1)–(T3), so we may endow the set A with this family as its topology. The topological space thus obtained is called a *topological subspace* or briefly a *subspace* of the space X. It follows from Theorem 1.6.5 that if the topology of the space X is induced by a metric ρ then the topology of the topological subspace

A is induced by the metric ρ_A of the metric subspace A of the metric space (X, ρ). Sometimes, for practical or traditional reasons, one speaks of 'subsets' of a topological space even though the discussion depends on their inherited subspace topology. Just as in the metric context, we agree on the convention that, whenever a notion applied to a subset $A \subset X$ is topological in character, then the 'subset' A is in fact deemed to be the topological 'subspace' of X. Conversely, certain subset properties may be ascribed to subspace e.g. when speaking of 'open' subspaces or 'closed' subspaces. The words 'subspace' and 'subset' will be treated below almost as synonyms within the meaning of this convention.

7.4.1. ASSERTION. *If X_0 is a subspace of the topological space X, then a set $A_0 \subset X_0$ is closed in X_0 if and only if $A_0 = X_0 \cap A$ where A is closed in X.* ■

7.4.2. ASSERTION. *If X_0 is an open (closed) subspace of a topological space X, then every open (closed) subset A of the subspace X_0 is open (closed) in the space X.* ■

7.4.3. ASSERTION. *The closure $\mathrm{cl}_{X_0} A$ of a subset $A \subset X_0$ in the subspace X_0 of a topological space X and the closure $\mathrm{cl}\, A$ of the subset A in the space X are related by the equation $\mathrm{cl}_{X_0} A = X_0 \cap \mathrm{cl}\, A$.* ■

7.4.4. ASSERTION. *If X_0 is a subspace of the topological space X and X_1 is a subset of X_0 then the two topologies defined on the set X_1, viz. the subspace topology inherited from the space X and the subspace topology inherited from the space X_0 coincide.* ■

For every topological space X and each subspace A the inclusion map $i_A: A \to X$ defined by $i_A(x) = x$ for $x \in A$ is a continuous map from the subspace A into the space X. A map $f: X \to Y$ is called a *homeomorphic imbedding* when f is the composition of a homeomorphism and an inclusion map; that is when there is a subspace B of Y and a homeomorphism $f': X \to B$ such that $f = i_B f'$. Evidently every homeomorphism is a homeomorphic imbedding. It follows from Assertion 7.4.4 that the composition of homeomorphic imbeddings is a homeomorphic imbedding. If there exists a homeomorphic imbedding $f: X \to Y$ of a space X into a space Y then we say that the space X is *imbeddable* in the space Y.

7.4.5. EXAMPLE. The discrete space of cardinality $\mathbf{c}$ is imbeddable in the Niemytzki plane L being in fact homeomorphic to the closed subspace $L_1 \subset L$. For each $\mathbf{m} \geq \aleph_0$ the discrete space $D(\mathbf{m})$ is imbeddable in the space $A(\mathbf{m})$ (see Example 7.2.14) as it is in fact homeomorphic to the subspace $A(\mathbf{m}) \setminus \{x_0\}$ of $A(\mathbf{m})$ where x_0 is the unique accumulation point of the space $A(\mathbf{m})$. ■

For every continuous map $f: X \to Y$ and any subspace A of the space X the restriction $f|A: A \to Y$, that is the map fi_A, is a continuous map of the subspace A into the space Y.

7.4.6. ASSERTION. *The restriction of an open (closed) map to an open (closed) subspace is an open (closed) map.* ■

7.4.7. ASSERTION. *The restriction of a homeomorphic imbedding is a homeomorphic imbedding.* ■

Inverse to the operation of restriction is the extension of maps. It is to be emphasized that this operation is not always feasible and its result is not in general unique. We now prove an important theorem on the existence of extensions and then we make a simple remark on uniqueness. The promised theorem is a generalization of Tietze's Theorem (3.1.4). The passage from metric spaces to topological spaces necessitates a completely new approach. A comparison of the two proofs is instructive. Of course the proof given here gives a more general theorem, but the proof of Section 3.1 affords a more direct route to a very important special case.

7.4.8. THEOREM (Tietze, Urysohn). *Let A be a closed subset of a normal space X. Every continuous function $f: A \to I$ has a continuous extension $f^*: X \to I$.*

PROOF. Since the interval I is homeomorphic to the interval $J = [-1, 1]$ it is enough to show that every continuous function $f: A \to J$ has a continuous extension $f^*: X \to J$. Substitution of the interval J for the interval I simplifies our arithmethic.

Observe, first of all, that for every continuous function $h: A \to J$ such that $|h(x)| \leq c \leq 1$ for $x \in A$, there exists a continuous function $g: X \to J$ satisfying the conditions

$$|g(x)| \leq \frac{1}{3}c \quad \text{for} \quad x \in X$$

and

$$|h(x) - g(x)| \leq \frac{2}{3}c \quad \text{for} \quad x \in A.$$

To see this note that $K = h^{-1}([-c, -\frac{1}{3}c])$ and $L = h^{-1}([\frac{1}{3}c, c])$ are disjoint and closed in A and hence also in X, so by Urysohn's Lemma (7.3.9) there is a continuous function $k: X \to I$ such that $k(K) \subset \{0\}$ and $k(L) \subset \{1\}$. It is easily checked that the map $g: X \to J$ defined by $g(x) = \frac{2}{3}c(k(x) - \frac{1}{2})$ fulfils the required conditions.

We now define by induction a sequence $g_1, g_2, \ldots$ of continuous maps from the space X to the interval J satisfying for $n = 1, 2, \ldots$ the conditions

$$(*) \qquad |g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \quad \text{for } x \in X$$

and

$$(**) \qquad |f(x) - \sum_{j=1}^{n} g_j(x)| \leq \left(\frac{2}{3}\right)^n \quad \text{for} \quad x \in A.$$

The existence of the map g_1 follows from our earlier observation by taking $h = f$ and $c = 1$. Suppose that the maps $g_1, g_2, \ldots, g_{n-1}$ have already been defined and satisfy both conditions above for $n < m$. Taking $h = f - \sum_{j=1}^{m-1} g_j | A$ and $c = (\frac{2}{3})^{m-1}$ and applying the earlier observation we obtain a map g_m with the desired properties.

We infer from inequality $(*)$ that the sequence $\{f_n\}$ of continuous maps from the space X into the interval J where $f_n(x) = \sum_{j=1}^{n} g_j(x)$ for $x \in X$ is uniformly convergent. It follows from Theorem 7.2.4 that the limit $f^*: X \to I$ of this sequence is a continuous map. By $(**)$ we have for each $x \in A$ that $f^*(x) = f(x)$ and so f^* is the desired extension of the function f. ■

Arguing as in the proof of Theorem 3.1.5 we obtain the following.

7.4.9. THEOREM. *Let A be a closed subset of a normal space X. Every continuous function $f: A \to \mathbf{R}$ has a continuous extension $f^*: X \to \mathbf{R}$.* ■

7.4.10. THEOREM. *Let A be a dense subset of a topological space X and let Y be a Hausdorff space. For every continuous map $f: A \to Y$ there is at most one continuous extension $f^*: X \to Y$.*

PROOF. Let $f_1^*, f_2^*: X \to Y$ both be continuous extensions of the map f. By Theorem 7.3.4 the set

$$B = \{x \in X : f_1^*(x) = f_2^*(x)\}$$

is closed in X. Since $A \subset B$ we have also $X = \operatorname{cl} A \subset B$ which implies that $f_1^* = f_2^*$. ■

We now check which topological properties pass down from a space to its subspaces (cf. Supplement 7.S.11). It is immediate from the definition of the subspace topology that if A is a subspace of the space X then for every base $\mathcal{B}$ of the space X (every local base $\mathcal{B}_x$ of the space X at a point $x \in A$) the family of sets of the form $A \cap U$ where $U \in \mathcal{B}$ ($U \in \mathcal{B}_x$) is a base of the subspace A (local base of the subspace A at the point x). We thus have the following consequence.

7.4.11. ASSERTION. *If a space X satisfies the first (second) axiom of countability and A is a subspace of the space X then the subspace A satisfies the first (second) axiom of countability.* ■

A reference to the Niemytzki plane, which is separable, but contains as a subspace a discrete space of cardinality $\mathfrak{c}$ (see Example 7.4.5) shows that a subspace of a separable space need not be separable.

7.4.12. THEOREM. *If X is a T_i-space for some $i \leq 3\frac{1}{2}$ and A is a subset of the space X then the subspace A is a T_i-space.*

PROOF. It follows immediately from Assertion 7.4.1 that every subspace of a T_1-space is a T_1-space.

The proofs for the remaining i are all similar. As an example we go through the proof that a subspace A of a regular space X is a regular space.

We have already pointed out that A is a T_1-space. Consider any point $x \in A$ and any closed set F of the subspace A with $x \notin F$. By Assertion 7.4.1 there is a set F_1 closed in X for which $F = A \cap F_1$; obviously $x \notin F_1$. By the regularity of X there are open sets $U_1, V_1 \subset X$ such that $x \in U_1$, $F_1 \subset V_1$ and $U_1 \cap V_1 = \emptyset$. The sets $U = A \cap U_1$ and $V = A \cap V_1$ are open in the subspace A and satisfy

$$x \in U, \quad F \subset V \quad \text{and} \quad U \cap V = \emptyset,$$

hence the subspace A is regular. ■

We shall show in Example 7.4.38 that a subspace of a normal space need not be normal. The following theorem, whose easy proof we leave for the reader, does hold.

7.4.13. THEOREM. *If X is a normal space and A is a closed subset of X, then the subspace A is normal.* ■

It is readily observed that subspaces of connected spaces are not in general connected. The properties of connected subspaces of topological spaces are similar to the properties of connected subspaces of metric spaces stated in Theorems 1.7.5, 1.7.6, 1.7.12 and 1.7.15–1.7.17. Since the proofs given in Section 1.7 continue to hold in the more general context considered here, we limit ourselves only to a statement of the appropriate generalizations.

7.4.14. THEOREM. *If $X = \bigcup_{t\in T} X_t$, where each of the subspaces X_t is connected for $t \in T$, and $\bigcap_{t\in T} X_t \neq \emptyset$, then the space X is connected.* ■

7.4.15. THEOREM. *If in a topological space X for each pair of points there exists a connected subspace containing the pair of points then the space X is connected.* ■

7.4.16. THEOREM. *If A is a connected subset of a topological space X, then the closure* $\operatorname{cl} A$ *of the set A in the space X is also a connected set.* ■

Let X be any topological space. Any subset of the space X which is connected and inclusion-maximal with respect to this property is called a *component* of the space X. In other words the set $S \subset X$ is a component of the space X if it is connected and for every connected set $C \subset X$ satisfying $S \subset C$ we have the equality $S = C$.

7.4.17. THEOREM. *The components of a space are closed and pairwise disjoint.* ■

7.4.18. THEOREM. *Every topological space is the union of its components.* ■

7.4.19. THEOREM. *For two points of a topological space to belong to the same component it is necessary and sufficient that there is a connected subspace containing the two points.* ■

7.4.20. COROLLARY. *A topological space is connected if and only if it has exactly one component.* ■

We now pass to the operation of topological product.

Consider first an arbitrary family of sets $\{X_t\}_{t\in T}$. Recall that the Cartesian product $X = \bigtimes_{t\in T} X_t$ is the set of all maps $x: T \to \bigcup_{t\in T} X_t$ with the property that $x(t) \in X_t$ for $t \in T$; the map $x \in X$ will be identified with the indexed set $\{x_t\}_{t\in T}$ where $x_t = x(t)$ is the t^{th}-coordinate of the element x. In cases where this identification leads to no confusion we shall denote the element $x = \{x_t\}_{t\in T}$ of the Cartesian product more simply as $\{x_t\}$. The elements of finite and of countable Cartesian products, that is finite sequences and infinite sequences, will be denoted correspondingly by $(x_1, x_2, \ldots, x_m)$ and $\{x_1, x_2, \ldots\}$. Recall that for every $t \in T$ the correspondence which sends the point $x = \{x_t\}$ of the Cartesian product $X = \bigtimes_{t\in T} X_t$ to the point x_t lying in X_t defines a map $p_t: X \to X_t$ which is known as the projection of the product $X = \bigtimes_{t\in T} X_t$ onto its t^{th}-factor.

Now let an arbitrary family of topological spaces $\{X_t\}_{t\in T}$ be given. The family $\mathcal{B}$ of subsets of the Cartesian product $X = \bigtimes_{t\in T} X_t$ comprising the sets of the form

$$(*) \qquad \bigcap_{i=1}^{m} p_{t_i}^{-1}(U_i), \quad \text{where } t_i \in T \text{ and } U_i \text{ is open in } X_{t_i}$$

for $i = 1,2,\ldots,m$ and $m = 1,2,\ldots$, has the properties (B1) and (B2) of Theorem 7.1.20, as may easily be verified. The topology on the set X generated by the base $\mathcal{B}$ is known as the *Tikhonov topology* and the set X endowed with this topology is called the *topological product* of the spaces $\{X_t\}_{t\in T}$. The family $\mathcal{B}$ is a base of the topological product X, it will be called the *canonical base* of the product.

Note that the set $(*)$ may also be expressed in the form $\bigtimes_{t\in T} U_t$ where $U_t = U_i$ for $t = t_i$ where $i = 1,2,\ldots,m$ and $U_t = X_t$ for $t \in T\setminus\{t_1,t_2,\ldots,t_m\}$. Thus the canonical base of the topological product $\bigtimes_{t\in T} X_t$ comprises sets consisting of maps $x \in \bigtimes_{t\in T} X_t$ whose values on indices belonging to a distinguished finite set $\{t_1,t_2,\ldots,t_m\} \subset T$ lie in distinguished open sets of the corresponding factors in the product.

We show first of all that the operation of topological product on topological spaces agrees with the operation of metric product for finite and infinite sequences of metric spaces, and also with the operation of topological product as earlier defined for metric spaces.

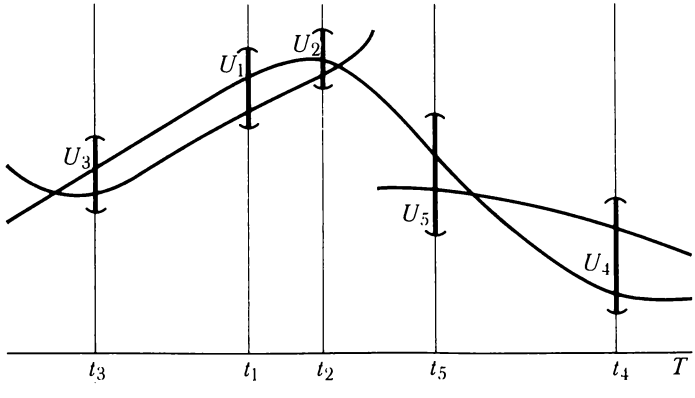


Fig.157. An element $\bigcap_{i=1}^{m} p_{t_i}^{-1}(U_i)$ of the canonical base of the topological product $\bigtimes_{t\in T} X_t$ consists of all maps which pass through the "gates" U_i.

7.4.21. THEOREM. *If for $i = 1,2,\ldots,m$ the topology of the space X_i is induced by a metric ρ_i, then the Tikhonov topology on the Cartesian product $X = \bigtimes_{i=1}^{m} X_i$ is induced by the metric of the metric product $\bigtimes_{i=1}^{m}(X_i,\rho_i)$.*

PROOF. It follows from Theorem 1.6.6 that every member of the canonical base of the topological product $\bigtimes_{i=1}^{m} X_i$ is an element of the topology $\mathcal{O}$ induced by the metric of the corresponding metric product, so every member of the Tikhonov topology $\mathcal{T}$ lies in $\mathcal{O}$, that is $\mathcal{T} \subset \mathcal{O}$.

It remains to prove that every set $U \in \mathcal{O}$ is a member of the Tikhonov topology $\mathcal{T}$. Of course we may assume that $U \neq \emptyset$. Consider an arbitrary point $x = (x_1, x_2, \ldots, x_m) \in U$ and choose a positive real number r such that $B(x;r) \subset U$, where $B(x;r)$ is a ball in the space X with the metric ρ of the metric product. From Lemma 1.2.4 it follows that the intersection $\bigcap_{i=1}^{m} p_i^{-1}(B_i(x_i; r/\sqrt{m}))$, where $B_i(x_i; r/\sqrt{m})$ is a ball in the space X_i with metric ρ_i, is contained in the ball $B(x,r)$. Since this intersection contains the point x, the set U is a union of members of the canonical base of the topological product X, that is $U \in \mathcal{T}$. ■

Using Lemma 6.1.3 a similar proof establishes the following.

7.4.22. THEOREM. *If for $i = 1, 2, \ldots$ the topology of the space X_i is induced by a metric ρ_i and the series $\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^2$ converges, then the Tikhonov topology on the Cartesian product $X = \bigtimes_{i=1}^{\infty} X_i$ is induced by the metric of the metric product $\bigtimes_{i=1}^{\infty} (X_i, \rho_i)$.* ■

7.4.23. THEOREM. *If for $i = 1, 2, \ldots$ the topology of the space X_i is induced by a metric ρ_i, then the Tikhonov topology on the Cartesian product $X = \bigtimes_{i=1}^{\infty} X_i$ is induced by the metric of the topological product $\bigtimes_{i=1}^{\infty} (X_i, \rho_i)$.*

PROOF. By definition the metric of the topological product $\bigtimes_{i=1}^{\infty} (X_i, \rho_i)$ is identical with the metric of the metric product $\bigtimes_{i=1}^{\infty} (X_i, \sigma_i)$ where for $i = 1, 2, \ldots$

$$\sigma_i(x_i, y_i) = \min(1/i, \rho_i(x_i, y_i)) \quad \text{for} \quad x_i, y_i \in X_i.$$

Since for $i = 1, 2, \ldots$ the topology of the space X_i is also induced by the metric σ_i, the proof is completed by appealing to Theorem 7.4.22. ■

7.4.24. ASSERTION. *If, for each $t \in T$, A_t is a subspace of the space X_t, then the two topologies defined on the set $\bigtimes_{t \in T} A_t$, viz. the product topology of the subspaces $\{A_t\}_{t \in T}$ and the subspace topology inherited from the product $\bigtimes_{t \in T} X_t$, coincide.* ■

7.4.25. THEOREM. *If, for each $t \in T$, A_t is a subspace of the space X_t, then the equation*

$$\operatorname{cl}\Big(\bigtimes_{t \in T} A_t\Big) = \bigtimes_{t \in T} \operatorname{cl} A_t$$

holds, where the closure operation refers to the product topology $\bigtimes_{t \in T} X_t$ and to the topologies of the spaces X_t, respectively.

PROOF. By Assertion 7.1.13 the point $x = \{x_t\}$ lies in $\operatorname{cl}(\bigtimes_{t \in T} A_t)$ if and only if for each member $\bigtimes_{t \in T} U_t$ of the canonical base which contains the point x we have $(\bigtimes_{t \in T} U_t) \cap (\bigtimes_{t \in T} A_t) \neq \emptyset$. Since $(\bigtimes_{t \in T} U_t) \cap (\bigtimes_{t \in T} A_t) = \bigtimes_{t \in T} (U_t \cap A_t)$, this means that for each $t \in T$ and each neighbourhood U_t of the point x_t in the space X_t we have $U_t \cap A_t \neq \emptyset$. The latter holds if and only if $x_t \in \operatorname{cl} A_t$ for each $t \in T$, that is if $x \in \bigtimes_{t \in T} \operatorname{cl} A_t$. ■

7.4.26. COROLLARY. *If, for each $t \in T$, A_t is a closed set in the space X_t, then the product $\bigtimes_{t \in T} A_t$ is closed in the topological product $\bigtimes_{t \in T} X_t$.* ■

7.4.27. COROLLARY. *If, for each $t \in T$, the set A_t is dense in the space X_t, then the product $\bigtimes_{t \in T} A_t$ is dense in the topological product $\bigtimes_{t \in T} X_t$.* ■

We note that if the set T is infinite and A_t is, for each $t \in T$, a non-empty proper subset of the space X_t, then the product $\bigtimes_{t\in T} A_t$ is not an open subset of the topological product $\bigtimes_{t\in T} X_t$ (cf. Assertion 6.1.15).

It is immediate from the definition of the Tikhonov topology that for each $t \in T$ the projection $p_t: X \to X_t$ is a continuous map; we call it the *projection of the topological product* $X = \bigtimes_{t\in T} X_t$ *onto its* t^{th}*-factor.*

7.4.28. THEOREM. *For each* $t \in T$ *the projection* $p_t: X \to X_t$ *of the topological product* $X = \bigtimes_{t\in T} X_t$ *onto its* t^{th}*-factor is an open map.*

PROOF. For every non-empty member $U = \bigtimes_{t\in T} U_t$ of the canonical base of the topological product $\bigtimes_{t\in T} X_t$ we have $p_t(U) = U_t$ so the openness of the map p_t follows from Assertion 7.2.8. ■

Example 7.2.10 shows that the projection $p_t: X \to X_t$ is not in general a closed map (cf. Theorem 7.5.14).

7.4.29. THEOREM. *A map* f *of a topological space* Y *into a topological product* $\bigtimes_{t\in T} X_t$ *is continuous if and only if the composition* $p_t f: Y \to X_t$ *is continuous for each* $t \in T$.

PROOF. It suffices to show that the continuity of the maps $p_t f$ for $t \in T$ implies the continuity of the map $f: Y \to \bigtimes_{t\in T} X_t$. For a member $U = \bigcap_{i=1}^{m} p_{t_i}^{-1}(U_i)$ of the canonical base of the topological product $\bigtimes_{t\in T} X_t$ the equation

$$f^{-1}(U) = \bigcap_{i=1}^{m} f^{-1} p_{t_i}^{-1}(U_i) = \bigcap_{i=1}^{m} (p_{t_i} f)^{-1}(U_i)$$

holds; so, if the compositions $p_t f$ are continuous, the map f is continuous by Theorem 7.2.3. ■

The next two theorems describe important properties of the topological product which are colloquially known as the commutativity and associativity of the operation.

7.4.30. THEOREM. *Suppose given a family of topological spaces* $\{X_t\}_{t\in T}$. *If* φ *is a bijective map of* T *onto itself then the topological products* $\bigtimes_{t\in T} X_t$ *and* $\bigtimes_{t\in T} X_{\varphi(t)}$ *are homeomorphic.*

PROOF. It follows from Theorem 7.4.29 that the map $f: \bigtimes_{t\in T} X_t \to \bigtimes_{t\in T} X_{\varphi(t)}$ sending the point $x = \{x_t\}$ to the point $f(x) = \{x'_t\}$ where $x'_t = x_{\varphi(t)}$ is continuous. Since f is a bijective map and the inverse map f^{-1} is determined in similar fashion by the map φ^{-1} we have that f is a homeomorphism. ■

The following is also a consequence of Theorem 7.4.29.

7.4.31. THEOREM. *Suppose given a family of topological spaces* $\{X_t\}_{t\in T}$. *If* $T = \bigcup_{s\in S} T_s$ *and* $T_s \cap T_{s'} = \emptyset$ *for* $s \neq s'$ *then the topological products* $\bigtimes_{t\in T} X_t$ *and* $\bigtimes_{s\in S}(\bigtimes_{t\in T_s} X_t)$ *are homeomorphic.* ■

Finally, from Theorem 7.4.29 it follows that if $\operatorname{card} T = \operatorname{card} S = \mathfrak{m} \geq 1$ and $X_t = X$ for all $t \in T$ and $X_s = X$ for all $s \in S$, then the topological products $\bigtimes_{t\in T} X_t$

and $\bigtimes_{s\in S} X_s$ are homeomorphic; any product of this form is called the *topological* $\mathbf{m}^{th}$ *power of the space* X and is denoted $X^{\mathbf{m}}$.

7.4.32. THEOREM. *For any topological space* X *and any cardinal numbers* $\mathbf{m}$ *and* $\mathbf{n}$ *satisfying* $1 \leq \mathbf{n} \leq \mathbf{m}$ *where* $\mathbf{m} \geq \aleph_0$ *the spaces* $X^{\mathbf{m}}$ *and* $(X^{\mathbf{n}})^{\mathbf{m}}$ *are homeomorphic.*

PROOF. The claim follows immediately from Theorem 7.4.31. ■

We now investigate which topological properties carry over from the factors to their topological product (cf. Supplement 7.S.11).

The reader will have no difficulty in proving the following theorem (cf. Exercise (f)).

7.4.33. THEOREM. *The topological product of countably many spaces satisfying the first (second) axiom of countability also satisfies the first (second) axiom of countability.* ■

7.4.34. THEOREM. *The topological product of countably many separable spaces is a separable space.*

PROOF. Consider the topological product $X = \bigtimes_{i=1}^{\infty} X_i$ where X_i is a separable space for $i = 1, 2, \ldots$ It may of course be assumed that the spaces X_i are all non-empty. Choose for $i = 1, 2, \ldots$ any countably dense set A_i in X_i and arrange its elements as a sequence $a_1^i, a_2^i, a_3^i, \ldots$ possibly with repetitions. The subset A of the topological product X consisting of points of the form $\{a_{n_1}^1, a_{n_2}^2, \ldots, a_{n_m}^m, a_1^{m+1}, a_1^{m+2}, \ldots\}$ where $m, n_1, n_2, \ldots, n_m$ are arbitrary natural numbers is countable. It follows immediately from the definition of the Tikhonov topology that the set A is dense in the space X, so the topological product X is a separable space. ■

Actually, it turns out that a product of $\mathbf{c}$ many separable spaces is a separable space (see Problems 7.P.20 and 7.P.21).

7.4.35. THEOREM. *The topological product of any family of* T_i*-spaces is a* T_i*-space for each* $i \leq 3\frac{1}{2}$.

PROOF. It follows immediately from Corollary 7.4.26 that the topological product of T_1-spaces is a T_1-space.

The proofs for the remaining values of i are much the same as one another. By way of an example we prove that the topological product of any family of completely regular spaces is completely regular.

Consider the topological product $X = \bigtimes_{t\in T} X_t$ of completely regular spaces X_t where $t \in T$. As indicated X is a T_1-space. Consider an arbitrary point $x = \{x_t\} \in X$ and its neighbourhood $W = \bigcap_{i=1}^{m} p_{t_i}^{-1}(U_i)$ taken from the canonical base of the product X. Obviously $x_{t_i} \in U_i \subset X_{t_i}$ for $i = 1, 2, \ldots, m$. Let $f_i \colon X_{t_i} \to I$ for $i = 1, 2, \ldots, m$ be continuous functions satisfying the conditions

$$f_i(x_i) = 0 \quad \text{and} \quad f_i(X_{t_i} \setminus U_i) \subset \{1\}.$$

The function

$$f = \max(f_1 p_{t_1}, f_2 p_{t_2}, \ldots, f_m p_{t_m})$$

taking the topological product X into the unit interval is continuous and satisfies the conditions $f(x) = 0$ and $f(X\backslash W) \subset \{1\}$. By Theorem 7.3.7 the topological product X is completely regular. ■

We now show that the topological product of two normal spaces need not be normal.

7.4.36. EXAMPLE. We saw in Section 7.3 that the Sorgenfrey line K is a normal space. The topological product $K \times K$ contains the closed set $D = \{(x, y) : y = -x\}$, which is a discrete space of cardinality $\mathfrak{c}$, and also contains a countable dense set – say the set C of points both of whose coordinates are rational. Repeating the argument of Example 7.3.8 we infer that $K \times K$ is not a normal space (cf. Exercise (g) of Section 7.3).

An alternative argument is possible. It follows from Theorem 7.4.10 that there are at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ continuous functions defined on the space $K \times K$, since each such function is uniquely determined by its restriction to the set C. If the space $K \times K$ were normal, by the Tietze-Urysohn Theorem (Theorem 7.4.8) each of the $2^{\mathfrak{c}}$ distinct continuous functions defined on the closed set D would have a continuous extension to the whole of $K \times K$. Since $\mathfrak{c} < 2^{\mathfrak{c}}$ the topological product $K \times K$ is not normal. Clearly this same approach shows that the Niemytzki plane is not normal. ■

We might add that an uncountable topological product of metrizable spaces need not be a normal space. For instance the space $\mathbf{N}^{\aleph_1}$ where $\mathbf{N}$ denotes the set of natural numbers with discrete topology is not normal (see Problem 7.P.23).

7.4.37. THEOREM. *The topological product of any family of connected spaces is connected.*

PROOF. We begin by showing that the topological product $X = X_1 \times X_2$ of two connected spaces X_1 and X_2 is connected. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be any two points of the space X; put $C_1 = \{(x_1, b_2) \in X : x_1 \in X_1\}$ and $C_2 = \{(a_1, x_2) \in X : x_2 \in X_2\}$. The correspondence taking the point $x_1 \in X_1$ to the point $(x_1, b_2) \in C_1$ is a homeomorphism so the subspace $C_1 \subset X$ is connected. Likewise the subspace $C_2 \subset X$ is connected. These two subspaces have the common point (a_1, b_2) so by Theorem 7.4.14 the union $C = C_1 \cup C_2$ is a connected space. But $a \in C_2$ and $b \in C_1$, so $a, b \in C$. Using Theorem 7.4.15 we deduce that the space X is connected.

Using induction and appealing to Theorem 7.4.31 and the connectedness of a product of two connected spaces we deduce that the topological product of finitely many connected spaces is connected.

To complete the proof it is enough to show that if $\operatorname{card} T \geq \aleph_0$ and if for each $t \in T$ the space X_t is non-empty and connected, then the topological product $X = \bigtimes_{t\in T} X_t$ is connected. For each $t \in T$ choose a point $a_t \in X_t$ and denote by $\mathcal{S}$ the family of all finite subsets of the set T. For each $S \in \mathcal{S}$ the subspace

$$A_S = \{\{x_t\} \in X : \ x_t = a_t \quad \text{for} \quad t \in T\backslash S\}$$

of the topological product X is homeomorphic to the finite topological product $\bigtimes_{t\in S} X_t$ and so is a connected space. Since $a = \{a_t\} \in \bigcap_{S\in\mathcal{S}} A_S$, we conclude from Theorem 7.4.14 that $A = \bigcup_{S\in\mathcal{S}} A_S$ is a connected subspace of the space X. From the definition of

the Tikhonov topology it follows that the set A is dense in X so the topological product X is connected by Theorem 7.4.16. ■

We now apply the topological product operation to construct an example of a normal space which contains a subspace that is not normal.

7.4.38. EXAMPLE. Let $X = A(\aleph_0)$, $Y = A(\mathbf{c})$ and let x_0 and y_0 be the unique accumulation points of the spaces X and Y respectively (see Example 7.2.14). The topological product $Z = X \times Y$ is a normal space. The reasoning is as follows. For every pair of disjoint closed sets $A, B \subset Z$ there is an open-and-closed set $G \times H \subset Z$ which contains the point $z_0 = (x_0, y_0)$ and is disjoint from one of the given pair of sets; but, since the open subspace $Z\backslash(G \times H)$ of Z is normal (a fact that is easily verified), the sets A and B may be expanded to disjoint open sets $U, V \subset Z$. We note that the normality of the space Z is also a consequence of Theorems 7.5.8 and 7.5.16 and Example 7.5.24.

We now prove that the subspace $Z_0 = Z\backslash\{z_0\}$ of Z is not normal. The sets

$$A_0 = \{(x, y_0) : x \in X\backslash\{x_0\}\} \quad \text{and} \quad B_0 = \{(x_0, y) : y \in Y\backslash\{y_0\}\}$$

are disjoint and closed in Z_0. Consider any open sets $U_0, V_0 \subset Z_0$ such that $A_0 \subset U_0$, $B \subset V_0$. By the definition of the Tikhonov topology for each $x \in X\backslash\{x_0\}$ there is a finite set $S(x) \subset Y\backslash\{y_0\}$ with the property that $\{x\} \times (Y\backslash S(x)) \subset U_0$. Now the union $S = \bigcup\{S(x) : x \in X\backslash\{x_0\}\} \subset Y$ is countable, while the space Y has cardinality $\mathbf{c}$ so there exists a point $y_1 \in Y\backslash(S \cup \{y_0\})$. The set $C = (X\backslash\{x_0\}) \times \{y_1\}$ is contained in U_0, and since $(x_0, y_1) \in B_0 \cap \operatorname{cl} C$, we have $U_0 \cap V_0 \neq \emptyset$. Thus the space Z_0 is not normal. We note however that by Theorem 7.4.12 the space Z_0 is completely regular. ■

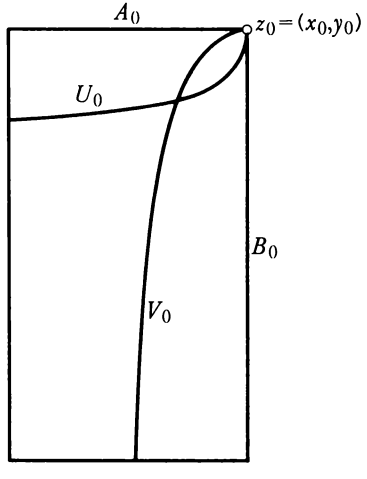


Fig.158. The subspace $Z_0 = Z\backslash\{z_0\}$ of the normal space Z is not normal (see Example 7.4.38).

We now discuss the embedding of spaces in topological products. We begin by stating some definitions.

Suppose given a topological space X, a family of topological spaces $\{Y_t\}_{t \in T}$ and a family of continuous maps $\{f_t\}_{t \in T}$ where $f_t : X \to Y_t$ for $t \in T$. The map f which sends

a point $x \in X$ to the point $f(x) = \{f_t(x)\} \in \bigtimes_{t\in T} Y_t$ is continuous by Theorem 7.4.29, since $p_t f = f_t$ for each $t \in T$; we call the map f the *diagonal* of the maps $\{f_t\}_{t\in T}$ and denote it $\triangle_{t\in T} f_t$, or $\triangle_{n=1}^{\infty} f_n$ when T is the set of natural numbers, or, in the finite case, $\triangle_{j=1}^{n} f_j$ or even $f_1 \triangle f_2 \triangle \ldots \triangle f_n$ when $T = \{1, 2, \ldots, n\}$.

Related to the notion of the diagonal of a family of maps are the notions of diagonal map and diagonal of a topological power. Let X be a topological space. Consider the power $X^{\mathfrak{m}} = \bigtimes_{t\in T} X_t$ where $X_t = X$ for each $t \in T$, $\operatorname{card} T = \mathfrak{m}$, and $\mathfrak{m}$ is an arbitrary cardinal number. The diagonal $d = \triangle_{t\in T} i_t \colon X \to X^{\mathfrak{m}}$ where $i_t = \mathrm{id}_X$ for each $t \in T$ is called the *diagonal map* and the set $\Delta = d(X) \subset X^{\mathfrak{m}}$ is called the *diagonal of the topological power* $X^{\mathfrak{m}}$.

It follows from Theorem 7.3.4 that if X is a Hausdorff space then the diagonal $\Delta = \bigcap_{t',t''\in T} \{x \in \bigtimes_{t\in T} X_t : p_{t'}(x) = p_{t''}(x)\}$ of the topological power $X^{\mathfrak{m}}$ is a closed subspace of the space $X^{\mathfrak{m}}$ (see Exercise g).

The *graph of the map* $f \colon X \to Y$ is the set $G(f) = \{(x, y) : f(x) = y\} \subset X \times Y$. The diagonal Δ of the topological power $X^2 = X \times X$ is thus the graph of the map $\mathrm{id}_X \colon X \to X$.

Suppose given a topological space X, a family of topological spaces $\{Y_t\}_{t\in T}$ and a family $\mathcal{F} = \{f_t\}_{t\in T}$ of continuous maps where $f_t \colon X \to Y_t$ for $t \in T$. We say that the family $\mathcal{F}$ *separates points* if for every pair of distinct points $x, y \in X$ there is a map $f_t \in \mathcal{F}$ such that $f_t(x) \neq f_t(y)$. If for every point $x \in X$ and every closed set $F \subset X$ with $x \notin F$ there is a map $f_t \in \mathcal{F}$ such that $f_t(x) \notin \operatorname{cl} f_t(F)$, then we say that the family *separates points from closed sets.* Evidently if X is a T_1-space then any family which separates points from closed sets also separates points.

7.4.39. LEMMA. *If a continuous map $f \colon X \to Y$ is injective and the singleton family $\{f\}$ separates points from closed sets then f is a homeomorphic imbedding.*

PROOF. It suffices to prove that for every closed set $F \subset X$ we have $f(F) = f(X) \cap \operatorname{cl} f(F)$. If $f(x) \notin f(F)$, then $x \notin F$ so $f(x) \notin \operatorname{cl} f(F)$. This implies $f(X) \cap \operatorname{cl} f(F) \subset f(F)$. The reverse inclusion is obvious. ■

We now prove an auxiliary result concerning the diagonal from which we shall deduce an important theorem about the imbedding of completely regular spaces in the topological powers of the unit interval.

7.4.40. THEOREM (on the diagonal). *If a family of continuous maps $\mathcal{F} = \{f_t\}_{t\in T}$, where $f_t \colon X \to Y_t$ for $t \in T$, separates points, then the diagonal map $f = \triangle_{t\in T} f_t \colon X \to \bigtimes_{t\in T} Y_t$ is injective. If moreover the family $\mathcal{F}$ separates points from closed sets, then f is a homeomorphic imbedding.*

In particular, *if there is $t \in T$ for which f_t is a homeomorphic imbedding, then f is a homeomorphic imbedding.*

PROOF. If the family $\mathcal{F}$ separates points, then for every pair of distinct points $x, y \in X$ there is a map $f_t \in \mathcal{F}$ with $f_t(x) \neq f_t(y)$, whence $f(x) \neq f(y)$ which means that the diagonal is injective.

If the family $\mathcal{F}$ separates points from closed sets, then the singleton family $\{f\}$ also has the property, since the condition $f(x) \in \operatorname{cl} f(F)$, where $F = \operatorname{cl} F \subset X$, implies

that

$$f_t(x) = p_t f(x) \in p_t(\operatorname{cl} f(F)) \subset \operatorname{cl} p_t f(F) = \operatorname{cl} f_t(F)$$

for each $t \in T$, thus $x \in F$. The proof is completed by appealing to Lemma 7.4.39. ■

7.4.41. COROLLARY. *For every cardinal number* $\mathbf{m}$ *the diagonal of the topological power* $X^{\mathbf{m}}$ *is homeomorphic to the space* X. ■

We call a topological space X a *universal space for spaces with topological property* P if the space X has property P and every space possessing property P is imbeddable in the space X.

Theorems on the existence of universal spaces are important and interesting. In principle they permit us to reduce the study of a class of spaces having some topological property to the study of subspaces of a fixed space; this is because, as explained in Section 7.2, two homeomorphic spaces may, from the topological point of view, be regarded as the same object.

The topological power $I^{\mathbf{m}}$ of the closed unit interval I where $\mathbf{m} \geq \aleph_0$, is called the *Tikhonov cube of weight* $\mathbf{m}$. From Theorem 7.4.23 and Example 6.1.19 it follows that the Tikhonov cube of weight $\aleph_0$ is homeomorphic to the Hilbert cube I^ω, where the latter has the topology induced by the metric ρ defined in Example 1.2.2. The Tikhonov cube $I^{\aleph_0}$ is also referred to as the *Hilbert cube*.

7.4.42. THEOREM. *For each cardinal number* $\mathbf{m} \geq \aleph_0$ *the Tikhonov cube* $I^{\mathbf{m}}$ *is a universal space for the completely regular spaces of weight not exceeding* $\mathbf{m}$.

PROOF. It follows from Theorem 7.4.35 that the Tikhonov cube $I^{\mathbf{m}}$ is a completely regular space. As may easily be verified the members of the canonical base of the topological product $I^{\mathbf{m}}$ whose distinguished open sets are intervals with rational endpoints also form a base of the space $I^{\mathbf{m}}$ and so $w(I^{\mathbf{m}}) \leq \mathbf{m}$.

We now prove that every completely regular space X satisfying $w(X) \leq \mathbf{m}$ is imbeddable in the Tikhonov cube $I^{\mathbf{m}}$. Denote by $\mathcal{B}$ an arbitrary base of the space X of cardinality $\leq \mathbf{m}$ and consider the set P of all pairs $(V_1, V_2) \in \mathcal{B} \times \mathcal{B}$ for which there is a continuous function $f: X \to I$ such that

$$(*) \qquad f(V_1) \subset [0, 1/2) \quad \text{and} \quad f(X \setminus V_2) \subset \{1\}.$$

Observe first of all that for each point $x \in X$ and every neighbourhood $V_2 \in \mathcal{B}$ of the point x there is a set $V_1 \in \mathcal{B}$ such that $(V_1, V_2) \in P$. Indeed, since X is a Tikhonov space there is a continuous function $f: X \to I$ such that $f(x) = 0$ and $f(X \setminus V_2) \subset \{1\}$. The set $f^{-1}([0, 1/2))$ is a neighbourhood of the point x, so there exists a set $V_1 \in \mathcal{B}$ such that $x \in V_1 \subset f^{-1}([0, 1/2)) \subset V_2$. Evidently the pair (V_1, V_2) belongs to P.

To each pair $(V_1, V_2) \in P$ assign a continuous function $f: X \to I$ satisfying the condition $(*)$ and denote by $\mathcal{F}$ the family of functions so obtained. As $\operatorname{card} P \leq \mathbf{m}$, we have $\operatorname{card} \mathcal{F} \leq \mathbf{m}$. By Theorem 7.4.40 it suffices for the proof to show that the family $\mathcal{F}$ separates points from closed sets. Consider any point $x \in X$ and a closed set $F \subset X$ with $x \notin F$. From the definition of a base it follows that there is a set $V_2 \in \mathcal{B}$ such that $x \in V_2 \subset X \setminus F$ and by the earlier observation there exists a set $V_1 \in \mathcal{B}$ such that $x \in V_1$

and $(V_1, V_2) \in \mathcal{P}$. In view of $(*)$ the function $f \in \mathcal{F}$ assigned to the pair (V_1, V_2) satisfies

$$\operatorname{cl} f(F) \subset \operatorname{cl} f(X \setminus V_2) \subset \{1\} \quad \text{and} \quad f(x) < \frac{1}{2}$$

and so $f(x) \notin \operatorname{cl} f(F)$ which means that $\mathcal{F}$ separates points from closed sets. ■

7.4.43. COROLLARY. *For every cardinal number* $\mathbf{m} \geq \aleph_0$ *we have* $w(I^{\mathbf{m}}) = \mathbf{m}$.

PROOF. By the last theorem the discrete space $D(\mathbf{m})$ is imbeddable in the Tikhonov cube $I^{\mathbf{m}}$ and since $w(D(\mathbf{m})) = \mathbf{m}$, we have $w(I^{\mathbf{m}}) = \mathbf{m}$. ■

The last corollary justifies the name "Tikhonov cube of weight $\mathbf{m}$".

Theorems 7.3.9, 7.3.11 and 7.4.42 and the metrizability of the Hilbert cube $I^{\aleph_0}$ imply the following.

7.4.44. THEOREM. *A topological space satisfying the second axiom of countability is metrizable if and only if it is regular.* ■

We complete this section with a brief discussion of the quotient space operation. Suppose given a topological space X and an equivalence relation R on the set X. The collection of equivalence classes of the relation R forms a partition of the set X into pairwise disjoint sets; we denote this collection by X/R. The map $q: X \to X/R$ taking each point $x \in X$ to the equivalence class $[x] \in X/R$ containing the point x, is known as the *quotient map*. It is not difficult to verify that the family $\mathcal{O}$ of subsets U of X/R for which $q^{-1}(U)$ is open in X satisfies the axioms (T1)–(T3), we may therefore use them to endow X/R with a topology. The topological space thus obtained is called the *quotient space* of the space X and its topology is called the *quotient topology*. It is obvious from the definition of the topology that the quotient map $q: X \to X/R$ is continuous.

7.4.45. ASSERTION. *A set* $A \subset X/R$ *is closed in the quotient space* X/R *if and only if its inverse image* $q^{-1}(A)$, *where* $q: X \to X/R$ *denotes the quotient map, is closed in the space* X. ■

7.4.46. THEOREM. *A map* f *of a quotient space* X/R *into a topological space* Y *is continuous if and only if the composition* fq, *where* $q: X \to X/R$ *denotes the quotient map, is a continuous map.*

PROOF. Necessity of the condition is obvious. To prove its sufficiency it is enough to notice that if the condition is fulfilled, then for every open set $U \subset Y$ the set $(fq)^{-1}(U) = q^{-1}(f^{-1}(U))$ is open in X, so the inverse image $f^{-1}(U)$ is an open set in the quotient space X/R. ■

Every continuous map $f: X \to Y$ of a topological space X onto a topological space Y determines an equivalence relation $R(f)$ on the set X defined by the condition

$$xR(f)y \quad \text{if and only if} \quad f(x) = f(y).$$

Evidently the equivalence classes of the relation $R(f)$ are the sets of form $f^{-1}(y)$ for $y \in Y$. The map f may be expressed in the form of a composition $\bar{f}q$, where $q: X \to X/R(f)$ is the quotient map and $\bar{f}$ is a map of the quotient space $X/R(f)$ onto the space Y defined by the formula $\bar{f}(f^{-1}(y)) = y$ for $y \in Y$; by Theorem 7.4.46 the map $\bar{f}$ is continuous.

The map $\bar{f}$ is bijective but is not in general a homeomorphism. If, for instance, f is a bijective map from the discrete space $X = D(\mathbf{c})$ onto the unit interval I, the quotient space $X/R(f)$ is discrete and so $\bar{f}$ is not a homeomorphism. We now show that if the map f is open or is closed, then $\bar{f}$ is a homeomorphism.

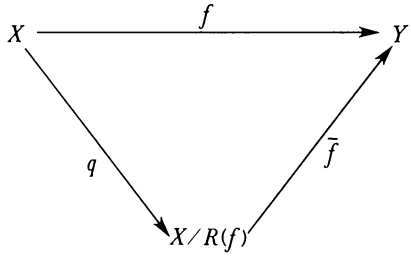


Fig.159. Every continuous map from a space X onto a space Y determines an equivalence relation $R(f)$ on the set X and a bijective map $\bar{f}$ of the quotient space $X/R(f)$ onto the space Y.

7.4.47. THEOREM. *If a map f of a topological space X onto a topological space Y is open, or is closed, then the map $\bar{f}: X/R(f) \to Y$ is a homeomorphism.*

PROOF. By Assertion 7.2.11 it is enough to prove that for every open set $U \subset X/R(f)$ the image $\bar{f}(U)$ is open in Y, or that for every closed set $F \subset X/R(f)$ the image $\bar{f}(F)$ is closed in Y. We observe that for every set $A \subset X/R(f)$ we have the equation $\bar{f}(A) = fq^{-1}(A)$; since q is a continuous map, when f is an open map the set $\bar{f}(U)$ is open, whereas when f is a closed map the set $\bar{f}(F)$ is closed. ■

We shall call an equivalence relation R on a space X an *open (closed)* relation if the quotient map $q: X \to X/R$ is an open (closed) map. The equation $q(A) = \bar{f}^{-1}f(A)$, which holds for every $A \subset X$, implies that if f is an open (closed) map, the relation $R(f)$ is open (closed).

7.4.48. ASSERTION. *For any equivalence relation R on a topological space X the following conditions are equivalent:*

(1) *the relation R is open (closed),*

(2) *for every open (closed) set $A \subset X$ the union of the equivalence classes of the relation R meeting A is open (closed) in X,*

(3) *for every closed (open) set $A \subset X$ the union of the equivalence classes of the relation R which are contained in A is closed (open) in X.* ■

It is well known that there is a bijective correspondence between equivalence relations on a set X and the partitions of the set into disjoint subsets. Sometimes the study of partitions or *decompositions*, as they are usually called in topology, is more convenient than the study of relations. Decompositions of a topological space corresponding to open (closed) relations are called *lower semicontinuous* (*upper semicontinuous*); when working with decompositions, especially upper semicontinuous ones, the term *pasting* is often used and one says that the quotient space X/R, where R is the equivalence relation corresponding to a decomposition $\mathcal{R}$, arises by pasting each member of $\mathcal{R}$ into one point.

7.4.49. EXAMPLE. Consider a space X and a finite family of pairwise disjoint closed subsets of X, say $A_1, A_2, \ldots, A_m$. Let R be the equivalence relation corresponding to the decomposition of the set X into the sets $A_1, A_2, \ldots, A_m$ and the singletons formed from the points of the complement $X \setminus \bigcup_{i=1}^{m} A_i$. For every closed set $A \subset X$ the union of the equivalence classes of R meeting A is $A \cup \bigcup\{A_i : A_i \cap A \neq \emptyset \text{ and } 1 \leq i \leq m\}$, and so is closed in A. The decomposition here is thus upper semicontinuous and the quotient map $q: X \to X/R$ is closed. The quotient space X/R may be obtained by pasting together the points of each set A_i into a point $q(A_i)$. It is easy to check that the spaces $X \setminus \bigcup_{i=1}^{m} A_i$ and $X/R \setminus \{q(A_1), q(A_2), \ldots, q(A_m)\}$ are homeomorphic. The quotient space obtained by pasting together into a point all the points of a closed set A of a space X is denoted by X/A. ■

7.4.50. EXAMPLE. *The topological cone over a space.* Let X be any topological space. Consider the topological product $X \times I$ and the closed subset $A = X \times \{1\}$. The quotient space $C_t(X) = (X \times I)/A$ is known as the *topological cone over the space* X. If X is a metric space with $\operatorname{diam} X \leq 1$ then the metric cone $C_m(X)$ defined in Example 3.2.13 is set-wise identical with the cone $C_t(X)$ over the topological space X; however the topology induced by the metric of the space $C_m(X)$ is in general distinct from the topology of the cone $C_t(X)$ which itself need not be metrizable (cf. Example 7.5.23). It is readily seen though that the identity map $f: C_t(X) \to C_m(X)$ is continuous.

By way of illustration we show that the cone $C_t(D(\aleph_0))$ over the discrete space of cardinality $\aleph_0$ does not satisfy the first axiom of countability. We may as well assume that $D(\aleph_0)$ is the set $\mathbf{N}$ of natural numbers with the discrete topology. Let $q: \mathbf{N} \times I \to C_t(\mathbf{N})$ be the quotient map corresponding to the pasting together into one point of the set $A = \mathbf{N} \times \{1\} \subset \mathbf{N} \times I$ and say $q(A) = \{v\}$. Consider an arbitrary sequence $V_1, V_2, \ldots$ of neighbourhoods of the point v in the space $C_t(\mathbf{N})$. For each natural number n we may choose a point $x_n \in V_n \setminus \{v\}$ such that $q^{-1}(x_n) = (n, r_n)$. The set $F = \bigcup_{n=1}^{\infty} \{(n, r_n)\}$ is closed in $\mathbf{N} \times I$ so its complement U is an open set. Since $r_n < 1$ for $n = 1, 2, \ldots$ we have $q^{-1}q(U) = U$; the set $V = q(U)$ is thus a neighbourhood of the point v in the space $C_t(\mathbf{N})$. It is easily seen that $x_n \notin V$ for $n = 1, 2, \ldots$ so the set V does not contain any member of the sequence $V_1, V_2, \ldots$ It follows that the space $C_t(\mathbf{N})$ does not have a countable local base at the point v. ■

7.4.51. EXAMPLE. Using the notion of a quotient space we may precisely define the model of the projective space described in Example 1.5.13. The continuous function defined there from the closed unit sphere $\bar{B}^m \subset \mathbf{R}^m$ onto the space P^m which we denote by f is readily seen to be closed, so by Theorem 7.4.47 the space P^m is homeomorphic to the space $\bar{B}^m/R(f)$. Using the notion of pasting we can say that the projective space P^m may be obtained from the ball $\bar{B}^m$ by pasting together into one point every pair of antipodal points lying on the boundary of the ball. ■

The next example is much more general (cf. Examples 5.3.6 and 5.3.9).

7.4.52. EXAMPLE. In Section 5.3 we defined the operation of pasting simplicial complexes. This led from a simplicial complex K, on which was defined an equivalence relation R, to a simplicial complex K/R; in parallel with it was defined a simplicial

map σ^0 of the vertices of the complex K into K/R and a corresponding simplicial map $\sigma: K \to K/R$ of the complex K onto the complex K/R. According to Theorem 2.5.5 the map σ determines a continuous map $|\sigma|: |K| \to |K/R|$ of the underlying space of the complex K into the underlying space of the complex K pasted according to R.

The operation of pasting simplicial complexes is intimately connected with the quotient space operation. In fact the polyhedron $|K/R|$ is homeomorphic with a quotient space of the polyhedron $|K|$. Indeed, since the polyhedra are compact (Corollary 2.3.4), the map $|\sigma|: |K| \to |K/R|$ is closed in view of Theorems 1.8.2, 1.8.3 and 1.8.4 (cf. Theorem 7.5.11), and so by Theorem 7.4.47 it follows that the polyhedron $|K/R|$ is homeomorphic with the quotient space $|K|/R(|\sigma|)$. The equivalence relation $R(|\sigma|)$ on the polyhedron $|K|$ may be described by means of the simplicial relation R. It is readily verified that if p and q are arbitrary points of the polyhedron $|K|$, and $r^0, r^1, \ldots, r^k$ and $s^0, s^1, \ldots, s^k$ are their barycentric coordinates in the polyhedron $|K|$, relative to the triangulation K with vertices $a_0, a_1, \ldots, a_k$, then $pR(|\sigma|)q$ if and only if, for each vertex a of the complex K, the following two sums are equal: the sum over all r^i for which $a_i R a$ holds, and the sum over all s^i for which $a_i R a$ holds. Evidently the relation $R(|\sigma|)$ restricted to the set $\{a_0, a_1, \ldots, a_k\}$ is identical with the relation R restricted to the same set. ■

Exercises

a) Check that the interior and boundary of a set A in a subspace X_0 of a topological space X may be expressed by the respective formulae:

$$\operatorname{int}_{X_0} A = X_0 \setminus \operatorname{cl}(X_0 \setminus A) \quad \text{and} \quad \operatorname{bd}_{X_0} A = X_0 \cap \operatorname{cl} A \cap \operatorname{cl}(X_0 \setminus A),$$

where the symbol cl refers to closure in the space X. Deduce that for every set $B \subset X$ the boundary of the intersection $X_0 \cap B$ calculated in the subspace X_0 is contained in the intersection of X_0 and the boundary of the set B calculated in the space X. Observe that for the interior this is not the case in general.

b) Let A_1 and A_2 be closed subsets of a topological space X and let $A_1 \cup A_2 = X$. Show that if the continuous maps $f_1: A_1 \to Y$ and $f_2: A_2 \to Y$ satisfy the condition $f_1 | A_1 \cap A_2 = f_2 | A_1 \cap A_2$, then the map $f: X \to Y$ defined by the formula

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in A_1, \\ f_2(x), & \text{if } x \in A_2, \end{cases}$$

is continuous.

c) Check that if $x_1, x_2, \ldots$ is a sequence of points in a Hausdorff space X and x_0 is its limit then the subspace $X_0 = \{x_0, x_1, x_2, \ldots\}$ of the space X either is finite, or is homeomorphic to the subspace $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ of the real line $\mathbf{R}$. Show that this need not be the case if X is only a T_1-space.

d) Show that every subspace of the Sorgenfrey line is separable.

e) Check that the interior and boundary of a set of the form $A \times B$ in the topological product $X \times Y$ may be expressed by the respective formulae:

$$\operatorname{int}(A \times B) = \operatorname{int} A \times \operatorname{int} B \quad \text{and} \quad \operatorname{bd}(A \times B) = (\operatorname{bd} A \times \operatorname{cl} B) \cup (\operatorname{cl} A \times \operatorname{bd} B).$$

f) Prove that if for each $t \in T$ the space X_t satisfies $w(X_t) > 1$ and $\operatorname{card} T > \aleph_0$ then the topological product $\bigtimes_{t\in T} X_t$ does not satisfy the first axiom of countability.

g) Show that a space X is a Hausdorff space if and only if the diagonal Δ of the topological power X^2 is a closed set.

Check that the diagonal Δ of the topological power X^2 is an open set if and only if the space X is discrete.

h) Let $f: X \to Y$ be a continuous map. Show that the graph $G(f)$ is the image of the space X under the homeomorphic imbedding $\mathrm{id}_X \triangle f: X \to X \times Y$ and that the restriction $p|G(f)$ of the projection $p: X \times Y \to X$ is a homeomorphism. Check that if Y is a Hausdorff space, the graph $G(f)$ is a closed set in the topological product $X \times Y$ (cf. Exercise (c) of Section 7.5).

i) Deduce from the Tietze-Urysohn Theorem that if A is a closed subset of a normal space X then for every continuous map $f: A \to I^{\mathbf{m}}$, where $\mathbf{m} \geq \aleph_0$, there exists a continuous extension $f^*: X \to I^{\mathbf{m}}$.

j) Suppose given two families of topological spaces $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ and a family of continuous maps $\{f_t\}_{t\in T}$ where $f_t: X_t \to Y_t$ for $t \in T$. Verify that the Cartesian product of the maps $\{f_t\}_{t\in T}$, that is the map $f = \bigtimes_{t\in T} f_t$ defined by the formula $f(\{x_t\}) = \{f_t(x_t)\}$ is a continuous map of the topological product $\bigtimes_{t\in T} X_t$ into the topological product $\bigtimes_{t\in T} Y_t$. Observe that for every family of maps $\{f_t\}_{t\in T}$, where $f_t: X \to Y_t$ for $t \in T$, the diagonal $\triangle_{t\in T} f_t$ is the composition of the diagonal map $d: X \to \bigtimes_{t\in T} X_t$, where $X_t = X$ for each $t \in T$, and the Cartesian product $\bigtimes_{t\in T} f_t: \bigtimes_{t\in T} X_t \to \bigtimes_{t\in T} Y_t$.

k) Let A be a closed subset of a topological space X and R an upper semicontinuous decomposition of the subspace A. Show that the decomposition of the space X into the members of R and singletons corresponding to the points of the complement $X \setminus A$ is upper semicontinuous.

l) Give an example of a closed map of the Niemytzki plane onto a T_1-space that is not a Hausdorff space. (Hint: Use Example 7.4.49.)

m) Give an example of a closed map of a subspace of the plane $\mathbf{R}^2$ onto a space which does not satisfy the first axiom of countability. (Hint: Use Example 7.4.50.)

7.5. Compact spaces and compactifications

Let X be a topological space and let $\{U_t\}_{t\in T}$ be a covering of the set X. If all the sets U_t are open in X then the covering $\{U_t\}_{t\in T}$ is said to be an *open* covering of the space X.

A topological space X is said to be *compact* if X is a Hausdorff space and every open covering of the space contains a finite covering, that is, for every family of open sets $\{U_t\}_{t\in T}$ such that $X = \bigcup_{t\in T} U_t$ there is a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $X = \bigcup_{i=1}^{m} U_{t_i}$.

We note that if the topology of the space X is induced by a metric ρ then the notion we have defined coincides with the notion of compactness considered in Section 1.8. This is a consequence of the Borel-Lebesgue Theorem (1.8.12) and of the fact that,

if in a metric space (X, ρ) there exists a sequence $\{x_n\}$ which contains no convergent subsequence, then every point $x \in X$ has a neighbourhood U_x containing only finitely many terms of the sequence, as then the open covering $\{U_x\}_{x \in X}$ of the space X cannot contain a finite covering.

7.5.1. THEOREM. *A Hausdorff space is compact if and only if every covering of the space X whose members belong to a fixed base $\mathcal{B}$ of the space contains a finite covering.*

PROOF. Clearly it is enough to prove sufficiency of the condition. Consider an arbitrary open covering $\{U_t\}_{t \in T}$ of the space X. For each point $x \in X$ choose an index $t(x) \in T$ such that $x \in U_{t(x)}$. Since $\mathcal{B}$ is a base of the space X there exists a set $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_{t(x)}$. By hypothesis the covering $\{B_x\}_{x \in X}$ of the space X contains a finite covering $\{B_{x_i}\}_{i=1}^m$. But then the family $\{U_{t(x_i)}\}_{i=1}^m$ is a finite covering of the space X contained in the covering $\{U_t\}_{t \in T}$. ■

A non-empty family $\{F_t\}_{t \in T}$ of subsets of a topological space X is said to be *centred* if $\bigcap_{i=1}^m F_{t_i} \neq \emptyset$ for every finite sequence of indices $t_1, t_2, \ldots, t_m \in T$. Alternatively one says that the family has the *finite intersection property.*

7.5.2. THEOREM. *A Hausdorff space X is compact if and only if every centred family $\{F_t\}_{t \in T}$ of closed subsets of the space X has non-empty intersection.*

PROOF. Suppose the space is compact and consider a family $\{F_t\}_{t \in T}$ of closed subsets of the space X such that $\bigcap_{t \in T} F_t = \emptyset$. The family $\{U_t\}_{t \in T}$ where $U_t = X \setminus F_t$ for $t \in T$ is an open covering of the space X; indeed we have

$$\bigcup_{t \in T} U_t = \bigcup_{t \in T} (X \setminus F_t) = X \setminus \bigcap_{t \in T} F_t = X.$$

By the compactness of the space X the covering $\{U_t\}_{t \in T}$ contains a finite covering $\{U_{t_i}\}_{i=1}^m$. We thus have

$$X = \bigcup_{i=1}^m U_{t_i} = \bigcup_{i=1}^m (X \setminus F_{t_i}) = X \setminus \bigcap_{i=1}^m F_{t_i},$$

whence it follows that $\bigcap_{i=1}^m F_{t_i} = \emptyset$. So if the family $\{F_t\}_{t \in T}$ of closed sets of X is centred, $\bigcap_{t \in T} F_t \neq \emptyset$.

Consider now a Hausdorff space with the property that every centred family $\{F_t\}_{t \in T}$ of closed sets of X has non-empty intersection. Let $\{U_t\}_{t \in T}$ be any open covering of X. The family $\{F_t\}_{t \in T}$ of closed sets of X where $F_t = X \setminus U_t$ has empty intersection so cannot be centred. Hence there is a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $\bigcap_{i=1}^m F_{t_i} = \emptyset$. The family $\{U_{t_i}\}_{i=1}^m$ is evidently a finite covering of the space X contained in the covering $\{U_t\}_{t \in T}$, so the space X is compact. ■

Theorem 7.5.2 implies the following.

7.5.3. THEOREM. *If X is a compact space and A is a closed subset of X, then the subspace A is compact.* ■

We now prove some theorems about compact subspaces of topological spaces.

We begin by asserting a simple fact which is immediate from the definition of a subspace.

7.5.4. ASSERTION. *If a subspace A of a space X is compact then every family $\{U_t\}_{t\in T}$ of open subsets of X with the property that $A \subset \bigcup_{t\in T} U_t$ contains a finite family $\{U_{t_i}\}_{i=1}^m$ such that $A \subset \bigcup_{i=1}^m U_{t_i}$.* ■

7.5.5. THEOREM. *Let U be an open set in a topological space X and $\{F_t\}_{t\in T}$ a family of closed sets of X. If for some $t_0 \in T$ the subspace F_{t_0} of the space X is compact and $\bigcap_{t\in T} F_t \subset U$ then there is a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $\bigcap_{i=1}^m F_{t_i} \subset U$.*

PROOF. It is enough to prove the existence of a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $\bigcap_{i=1}^m (F_{t_0} \cap F_{t_i}) \subset F_{t_0} \cap U$, that is to prove the theorem for the open subset $F_{t_0} \cap U$ in the space F_{t_0} and for the family $\{F_{t_0} \cap F_t\}_{t\in T}$ of closed subsets of this space. In other words, it suffices to consider the case when the space X is compact. Under this assumption the subspace $A = X\backslash U$ of the space X is compact and contained in the union $\bigcup_{t\in T} U_t$ where $U_t = X\backslash F_t$ for $t \in T$. Appealing to Assertion 7.5.4 we obtain a finite family $\{U_{t_i}\}_{i=1}^m$ such that $A \subset \bigcup_{i=1}^m U_{t_i}$; evidently $\bigcap_{i=1}^m F_{t_i} \subset U$. ■

7.5.6. THEOREM. *If A is a compact subspace of a regular space X then for every closed set $B \subset X$ disjoint from A there are open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. In the case when the subspace B of the space X is also compact, it is enough to suppose that X is a Hausdorff space.*

PROOF. Since X is a regular space there exist for each point $x \in A$ two open sets $U_x, V_x \subset X$ such that

$$(*) \qquad x \in U_x, \quad B \subset V_x \quad \text{and} \quad U_x \cap V_x = \emptyset.$$

The family $\{U_x\}_{x\in A}$ has the property that $A \subset \bigcup_{x\in A} U_x$ and so by Assertion 7.5.4 contains a finite family $\{U_{x_i}\}_{i=1}^m$ such that $A \subset \bigcup_{i=1}^m U_{x_i}$. It is easy to check that the sets $U = \bigcup_{i=1}^m U_{x_i}$ and $V = \bigcap_{i=1}^m V_{x_i}$ fulfil the claim of the theorem.

Note that if B is a singleton, then in the proof of the first part of the theorem it is enough to suppose that X is a Hausdorff space. When X is a Hausdorff space and the subspace B of X is compact the open sets satisfying $(*)$ may be obtained by virtue of the last observation upon substitution in the first part of the theorem B for A and x for B. ■

7.5.7. THEOREM. *If a subspace A of a Hausdorff space X is compact, then A is a closed set.*

PROOF. By the second part of Theorem 7.5.6 for each point $x \in X\backslash A$ there is an open set $V_x \subset X$ such that $x \in V_x$ and $A \cap V_x = \emptyset$. The set $X\backslash A$ is thus equal to the union $\bigcup_{x\in X\backslash A} V_x$ and is open, so the set A is closed. ■

Theorem 7.5.3 and the second part of Theorem 7.5.6 imply the following.

7.5.8. THEOREM. *Every compact space is normal.* ■

We now prove four theorems about continuous maps on a compact space.

7.5.9. THEOREM. *If f is a continuous map of a compact space X onto a Hausdorff space Y, then Y is also compact.*

PROOF. Let $\{U_t\}_{t\in T}$ be any open covering of the space Y. The family $\{f^{-1}(U_t)\}_{t\in T}$ is an open covering of the space X and so there is a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $X = \bigcup_{i=1}^{m} f^{-1}(U_{t_i})$. The family $\{U_{t_i}\}_{i=1}^{m}$ is a finite covering of the space Y contained in the covering $\{U_t\}_{t\in T}$; since Y is a Hausdorff space it is compact. ■

7.5.10. THEOREM. *If f is a continuous map of a compact space X into a Hausdorff space Y then for every set $A \subset X$ the equation* $\operatorname{cl} f(A) = f(\operatorname{cl} A)$ *holds.*

PROOF. By Theorems 7.5.3, 7.5.9 and 7.5.7 it follows that the set $f(\operatorname{cl} A)$ is closed in the space Y and so $\operatorname{cl} f(A) \subset f(\operatorname{cl} A)$. The reverse inclusion follows from Theorem 7.2.3. ■

7.5.11. THEOREM. *Every continuous map of a compact space into a Hausdorff space is closed.* ■

Theorems 7.5.11 and 7.2.11 imply the following.

7.5.12. THEOREM. *Every continuous bijective map of a compact space onto a Hausdorff space is a homeomorphism.* ■

We turn now to problems concerned with topological products. We begin with an interesting characterization of compact spaces.

7.5.13. LEMMA. *Let X be a compact space and Y any topological space. If the product $X \times \{y\}$ for $y \in Y$ is contained in an open set W of the topological product $X \times Y$ then there is a neighbourhood V of the point y in the space Y such that $X \times V \subset W$.*

PROOF. For every point $x \in X$ there are open sets $U_x \subset X$ and $V_x \subset Y$ such that $(x, y) \in W_x = U_x \times V_x \subset W$. Since $X \times \{y\} \subset \bigcup_{x\in X} W_x$, there exists by Assertion 7.5.4 a finite sequence of points $x_1, x_2, \ldots, x_m \in X$ such that $X \times \{y\} \subset \bigcup_{i=1}^{m} W_{x_i}$. It may readily be checked that the set $V = \bigcap_{i=1}^{m} V_{x_i}$ fulfils the claim of the lemma. ■

7.5.14. THEOREM (Kuratowski). *A Hausdorff space X is compact if and only if for every topological space Y the projection $p\colon X \times Y \to Y$ is a closed map.*

PROOF. Let X be a compact space and Y any topological space. Consider a closed set $F \subset X \times Y$ and a point $y \in Y \setminus p(F)$. Evidently $X \times \{y\} \subset W = (X \times Y) \setminus F$, so by Lemma 7.5.13 there is a neighbourhood V of the point y in the space Y such that $(X \times V) \cap F = \emptyset$. Now $V \cap p(F) = \emptyset$, so we have, since y was arbitrary, that the set $p(F)$ is closed in Y, that is, the projection p is a closed map.

Consider now a Hausdorff space X with the property that for every topological space Y the projection $p\colon X \times Y \to Y$ is a closed map. Suppose the space X is not compact; by Theorem 7.5.2 there is then a centred family $\{F_t\}_{t\in T}$ of closed sets of the space X such that $\bigcap_{t\in T} F_t = \emptyset$. Consider the set $Y = X \cup \{y_0\}$ where $y_0 \notin X$. For each $y \in Y \setminus \{y_0\}$ take $\mathcal{B}_y = \{\{y\}\}$ and let $\mathcal{B}_{y_0}$ be the family of all sets of the form

$\{y_0\} \cup \bigcap_{i=1}^{m} F_{t_i}$ where $t_1, t_2, \ldots, t_m \in T$ is any finite sequence of indices. The system $\{\mathcal{B}_y\}_{y \in Y}$ has properties (BP1)–(BP3); we consider the topology on Y generated by the neighbourhood system $\{\mathcal{B}_y\}_{y \in Y}$ as in Theorem 7.1.26. The projection $p(F)$ of the set $F = \operatorname{cl}\{(x,x) : x \in X\} \subset X \times Y$ is closed in the space Y. Now $X \subset p(F)$, so $Y = \operatorname{cl} X \subset p(F)$ and thus $y_0 \in p(F)$, which means that there exists a point $x_0 \in X$ with $(x_0, y_0) \in F$. For every neighbourhood U of the point x_0 and each $t \in T$ we have $(U \times (\{y_0\} \cup F_t)) \cap \{(x,x) : x \in X\} \neq \emptyset$ which implies that $U \cap F_t \neq \emptyset$. But the set F_t is closed so $x_0 \in F_t$ for each $t \in T$ and so $\bigcap_{t \in T} F_t \neq \emptyset$. This contradiction proves the compactness of the space X. ■

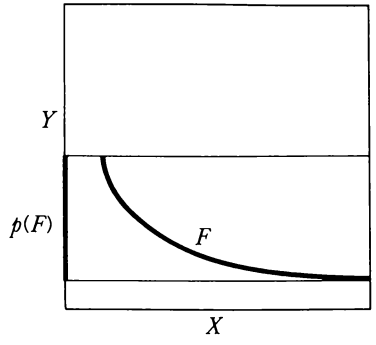


Fig.160. Projection parallel to a compact axis is a closed map (see Theorem 7.5.14).

It is not difficult to observe that the space Y constructed in the second part of the proof of Kuratowski's Theorem is normal (a slightly more complicated argument (see [4], p. 233) even permits the replacement of Y by a compact space) and so to show the compactness of a Hausdorff space X it suffices to prove that for every normal space Y (or even for every compact space Y) the projection $p\colon X \times Y \to Y$ is a closed map.

The next theorem is one of the most important results in general topology and finds application in many branches of mathematics.

7.5.15. LEMMA. *Let X be any topological space and let $\mathcal{F}_0$ a centred family of subsets of the space X. In the collection $\mathbf{A}$ of all centred families of X containing $\mathcal{F}_0$ ordered by inclusion there exists a maximal family.*

PROOF. By the Kuratowski-Zorn Theorem it suffices to prove that every linearly ordered subcollection $\mathbf{A}_0$ of the collection $\mathbf{A}$ possesses an upper bound. The upper bound is in fact the family $\mathcal{A}_0 = \bigcup \mathbf{A}_0$. Since for every $\mathcal{A} \in \mathbf{A}_0$ we have $\mathcal{A} \subset \mathcal{A}_0$, it is enough to show that $\mathcal{A}_0 \in \mathbf{A}_0$, that is, that the family $\mathcal{A}_0$ is centred. The family $\mathcal{A}_0$ is non-empty since $\mathcal{F}_0 \subset \mathcal{A}_0$ and $\mathcal{F}_0$ is a non-empty family. Consider a finite sequence $A_1, A_2, \ldots, A_m$ of members of $\mathcal{A}_0$; we show that $\bigcap_{i=1}^{m} A_i \neq \emptyset$. From the definition of $\mathcal{A}_0$ it follows that there exist families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m \in \mathbf{A}_0$ such that $A_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$. Since $\mathbf{A}_0$ is linearly ordered by inclusion, there exists a natural number $j \leq m$ such that $\mathcal{A}_i \subset \mathcal{A}_j$ for $i = 1, 2, \ldots, m$. Evidently $A_i \in \mathcal{A}_j$ for $i = 1, 2, \ldots, m$ and so $\bigcap_{i=1}^{m} A_i \neq \emptyset$, since the family $\mathcal{A}_j$ is centred. ■

7.5.16. THEOREM (Tikhonov). *The topological product of any family of compact spaces is compact.*

PROOF. Consider the topological product $X = \bigtimes_{t\in T} X_t$ of compact spaces X_t. Let $\mathcal{F}_0$ be any centred family consisting of closed sets of the space X. By Lemma 7.5.15 the family $\mathcal{F}_0$ is contained in a maximal centred family $\mathcal{A}$ of subsets of the space X. For the proof that the intersection of $\mathcal{F}_0$ is non-empty, it is enough to show that there is a point $x \in \bigtimes_{t\in T} X_t$ such that $x \in \operatorname{cl} A$ for each $A \in \mathcal{A}$.

It follows from the maximality of $\mathcal{A}$ that

(*) if $A_1, A_2, \ldots, A_m \in \mathcal{A}$ then $A_1 \cap A_2 \cap \ldots \cap A_m \in \mathcal{A}$.

Suppose otherwise, then by adjoining the set $A_1 \cap A_2 \cap \ldots \cap A_m$ to $\mathcal{A}$ we should obtain a centred family of which $\mathcal{A}$ is a proper subfamily. Similarly

(**) if $A_0 \subset \bigtimes_{t\in T} X_t$ and $A_0 \cap A \neq \emptyset$ for each $A \in \mathcal{A}$, then $A_0 \in \mathcal{A}$.

Since $\mathcal{A}$ is a centred family we have for each $t \in T$ that the family $\mathcal{A}_t = \{\operatorname{cl} p_t(A) : A \in \mathcal{A}\}$, consisting of closed subsets of the space X_t, is also a centred family. The compactness of the space X_t implies that there exists a point $x_t \in \bigcap_{A\in\mathcal{A}} \operatorname{cl} p_t(A)$. For each neighbourhood U of the point $x_t \in X_t$ we thus have $U \cap p_t(A) \neq \emptyset$, or, equivalently, $A \cap p_t^{-1}(U) \neq \emptyset$ for every $A \in \mathcal{A}$. By (**) we deduce that $p_t^{-1}(U) \in \mathcal{A}$ and so by (*) the family $\mathcal{A}$ contains all the finite intersections $\bigcap_{i=1}^m p_{t_i}^{-1}(U_i)$, where $t_i \in T$ and U_i is an open neighbourhood of the point x_{t_i} in the space X_{t_i} for $i = 1, 2, \ldots, m$, that is, all the members of the canonical base of the topological product X containg the point $x = \{x_t\}$. Since $\mathcal{A}$ is a centred family, $x \in \operatorname{cl} A$ for each $A \in \mathcal{A}$.

We have thus shown that every centred family of closed sets in X has non-empty intersection. To complete the proof it is enough to appeal to Theorems 7.4.35 and 7.5.2. ■

Theorems 7.5.8, 7.5.16 and 7.4.42 have two immediate consequences.

7.5.17. THEOREM. *For each cardinal number* $\mathfrak{m} \geq \aleph_0$ *the Tikhonov cube* $I^{\mathfrak{m}}$ *is a universal space for the compact spaces of weight not exceeding* $\mathfrak{m}$. ■

7.5.18. THEOREM. *A topological space is completely regular if and only if it is imbeddable in a compact space.* ■

Theorems 7.5.17, 6.3.4 and 7.1.23 yield the following result.

7.5.19. THEOREM. *A compact space is metrizable if and only if it satisfies the second axiom of countability.* ■

We briefly discuss quotient spaces of compact spaces.

7.5.20. THEOREM (Alexandrov). *For every closed equivalence relation R on a compact space X, there is, up to homeomorphism, exactly one Hausdorff space Y and one continuous map $f: X \to Y$ of the space X onto the space Y such that $R = R(f)$ - namely, the quotient space X/R and the quotient map $q: X \to X/R$. Moreover Y is a compact space.*

Conversely, *for every continuous map $f: X \to Y$ of a compact space X onto a Hausdorff space Y the equivalence relation $R(f)$ on the space X is closed.*

PROOF. If R is a closed equivalence relation on a compact space X the quotient map $q: X \to X/R$ is closed, so that by Theorems 7.5.8 and 7.3.14 the quotient space X/R is a Hausdorff space; furthermore by Theorem 7.5.9 the space X/R is compact. Now consider any Hausdorff space Y for which there is a continuous map $f: X \to Y$ of the space X onto the space Y such that $R(f) = R$. Since by Theorem 7.5.11 the map f is closed, it follows from Theorem 7.4.47 that the map $\bar{f}: X/R(f) \to Y$ is a homeomorphism, that is, the space Y is homeomorphic to the quotient space X/R.

The second part of the theorem follows from Theorem 7.5.11. ■

7.5.21. THEOREM. *For every closed equivalence relation R on a compact space X the inequality $w(X/R) \leq w(X)$ holds.*

PROOF. By Theorem 7.5.20 it is enough to prove that for every continuous map $f: X \to Y$ of a compact space X onto a Hausdorff space X the inequality $w(Y) \leq w(X)$ holds. Let $w(X) = \mathbf{m}$; clearly we may assume that $\mathbf{m} \geq \aleph_0$. Consider a base $\{U_t\}_{t \in T}$ of the space X such that $\operatorname{card} T = \mathbf{m}$ and denote by $\mathcal{S}$ the family of all finite subsets of the set T. Since $\operatorname{card} \mathcal{S} = \mathbf{m}$ it is enough to show that the family of open sets $\{W_S\}_{S \in \mathcal{S}}$ where

$$W_S = Y \setminus f(X \setminus \bigcup_{t \in S} U_t)$$

is a base of the space Y. Consider an arbitrary point $y \in Y$ and a neighbourhood $W \subset Y$ of the point y. The inverse image $f^{-1}(y)$ is a compact subspace of the space X contained in the open set $f^{-1}(W)$. Appealing to Assertion 7.5.4 we may easily find a finite set $S \in \mathcal{S}$ for which

$$f^{-1}(y) \subset \bigcup_{t \in S} U_t \subset f^{-1}(W).$$

Evidently $y \in W_S$ and moreover

$$Y \setminus W = f(X \setminus f^{-1}(W)) \subset f(X \setminus \bigcup_{t \in S} U_t) = Y \setminus W_S,$$

that is $W_S \subset W$. The family $\{W_S\}_{S \in \mathcal{S}}$ is thus a base of the space Y. ■

From Theorems 7.5.20, 7.5.21 and 7.5.19 we obtain the following.

7.5.22. THEOREM. *For every closed equivalence relation R on a compact metrizable space X the quotient space X/R is metrizable.* ■

7.5.23. EXAMPLE. It follows from the theorems of Tikhonov and Alexandrov that if the space X is compact, the cone $C_t(X)$ over the space X is a compact space (see Example 7.4.50). If X is a compact metrizable space with $\operatorname{diam} X \leq 1$, then by Theorem 7.5.12 the cone $C_t(X)$ over the topological space X is homeomorphic to the metric cone $C_m(X)$ over the metric space X since the identity map $f: C_t(X) \to C_m(X)$ is continuous; in particular $C_t(X)$ is a metrizable space (the latter also follows from Theorem 7.5.22). ■

We now give two examples of non-metrizable compact spaces.

7.5.24. EXAMPLE. We show that the space $A(\mathbf{m})$ (see Example 7.2.14) is compact for every $\mathbf{m} \geq \aleph_0$. Let x_0 be the unique accumulation point of the space $A(\mathbf{m})$. Consider

any open covering $\{U_t\}_{t\in T}$ of the space $A(\mathbf{m})$ and choose an index $t_0 \in T$ such that $x_0 \in U_{t_0}$. In view of the definition of the topology of the space $A(\mathbf{m})$ the set $A(\mathbf{m})\backslash U_{t_0}$ is finite. Thus there is a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $A(\mathbf{m})\backslash U_{t_0} \subset \bigcup_{i=1}^m U_{t_i}$. The family $\{U_{t_i}\}_{i=0}^m$ is a finite covering of the space $A(\mathbf{m})$ contained in the covering $\{U_t\}_{t\in T}$, so $A(\mathbf{m})$ is compact. As we know (see Example 7.1.27), for $\mathbf{m} > \aleph_0$ the space $A(\mathbf{m})$ is not metrizable. ■

7.5.25. EXAMPLE. *The Alexandrov double circumference.* Consider the two concentric circles in the plane $\mathbf{R}^2$ given by $C_i = \{(x^1, x^2) \in \mathbf{R}^2 : (x^1)^2 + (x^2)^2 = i\}$ for $i = 1, 2$, and their union $X = C_1 \cup C_2$. Let p denote the central projection of the set C_1 onto the set C_2 from the origin $(0,0)$. For each $z \in C_2$ take $\mathcal{B}_z = \{\{z\}\}$ and for each $z \in C_1$ let $\mathcal{B}_z = \{U_n(z) : n = 1, 2, \ldots\}$, where $U_n(z) = V_n(z) \cup p(V_n(z)\backslash\{z\})$ and $V_n(z)$ is the arc of C_1 centred at z and of length $1/n$. It may easily be checked that the collection $\{\mathcal{B}_z\}_{z\in X}$ of subsets of X has properties (BP1)–(BP4) and so by Assertion 7.3.3 the set X with topology generated by the neighbourhood system $\{\mathcal{B}_z\}_{z\in X}$ is a Hausdorff space; evidently the space X satisfies the first axiom of countability. We show that X is a compact space.

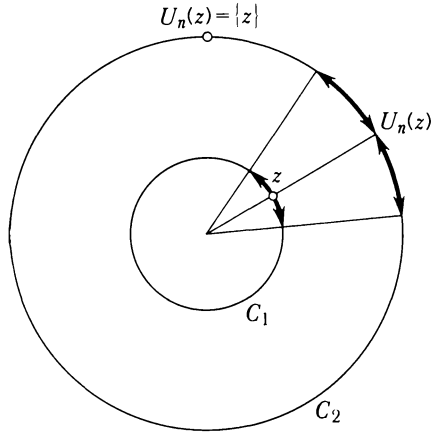


Fig.161. The sets $U_n(z)$ from a local base of the point z in the Alexandrov double circumference.

Note first of all that the subspace C_1 of the space X is just S^1 with its usual topology so is a compact space. Consider an arbitrary open covering $\{U_t\}_{t\in T}$ of the space X using members from the base $\mathcal{B} = \bigcup_{z\in X} \mathcal{B}_z$. By Assertion 7.5.4 the family $\{U_t\}_{t\in T}$ contains a finite family $\{U_{t_i}\}_{i=1}^m$ such that

$$C_1 \subset U_{t_1} \cup U_{t_2} \cup \ldots \cup U_{t_m}.$$

We may remove those of the sets U_{t_i} which are singletons without upsetting the last inclusion, that is we may assume that $U_{t_i} = U_{n_i}(z_i)$ for $i = 1, 2, \ldots, m$. It is easy to see that

$$X\backslash\{p(z_1), \ldots, p(z_m)\} \subset U_{t_1} \cup U_{t_2} \cup \ldots \cup U_{t_m}.$$

Adjoining to the family $\{U_{t_i}\}_{i=1}^m$ arbitrary members $U_{t_{m+1}}, U_{t_{m+2}}, \ldots, U_{t_{2m}}$ of the given covering so as to cover respectively the points $p(z_1), p(z_2), \ldots, p(z_m)$ we obtain a

finite covering of the space X contained in $\{U_t\}_{t\in T}$. Hence the space X is compact by Theorem 7.5.1.

The subspace C_2 of the space X is a discrete space of cardinality $\mathfrak{c}$ so X does not satisfy the second axiom of countability and by Theorem 7.5.19 is not therefore metrizable. The space is known as the *Alexandrov double circumference.* ■

Localization of the notion of compactness leads to the class of locally compact spaces. We call a topological space X *locally compact* if every point $x \in X$ has a neighbourhood U whose closure $\operatorname{cl} U$ is a compact subspace of the space X. Of course every compact space is locally compact. An example of a locally compact space which is not compact is provided by any infinite discrete space.

7.5.26. THEOREM. *Every locally compact space is completely regular.*

PROOF. Let X be a locally compact space and let x be any point of the space X. Consider a neighbourhood U of x whose closure $\operatorname{cl} U$ is compact. The singleton $\{x\}$ is closed in the subspace $\operatorname{cl} U$, so by Theorem 7.4.2 is also closed in the whole space X. From Theorem 7.3.1 it follows that X is a T_1-space. Consider now an arbitrary closed set $F \subset X$ such that $x \notin F$. The set $F_0 = (\operatorname{cl} U \setminus U) \cup (F \cap \operatorname{cl} U)$ is a closed subset of the subspace $\operatorname{cl} U$ and does not contain x, so by Theorem 7.5.8 there exists a continuous function $f_0 : \operatorname{cl} U \to I$ such that $f_0(x) = 0$ and $f_0(F_0) \subset \{1\}$. It is easily checked that the function $f : X \to I$ defined by the formula

$$f(y) = \begin{cases} f_0(y), & \text{if } y \in \operatorname{cl} U, \\ 1, & \text{otherwise,} \end{cases}$$

is continuous (cf. Exercise (b) of Section 7.4). Since $f(x) = 0$ and $f(F) \subset \{1\}$, the space X is completely regular. ■

7.5.27. THEOREM. *If X is a locally compact space and A is an open, or a closed subset of X, then the subspace A is locally compact.*

PROOF. If the set A is open, then by Theorem 7.5.26 for each point $x \in A$ there is a neighbourhood V of the point x in the space X such that $\operatorname{cl} V \subset A$. The intersection $W = V \cap U$ of this neighbourhood with a neighbourhood U whose closure $\operatorname{cl} U$ is a compact subspace of the space X is a neighbourhood of the point x in the subspace A. Since $W \subset \operatorname{cl} W \subset \operatorname{cl} V \subset A$ the closure of the set W in the subspace A is identical with the closure $\operatorname{cl} W$ of the set in the space X. It follows from the inclusion $\operatorname{cl} W \subset \operatorname{cl} U$ and Theorem 7.5.3 that this closure is a compact subspace of the space A.

If A is a closed set, then for every point $x \in A$ the set $W = A \cap U$, where U is a neighbourhood of the point x such that $\operatorname{cl} U$ is a compact subspace of the space X, is a neighbourhood of the point x in the space A. The closure of the set W in the subspace A is the set $A \cap \operatorname{cl} W$. Since $\operatorname{cl} W \subset \operatorname{cl} U$, this closure is a compact subspace of the space A. ■

7.5.28. COROLLARY. *If X is a locally compact space and A is the intersection of an open set of X with a closed set of X, then the subspace A is locally compact.* ■

The subspace of the real line consisting of the rational numbers is not locally compact, so local compactness is not passed down to arbitrary subspaces (cf. Exercise (f)).

We note that it follows from Theorem 7.5.27 that the space Z_0 of Example 7.4.38 is locally compact. Thus there exist locally compact spaces which are not normal.

7.5.29. THEOREM. *The topological product of any family of locally compact spaces, of which all but a finite number are compact, is a locally compact space.*

PROOF. By Theorems 7.4.31 and 7.5.16 it is enough to show that the product of a finite number of locally compact spaces is locally compact. Consider the locally compact spaces $X_1, X_2, \ldots, X_m$ and an arbitrary point $x = (x_1, x_2, \ldots, x_m)$ in the topological product $X = X_1 \times X_2 \times \ldots \times X_m$. For $i = 1, 2, \ldots, m$ the point x_i has a neighbourhood $U_i \subset X_i$ whose closure $\operatorname{cl} U_i$ is a compact subspace of the space X_i. The set $U = U_1 \times U_2 \times \ldots \times U_m$ is a neighbourhood of x in the space X. By Theorems 7.4.25 and 7.5.16 the closure $\operatorname{cl} U$ is a compact subspace of the compact space X. ■

The reader will have no difficulty in checking that the topological $\aleph_0$ power of the discrete space of cardinality $\aleph_0$ is not locally compact (cf. Problem 6.P.23 and Exercise (f)).

7.5.30. THEOREM. *If f is an open map of a locally compact space X onto a Hausdorff space Y then the space Y is also locally compact.*

PROOF. Consider any point $y \in Y$. Let x be any point of the set $f^{-1}(y)$ and U a neighbourhood of x such that $\operatorname{cl} U$ is a compact subspace of the space X. The set $V = f(U)$ is a neighbourhood of the point y and since by Theorem 7.5.10 $\operatorname{cl} V = f(\operatorname{cl} U)$ we have by Theorem 7.5.9 that the closure $\operatorname{cl} V$ of the neighbourhood V is a compact subspace of the space Y. ■

The analogue of the last theorem for closed maps is not true. The reader may easily check that the quotient space $\mathbf{R}/\mathbf{N}$ obtained from the real line by pasting into one point the set $\mathbf{N}$ of natural numbers is not locally compact.

We now discuss the expansion of a topological space to a compact space. A *compactification* of a space X is any pair (Y, c) where Y is a compact space and c is a homeomorphic imbedding of the space X into the space Y with the property that $\operatorname{cl} c(X) = Y$. Thus if a topological space X has a compactification, it is imbeddable in a compact space. Conversely, too, every topological space which is imbeddable in a compact space has a compactification. For, if $f: X \to Z$ is a homeomorphic imbedding of the space X into a compact space Z, then the pair (Y, c) where $Y = \operatorname{cl} f(X)$ and $c: X \to Y$ is defined by taking $c(x) = f(x)$ for $x \in X$, constitutes a compactification of the space X. Theorems 7.5.18 and 7.4.42 thus have the following two theorems as their consequence.

7.5.31. THEOREM. *A topological space X has a compactification if and only if X is completely regular.* ■

7.5.32. THEOREM. *Every completely regular space X has a compactification (Y, c) satisfying $w(Y) = w(X)$.* ■

In the sequel by a compactification of a space X we shall mean only a space Y in which X may be imbedded as a dense subspace. We shall denote a compactification of the space X by the symbol cX when $c: X \to cX$ is a homeomorphic imbedding of the space X into the compact space cX for which $\operatorname{cl} c(X) = cX$.

We shall call two compactifications c_1X and c_2X of a completely regular space X *equivalent* if there is a homeomorphism $f: c_1X \to c_2X$ such that $fc_1(x) = c_2(x)$ for each $x \in X$. Two compactifications are thus equivalent if they are homeomorphic and the space X is identically imbedded in each. It is not difficult to see that the equivalence of compactifications is an equivalence relation.

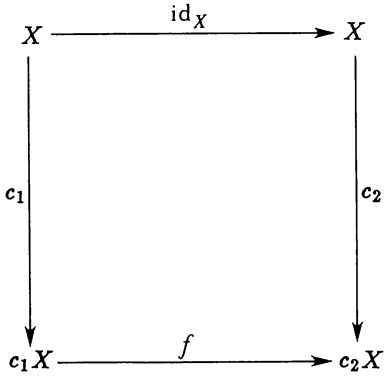


Fig.162. The compactifications c_1X and c_2X of the space X are equivalent when $fc_1 = c_2 \operatorname{id}_X$.

In the sequel we shall often identify equivalent compactifications and we shall regard the whole of an equivalence class of equivalent compactifications as though it was one compactification. This convention is indispensable if we are to study the family of all compactifications of a fixed completely regular space X. It is easy to check that if cX is a compactification of a compact space X, then c is a homeomorphism of X onto cX. It follows that every compactification of a compact space X is equivalent to the compactification (X, id_X) which is identified with the space X. Thus every compact space has exactly one compactification namely itself.

7.5.33. LEMMA. *If Y is a regular space and X an arbitrary dense subset of Y then*

$$w(Y) \leq 2^{\mathfrak{m}} \quad \textit{and} \quad \operatorname{card} Y \leq 2^{2^{\mathfrak{m}}}, \quad \textit{where} \quad \mathfrak{m} = \operatorname{card} X.$$

PROOF. Let $\mathcal{B} = \{\operatorname{int} \operatorname{cl} B : B \subset X\}$. Since by Theorem 7.1.18 for every open set $U \subset Y$ we have the equation $\operatorname{cl} U = \operatorname{cl}(U \cap X)$, from the regularity of the space Y it follows that $\mathcal{B}$ is a base of that space. Clearly $\operatorname{card} \mathcal{B} \leq 2^{\mathfrak{m}}$, so $w(Y) \leq 2^{\mathfrak{m}}$.

For the proof of the second inequality it is enough to observe that the correspondence assigning to each point of the space Y those members of the base $\mathcal{B}$ which contain the point is an injective map of the set Y into the family of all subsets of the set $\mathcal{B}$. ■

From Lemma 7.5.33 and Theorem 7.5.17 it follows that for every completely regular space X all its compactifications are, up to equivalence, subspaces of the Tikhonov cube

$I^{2^{\mathbf{m}}}$ where $\mathbf{m} = \max(\aleph_0, \operatorname{card} X)$. Thus for every completely regular space X we may validly speak of the family of all its compactifications.

Let X be any completely regular space and $\{c_t X\}_{t\in T}$ the family of all compactifications of the space X. Consider the topological product $\bigtimes_{t\in T} c_t X$ and the diagonal $c = \triangle_{t\in T} c_t : X \to \bigtimes_{t\in T} c_t X$. By Theorems 7.5.3 and 7.5.16 the subspace $\beta X = \operatorname{cl} c(X)$ of the space $\bigtimes_{t\in T} c_t X$ is compact and from Theorem 7.4.40 it follows that the map $\beta : X \to \beta X$ assigning to each point $x \in X$ the point $\beta(x) = c(x) \in \beta X$ is a homeomorphic imbedding. The space βX is thus a compactification of the space X; it is called the *Stone-Čech compactification* or the *maximal compactification* of the space X (see Supplement 7.S.24).

7.5.34. THEOREM. *For every compactification cX of a completely regular space X there exists a continuous map $c^* : \beta X \to cX$ such that $c^*\beta = c$.*

PROOF. Let $\{c_t X\}_{t\in T}$ be the family of all compactifications of the space X and let $cX = c_{t_0} X$. The map $c^* = p_{t_0} | \beta X : \beta X \to c_{t_0} X$ where $p_{t_0} : \bigtimes_{t\in T} c_t X \to c_{t_0} X$ is the projection evidently satisfies the equation $c^*\beta = c$. ■

7.5.35. THEOREM. *Let X be a completely regular space. For every compact space Z and for every continuous map $f : X \to Z$ there exists a continuous map $f^* : \beta X \to Z$ such that $f^*\beta = f$.*

PROOF. It follows from Theorem 7.4.40 that the diagonal $c = \beta \triangle f : X \to \beta X \times Z$ is a homeomorphic imbedding and so $cX = \operatorname{cl} c(X) \subset \beta X \times Z$ is a compactification of the space X. By Theorem 7.5.34 there exists therefore a continuous map $c^* : \beta X \to cX$ such that $c^*\beta = c$. Let $p : \beta X \times Z \to Z$ be the projection and $f^* = (p | cX)c^* : \beta X \to Z$. For each $x \in X$ we have $f^*\beta(x) = p(c^*(\beta(x))) = p(c(x)) = p((\beta(x), f(x))) = f(x)$ and so $f^*\beta = f$. ■

7.5.36. COROLLARY. *Let X be a completely regular space. For every continuous function $f : X \to I$ there exists a continuous function $f^* : \beta X \to I$ such that $f^*\beta = f$.* ■

The last corollary and Urysohn's Lemma (7.3.9) imply the following.

7.5.37. COROLLARY. *If X is a normal space then for every pair of disjoint closed sets $A, B \subset X$ the closures $\operatorname{cl}\beta(A)$ and $\operatorname{cl}\beta(B)$ of the sets $\beta(A)$ and $\beta(B)$ in the Stone-Čech compactification βX are disjoint.* ■

The property of the Stone-Čech compactification described in Corollary 7.5.36 characterizes this compactification (cf. Problem 7.P.37); the following in fact holds.

7.5.38. THEOREM. *If a compactification cX of a completely regular space X has the property that for every continuous function $f : X \to I$ there exists a continuous function $f^* : cX \to I$ such that $f^*c = f$, then the compactifications cX and βX are equivalent.*

PROOF. In view of Theorem 7.5.17 the Stone-Čech compactification βX may be regarded as a subspace of the Tikhonov cube $I^{\mathbf{m}} = \bigtimes_{t\in T} I_t$ where $I_t = I$ for $t \in T$ and $\operatorname{card} T = \mathbf{m} = w(\beta X)$. For each $t \in T$ consider the composition $f_t = (p_t | \beta X)\beta : X \to I_t$

where $p_t: \bigtimes_{t\in T} I_t \to I_t$ is the projection; by hypothesis there exists a continuous function $f_t^*: cX \to I_t$ such that $f_t^* c = f_t$. It is easily seen that the diagonal $f = \triangle_{t\in T} f_t^*$ satisfies the equation $fc = \beta$ and so $f(cX) = f(\mathrm{cl}\, c(X)) \subset \mathrm{cl}\, f(c(X)) = \mathrm{cl}\, \beta(X) = \beta X$, that is $f: cX \to \beta X$. Let $c^*: \beta X \to cX$ be a continuous map such that $c^*\beta = c$. Observe that for each point $\beta(x) \in \beta(X)$ we have $fc^*\beta(x) = fc(x) = \beta(x)$ so $fc^* | \beta(X) = \mathrm{id}_{\beta X} | \beta(X)$; hence by Theorem 7.4.10 it follows that $fc^* = \mathrm{id}_{\beta X}$. We similarly arrive at the conclusion that $c^* f = \mathrm{id}_{\beta X}$. Thus the map $f: cX \to \beta X$ is a homeomorphism and the compactifications cX and βX are equivalent. ■

Theorem 7.4.8, Corollary 7.5.36 and Theorem 7.5.38 imply the following.

7.5.39. COROLLARY. *If X is a normal space, then for every closed set $A \subset X$ the closure $\mathrm{cl}\, \beta(A)$ of the set $\beta(A)$ in the Stone-Čech compactification βX is a compactification of the space A equivalent to the Stone-Čech compactification of the space A.* ■

It is sometimes convenient to identify a completely regular space X with the homeomorphic subspace $\beta(X)$ of the Stone-Čech compactification βX, or with an appropriate subspace of some other compactification of X. Observe that under such an identification the map f^* studied above is then just a continuous extension of the map f from the set X over the corresponding compactification.

We might note moreover that the Stone-Čech compactification βX is in general very dissimilar to the space X, that is the difference $\beta X \setminus \beta(X)$ is, speaking imprecisely, very large. In particular the Stone-Čech compactification $\beta\mathbf{N}$ of the space of natural numbers $\mathbf{N}$, where $\mathbf{N}$ is given the discrete topology, has cardinality $2^{\mathfrak{c}}$ and weight $\mathfrak{c}$ (see Problem 7.P.39) and so is not metrizable (compare Theorem 7.5.19). Using this fact and Corollary 7.5.39 it is easily shown that if a metrizable space X is not compact then its Stone-Čech compactification βX is not metrizable.

Theorem 7.5.34 justifies the name "maximal compactification" which is sometimes used when referring to the Stone-Čech compactification. The natural question arises whether for each completely regular space X there also exists a minimal compactification, that is a compactification ωX with the property that for every compactification cX of the space X there exists a continuous map $c_*: cX \to \omega X$ such that $c_* c = \omega$. Evidently such a compactification exists for every compact space, since every compact space is its own only compactification. We show that also every locally compact space has a minimal compactification. It turns out that the existence of a minimal compactification characterizes the locally compact spaces (see Problem 7.P.38; cf. Supplement 7.S.24).

Let X be any non-compact locally compact space. We consider the set $\omega X = X \cup \{\Omega\}$, where $\Omega \notin X$. For the open sets of ωX we take all sets of the form $\{\Omega\} \cup (X \setminus F)$ where F is a compact subspace of the space X and also all the open sets of X. It is easily verified that the set ωX with the family of open sets just defined is a Hausdorff space, that the map $\omega: X \to \omega X$ defined by the formula $\omega(x) = x$ is a homeomorphic imbedding and that $\mathrm{cl}\, \omega(X) = \omega X$. We show that ωX is compact. Consider an arbitrary open covering $\{U_t\}_{t\in T}$ of the space ωX and choose an index $t_0 \in T$ such that $\Omega \in U_{t_0}$. From the definition of the topology of ωX the set $F = X \setminus U_{t_0}$ is a compact subspace

of the space X. By Assertion 7.5.4 there therefore exists a finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ such that $F \subset \bigcup_{i=1}^{m} U_{t_i}$. The family $\{U_{t_i}\}_{i=0}^{m}$ is a finite covering of the space ωX contained in the covering $\{U_t\}_{t\in T}$ hence the space ωX is compact. The space ωX is thus a compactification of the space X; it is variously called the *one-point compactification* or the *Alexandrov compactification* or the *minimal compactification* of the space X.

7.5.40. THEOREM. *For every compactification cX of a locally compact, non-compact space X there exists a continuous map $c_*: cX \to \omega X$ such that $c_* c = \omega$.*

PROOF. Observe first of all that the $c(X)$ is open in cX (cf. Exercise (f)). Indeed every point $x \in X$ has a neighbourhood U whose closure $\operatorname{cl} U$ is compact. Since the set $c(U)$ is open in $c(X)$ there exists a neighbourhood W of $c(x)$ in the space cX such that $W \cap c(X) = c(U)$. It follows from Theorem 7.1.18 that $\operatorname{cl} W = \operatorname{cl}(W \cap c(X)) = \operatorname{cl} c(U)$ and, since the set $c(\operatorname{cl} U) \subset cX$ is compact, we have $W \subset \operatorname{cl} W = \operatorname{cl} c(U) \subset c(\operatorname{cl} U) \subset c(X)$ which proves that $c(X)$ is open in cX.

To complete the proof it is enough to notice that the map $c_*: cX \to \omega X$ defined by the formula

$$c_*(x) = \begin{cases} \omega(c^{-1}(x)), & \text{if } x \in c(X), \\ \Omega, & \text{if } x \in cX \backslash c(X), \end{cases}$$

satisfies $c_* c = \omega$ and is continuous since the inverse image of every open set of ωX is open in cX either by way of being an open subset of the open subspace $c(X)$, or by way of being the complement of a compact subspace. ■

Theorems 7.5.32, 7.5.40, 7.5.20 and 7.5.21 imply the following.

7.5.41. THEOREM. *Every locally compact, non-compact space X satisfies the equation $w(\omega X) = w(X)$.* ■

Theorems 7.5.19 and 7.5.41 yield the following.

7.5.42. COROLLARY. *Let X be a locally compact, non-compact metrizable space. The compactification ωX of the space X is metrizable if and only if the space X satisfies the second axiom of countability.* ■

We now give some examples of compactifications.

7.5.43. EXAMPLE. The circle S^1 and the unit interval I are compactifications of the real line $\mathbf{R}$; the circle is the one-point compactification of the real line. Another compactification of the real line is the space X of Example 3.1.22.

The one-point compactification of the discrete space $D(\mathfrak{m})$ for $\mathfrak{m} \geq \aleph_0$ is the space $A(\mathfrak{m})$ (see Example 7.2.14) so often used in earlier examples. The Alexandrov double circumference (see Example 7.5.25) is a compactification of the discrete space $D(\mathfrak{c})$. ■

Let $C(X)$ denote the set of real-valued continuous functions defined on a topological space X. A set $R \subset C(X)$ will be called a *ring of functions*, if for every $f, g \in R$ the functions $f + g$, $f - g$ and $f \cdot g$ are also in R. It follows from the discussion of Section

7.2 that the set $C(X)$ is a ring of functions and is closed under uniform convergence, that is, the limit of every uniformly convergent sequence of functions of $C(X)$ is also in $C(X)$. Moreover, the ring $C(X)$ contains all the constant functions and, when X is completely regular, the ring $C(X)$ separates points, that is, for every pair of distinct points $x, y \in X$ there is a function $f \in C(X)$ such that $f(x) \neq f(y)$. Similar properties apply to the ring $C^*(X) \subset C(X)$ consisting of the bounded functions of $C(X)$. Of course, if X is a compact space, $C^*(X) = C(X)$. We now show that if X is a compact space then the properties above characterize the ring $C(X)$, that is, every ring R contained in $C(X)$ which has these properties is identical with the ring $C(X)$. A proof of this theorem is preceded by three lemmas. Lemma 7.5.45 is a particular case of a theorem known in analysis as the Weierstrass approximation theorem, which is concerned with the approximation by polynomials of continuous functions defined on the unit interval I. The target Theorem 7.5.47 is a far-reaching generalization of Weierstrass' Theorem.

7.5.44. LEMMA (Dini). *Suppose given a compact space X and a sequence $\{f_n\}$ of functions in $C(X)$ such that $f_n(x) \leq f_{n+1}(x)$ for each $x \in X$ and $n = 1, 2, \ldots$ If a function $f_0 \in C(X)$ has the property that $\lim_n f_n(x) = f_0(x)$ for each $x \in X$ then the sequence $\{f_n\}$ is uniformly convergent to the function f_0.*

PROOF. Let ϵ be an arbitrary positive number and let $F_n = \{x \in X : f_0(x) - f_n(x) \geq \epsilon\}$ for $n = 1, 2, \ldots$ The sets $F_1, F_2, \ldots$ are closed in the space X and form a decreasing sequence. Since $\bigcap_{n=1}^{\infty} F_n = \emptyset$, the family $\{F_n\}_{n=1}^{\infty}$ is not centred, that is, there exists an index k such that $F_k = \emptyset$. For $n \geq k$ and each $x \in X$ we thus have $0 \leq f_0(x) - f_n(x) \leq f_0(x) - f_k(x) < \epsilon$, which completes the proof of the lemma. ■

7.5.45. LEMMA. *There exists a sequence of polynomials $\{p_n\}$ uniformly convergent on the interval I to the function $\sqrt{r}$ for $0 \leq r \leq 1$.*

PROOF. Consider the sequence of polynomials $p_1, p_2, \ldots$ defined by the recurrence relation

$$(*) \qquad p_1(r) = 0, \quad p_{n+1}(r) = p_n(r) + \frac{1}{2}[r - (p_n(r))^2] \quad \text{for } r \in I \text{ and } n = 2, 3, \ldots$$

We prove now by induction that

$$(**) \qquad p_n(r) \leq \sqrt{r} \quad \text{for} \quad r \in I \quad \text{and} \quad n = 1, 2, \ldots.$$

The inequality $(**)$ is true for $n = 1$. Suppose that $p_n(r) \leq \sqrt{r}$ for $r \in I$. Now

$$\sqrt{r} - p_{n+1}(r) = \sqrt{r} - p_n(r) - \frac{1}{2}[r - (p_n(r))^2] = (\sqrt{r} - p_n(r))[1 - \frac{1}{2}(\sqrt{r} + p_n(r))],$$

and by the inductive hypotheses, since $r \leq 1$, we have

$$\sqrt{r} - p_{n+1}(r) \geq (\sqrt{r} - p_n(r))(1 - \frac{1}{2}2\sqrt{r}) \geq 0,$$

which completes the proof of inequality $(**)$. From $(*)$ and $(**)$ it follows that $p_n(r) \leq p_{n+1}(r)$. Appealing to $(**)$ we conclude that for each $r \in I$ the sequence $\{p_n(r)\}$ converges. Passing to the limit in the second equation of $(*)$ we infer that $\lim_n p_n(r) = \sqrt{r}$. To complete the proof it suffices to apply Lemma 7.5.44. ■

7.5.46. LEMMA. *Suppose given a topological space X and a ring of functions $R \subset C^*(X)$. If the ring R is closed under uniform convergence and contains all the constant functions, then for every pair of functions $f, g \in R$ the functions* $\max(f,g)$ *and* $\min(f,g)$ *belong to the ring R.*

PROOF. Since

$$\min(f,g) = \frac{1}{2}(f+g-|f-g|) \quad \text{and} \quad \max(f,g) = \frac{1}{2}(f+g+|f-g|),$$

it is enough to prove that for every function $f \in R$ the function $|f|$ belongs to R. There exists a positive number c such that $|f(x)| \leq c$ for each $x \in X$. Evidently it is enough to prove that $(1/c)|f| \in R$ so we may assume that $|f(x)| \leq 1$ for each $x \in X$. From Lemma 7.5.45 we infer that the sequence $\{p_n(f^2(x))\}$, all of whose terms belong to R, is uniformly convergent to the function $\sqrt{f^2} = |f|$ and so $|f| \in R$. ■

7.5.47. THEOREM (Weierstrass, Stone). *Suppose given a compact space X and a ring of functions $R \subset C(X)$. If the ring R is closed under uniform convergence, contains all the constant functions and separates points, then it coincides with the ring $C(X)$.*

PROOF. It is enough to show that for every function $f \in C(X)$ and every positive real number ϵ there is a function $f_\epsilon \in R$ such that $|f(x) - f_\epsilon(x)| < \epsilon$ for each $x \in X$.

For every pair of distinct points $a, b \in X$ there exists a function $h \in R$ such that $h(a) \neq h(b)$. The function $g \in C(X)$ defined by the formula

$$g(x) = \frac{h(x) - h(a)}{h(b) - h(a)} \quad \text{for } x \in X$$

belongs to R and satisfies $g(a) = 0$ and $g(b) = 1$. For the function $f_{a,b} \in R$ defined by the formula

$$f_{a,b}(x) = (f(b) - f(a))g(x) + f(a) \quad \text{for} \quad x \in X$$

we have

$$f_{a,b}(a) = f(a) \quad \text{and} \quad f_{a,b}(b) = f(b).$$

The sets

$$U_{a,b} = \{x \in X : f_{a,b}(x) < f(x) + \epsilon\} \quad \text{and} \quad V_{a,b} = \{x \in X : f_{a,b}(x) > f(x) - \epsilon\}$$

are respectively neighbourhoods of a and b. For a fixed point b the family $\{U_{a,b}\}_{a \in X}$ is an open covering of the space X and so contains a finite covering $\{U_{a_i,b}\}_{i=1}^m$. By Lemma 7.5.46 the function $f_b = \min(f_{a_1,b}, f_{a_2,b}, \ldots, f_{a_m,b})$ lies in R. Moreover $f_b(x) < f(x) + \epsilon$ for each $x \in X$ and $f_b(x) > f(x) - \epsilon$ for each $x \in V_b = \bigcap_{i=1}^m V_{a_i,b}$. The set V_b is a neighbourhood of the point b. The family $\{V_b\}_{b \in X}$ is an open covering of the space X and so contains a finite covering $\{V_{b_j}\}_{j=1}^n$. By Lemma 7.5.46 the function $f_\epsilon = \max\{f_{b_1}, f_{b_2}, \ldots, f_{b_n}\}$ lies in R. It is easily verified that $|f(x) - f_\epsilon(x)| < \epsilon$ for each point $x \in X$. ■

The significance of the Stone-Weierstrass Theorem rests, for instance, on the fact that it provides a method for uniformly approximating real-valued continuous functions on a compact space. Thus, every function $f \in C(X)$ may be approximated to any

desired accuracy by means of polynomials of several variables formed out of the members of a fixed family of continuous functions which separate points. Since for every closed interval $A = [a, b] \subset \mathbf{R}$ the inclusion map $i_A: A \to \mathbf{R}$ separates points, Theorem 7.5.47 implies the earlier mentioned classical theorem due to Weierstrass.

We close the section with a proof of Stone's representation theorem for Boolean algebras. This asserts that any Boolean algebra is isomorphic to the Boolean algebra consisting of the open-and-closed sets of some compact space. It turns out that the relevant space is *zero-dimensional*, that is, it is a T_1-space with a base consisting of open-and-closed sets. In what follows we shall assume only that the reader is familiar with the concept of a Boolean algebra and the concept of an isomorphism between Boolean algebras.

Suppose given a Boolean algebra $(A, \cup, \cap, -, \wedge, \vee)$. A proper subset Δ of A is known as an *ideal* in the Boolean algebra A if it satisfies the following conditions:

(I1) $\wedge \in \Delta$,

(I2) *if* $a, b \in \Delta$ *then* $a \cup b \in \Delta$,

(I3) *if* $a \leq b$ *and* $b \in \Delta$ *then* $a \in \Delta$.

Since $\Delta \neq A$, it follows from (I3) that $\vee \notin \Delta$. An ideal Δ in a Boolean algebra A is called *maximal* if Δ is not a proper subset of any ideal in the algebra A. It is easily verified that the union of a family of ideals in the Boolean algebra A linearly ordered under the inclusion $\subset$ is again an ideal in the algebra. Thus it follows from the Kuratowski-Zorn Theorem that every ideal in a Boolean algebra A is contained in some maximal ideal in the algebra; this ideal is not in general uniquely determined. It is not difficult also to check that an ideal Δ is maximal if and only if it satisfies the condition

(MI) *if* $a \cap b \in \Delta$ *then* $a \in \Delta$ *or* $b \in \Delta$.

7.5.48. THEOREM (Stone). *For every Boolean algebra A there exists a zero-dimensional compact space X such that the algebra A is isomorphic to the Boolean algebra consisting of the open-and-closed subsets of the space X.*

PROOF. Let X be the family of all maximal ideals in the algebra A. For each $a \in A$ put $V(a) = \{\Delta \in X : a \notin \Delta\}$. From conditions (I2) and (I3) it follows that if Δ is an ideal then $a \cup b \in \Delta$, if and only if, $a \in \Delta$ and $b \in \Delta$; on the other hand from conditions (MI) and (I3) it follows that if Δ is a maximal ideal then $a \cap b \in \Delta$ if and only if $a \in \Delta$ or $b \in \Delta$, so

$$(*) \qquad V(a \cup b) = V(a) \cup V(b) \quad \text{and} \quad V(a \cap b) = V(a) \cap V(b).$$

From the second part of $(*)$ and the equation $V(\vee) = X$ it follows that the family $\mathcal{B} = \{V(a) : a \in A\}$ has the properties (B1) and (B2) of Theorem 7.1.20.

Endow X with the topology generated by the base $\mathcal{B}$. Since, as is easy to see,

$$V(-a) = X \setminus V(a),$$

the members of the base $\mathcal{B}$ are open-and-closed in X. If Δ_1 and Δ_2 are distinct points of the space X, the inclusion $\Delta_1 \subset \Delta_2$ is ruled out and so there exists $a \in \Delta_1 \setminus \Delta_2$. It follows from conditions (I1), (I2) and (MI) that $-a \in \Delta_2 \setminus \Delta_1$ and so $\Delta_1 \in V(-a)$ while

$\Delta_2 \in V(a)$. Since $V(-a) \cap V(a) = \emptyset$, the space X is a Hausdorff space. Consider now an arbitrary covering $\{V(a_t) : t \in T\}$ of the space X by members of the base $\mathcal{B}$. The set $\Delta \subset A$ consisting of those $x \in A$ for which there exists a finite number of indices $t_1, t_2, \ldots, t_m \in T$ such that $x \leq a_{t_1} \cup a_{t_2} \cup \ldots \cup a_{t_m}$ satisfies conditions (I1)–(I3). Since $a_t \in \Delta$ for each $t \in T$ and every maximal ideal in the algebra A belongs to one of the sets $V(a_t)$, that is to say, it does not contain a_t, we have $\Delta = A$. It follows that for some finite sequence of indices $t_1, t_2, \ldots, t_m \in T$ we have $\vee = a_{t_1} \cup a_{t_2} \cup \ldots \cup a_{t_m}$ which implies that $V(a_{t_1}) \cup V(a_{t_2}) \cup \ldots \cup V(a_{t_m}) = X$. The space X is thus compact by Theorem 7.5.1.

It follows from the compactness of the space X that every open-and-closed set of X is the union of a finite number of members of the base $\mathcal{B}$ and so is itself a member of the base $\mathcal{B}$. The correspondence assigning to a member $a \in A$ the set $V(a) \subset X$ is thus a map of the algebra A onto the family of all open-and-closed subsets of the space X. In view of $(*)$ and $(**)$ to show that the map is an isomorphism it is enough to check that $V(a) \neq \emptyset$ for $a \neq \wedge$, which amounts to the observation that the set $\{x \in A : x \leq -a\}$ is an ideal and that all maximal ideals containing this ideal belong to $V(a)$. ■

The space X of Theorem 7.5.48 is called the *Stone space* of the Boolean algebra A (see Problem 7.P.44).

Exercises

a) Show that, if A is a compact subspace of a completely regular space X, then for every closed set $B \subset X$ disjoint from the set A there is a continuous function $f: X \to I$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.

b) Observe that the compactness of a topological product of finitely many compact spaces follows from Theorem 7.5.14.

c) Show that a map $f: X \to Y$ of a topological space X into a compact space Y is continuous if and only if its graph $G(f)$ is a closed set in the topological product $X \times Y$.

d) Show that if a space X is compact, then an equivalence relation R on the space X is closed if and only if R is a closed set in the topological product $X \times X$. Check that the assumption of compactness is essential.

e) Give an example of a closed equivalence relation R on a compact space X for which $\chi(X/R) > \chi(X)$ (cf. Theorem 7.5.21).

f) Prove that, if a subspace A of a Hausdorff space X is locally compact, then A is the intersection of an open set in X with a closed set of X (cf. Corollary 7.5.28).

g) Show that every metrizable, locally compact space is completely metrizable.

h) Give an example of a compactification $cD(\aleph_0)$ of the discrete space of cardinality $\aleph_0$ for which the subspace $cD(\aleph_0) \backslash c(D(\aleph_0))$ of the space $cD(\aleph_0)$ is homeomorphic to the closed interval I. Give an example of a compactification of the real line which has the same property.

i) Let c_1X and c_2X be compactifications of a completely regular space X. Show that if a continuous function $f: c_1X \to c_2X$ satisfies $fc_1 = c_2$ then $f(c_1(X)) = c_2(X)$ and $f(c_1X \backslash c_1(X)) = c_2X \backslash c_2(X)$.

j) Suppose $c: D(\mathbf{c}) \to cD(\mathbf{c}) = X$ is a homeomorphic imbedding taking the space $D(\mathbf{c})$ onto the subspace C_2 of the Alexandrov double circumference X. Give an example of a continuous function $f: D(\mathbf{c}) \to I$ for which there does not exist a continuous function $f^*: cD(\mathbf{c}) \to I$ with $f^*c = f$.

k) Let X be any completely regular space and let $F = \triangle_{f\in\mathcal{F}} f: X \to \bigtimes_{f\in\mathcal{F}} \mathbf{R}_f$ where $\mathcal{F} = C^*(X)$ and $\mathbf{R}_f = \mathbf{R}$ for $f \in \mathcal{F}$. Prove that the closure $\operatorname{cl} F(X)$ of the image of the space X in the topological product $\bigtimes_{f\in\mathcal{F}} \mathbf{R}_f$ is the Stone-Čech compactification of the space X.

l) Show that the assumption of compactness on the space X in the Stone-Weierstrass Theorem cannot be omitted.

m) Prove that the sets G_n of Example 6.4.4 are dense in the function space X of that Example by an appeal to the Weierstrass Theorem. (Hint: For any function $g \in X$ and any positive real number ϵ choose a polynomial $p \in X$ with $\sigma(g,p) < \frac{1}{2}\epsilon$ and a function h with a steep "saw-tooth" graph of height $\frac{1}{2}\epsilon$ so that the function $f = p + h$ belongs to G_n.)

7.6. Metrization of topological spaces. Paracompact spaces

We shall say that a family $\{A_t\}_{t\in T}$ of subsets of a topological space X is *locally finite* if every point $x \in X$ has a neighbourhood $U \subset X$ such that the set $\{t \in T : U \cap A_t \neq \emptyset\}$ is finite. Likewise we call a family $\{A_t\}_{t\in T}$ *discrete* if every point $x \in X$ has a neighbourhood U intersecting at most one set A_t. Cleary every discrete family and every finite family is locally finite.

7.6.1. THEOREM. *For every locally finite family $\{A_t\}_{t\in T}$ it is the case that* $\operatorname{cl}(\bigcup_{t\in T} A_t) = \bigcup_{t\in T} \operatorname{cl} A_t$.

PROOF. As $\operatorname{cl} A_t \subset \operatorname{cl}(\bigcup_{t\in T} A_t)$ for each $t \in T$, we have $\bigcup_{t\in T} \operatorname{cl} A_t \subset \operatorname{cl}(\bigcup_{t\in T} A_t)$. To prove the reverse inclusion consider any point $x \in \operatorname{cl}(\bigcup_{t\in T} A_t)$. Since the family $\{A_t\}_{t\in T}$ is locally finite, there is a neighbourhod U of x such that the set $T_0 = \{t \in T : U \cap A_t \neq \emptyset\}$ is finite. Evidently $x \notin \operatorname{cl}(\bigcup_{t\in T\setminus T_0} A_t)$ and since $\operatorname{cl}(\bigcup_{t\in T} A_t) = \operatorname{cl}(\bigcup_{t\in T_0} A_t) \cup \operatorname{cl}(\bigcup_{t\in T\setminus T_0} A_t)$, we have $x \in \operatorname{cl}(\bigcup_{t\in T_0} A_t) = \bigcup_{t\in T_0} \operatorname{cl} A_t \subset \bigcup_{t\in T} \operatorname{cl} A_t$. We have thus shown that $\operatorname{cl}(\bigcup_{t\in T} A_t) \subset \bigcup_{t\in T} \operatorname{cl} A_t$. ■

7.6.2. COROLLARY. *For every locally finite collection of closed sets $\{F_t\}_{t\in T}$ the union $\bigcup_{t\in T} F_t$ is a closed set.* ■

The definitions of local finiteness and discreteness imply the following.

7.6.3. ASSERTION. *For every locally finite (discrete) family $\{A_t\}_{t\in T}$ the family of closures $\{\operatorname{cl} A_t\}_{t\in T}$ is locally finite (discrete).* ■

Before we come to state the next theorem which describes one of the most important properties of metrizable spaces, we need two definitions (see Supplement 7.S.26).

We shall call a family of subsets of a topological space *σ-locally finite* (respectively, *σ-discrete*) if it may be expressed as a union of countably many locally finite (respectively, discrete) families.

Given two coverings $\mathcal{A} = \{A_t\}_{t\in T}$ and $\mathcal{B} = \{B_s\}_{s\in S}$ of a topological space, if for every index $t \in T$ there is an index $s \in S$ such that $A_t \subset B_s$, then we say that the covering $\mathcal{A}$ *refines* (or is a *refinement*) of covering $\mathcal{B}$.

7.6.4. THEOREM (Stone). *Every open covering of a metrizable space has an open refinement which is both locally finite and σ-discrete.*

PROOF. Let $\{U_t\}_{t\in T}$ be an open covering of a metrizable space X whose topology is induced by a metric ρ on the set X. Fix any well ordering $<$ on the set T and for $n = 1, 2, \ldots$ define recursively a family $\mathcal{V}_n = \{V_{t,n}\}_{t\in T}$ of open sets of the space X by taking $V_{t,n} = \bigcup B(c; 1/2^n)$ where the union extends over all points $c \in X$ satisfying the conditions:

$$(*) \qquad c \in X \setminus [\bigcup_{s<t} U_s \cup \bigcup_{m<n} \bigcup_{s\in T} V_{s,m}],$$

$$(**) \qquad B(c; 3/2^n) \subset U_t.$$

Let x be any point of the space X. Consider the smallest element $t \in T$ for which $x \in U_t$ and a natural number n such that $B(x; 3/2^n) \subset U_t$. Then either $x \in V_{s,m}$ for some $m < n$ and some $s \in T$, or $x \in V_{t,n}$ and so the family $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ is an open covering of the space X; it follows from $(**)$ that the covering is a refinement of the covering $\{U_t\}_{t\in T}$.

We may show that for $n = 1, 2, \ldots$

$$\text{if } x_1 \in V_{t_1,n}, \quad x_2 \in V_{t_2,n} \text{ and } t_1 \neq t_2, \text{ then } \rho(x_1, x_2) > 1/2^n,$$

from which it will follow that every ball of radius $1/2^{n+1}$ intersects at most one member of the family $\mathcal{V}_n$, that is, the family $\mathcal{V}_n$ is discrete. We may suppose that $t_1 < t_2$. It follows from the definition of the sets $V_{t_1,n}$ and $V_{t_2,n}$ that there are points c_1 and c_2 satisfying analogues of $(*)$ and $(**)$ such that $x_i \in B(c_i; 1/2^n) \subset V_{t_i,n}$ for $i = 1, 2$. In particular $B(c_1; 3/2^n) \subset U_{t_1}$ and $c_2 \notin U_{t_1}$ and so $\rho(c_1, c_2) \geq 3/2^n$, hence $\rho(x_1, x_2) \geq \rho(c_1, c_2) - \rho(c_1, x_1) - \rho(c_2, x_2) > 1/2^n$.

For every point $x \in X$ there exist s, k and m such that $B(x; 1/2^k) \subset V_{s,m}$; to complete the proof it is thus enough to show that for each $s \in T$ and for $k, m = 1, 2, \ldots$

$$\text{if } B(x; 1/2^k) \subset V_{s,m}, \text{ then } B(x; 1/2^{m+k}) \cap V_{t,n} = \emptyset \text{ for } n \geq m + k \text{ and } t \in T,$$

since from this it will follow that the neighbourhood $B(x; 1/2^{m+k})$ of the point x intersects at most $m + k - 1$ members of the covering $\mathcal{V}$. From condition $(*)$ we infer that the points c in the definition of the set $V_{t,n}$ do not belong to $V_{s,m}$ for $n \geq m + k$; since $B(x, 1/2^k) \subset V_{s,m}$ we have $\rho(x, c) \geq 1/2^k$ for each such point. From the inequalities $m + k \geq k + 1$ and $n \geq k + 1$ we conclude that $B(x; 1/2^{m+k}) \cap B(c; 1/2^n) = \emptyset$ and hence that $B(x; 1/2^{m+k}) \cap V_{t,n} = \emptyset$. ■

Theorem 7.6.4 and Lemma 6.3.2 imply the following.

7.6.5. THEOREM. *Every metrizable space has a σ-discrete base.* ■

7.6.6. COROLLARY. *Every metrizable space has a σ-locally finite base.* ■

It turns out that the existence of a σ-locally finite base is sufficient for the metrizability of a regular space. We first prove a lemma which strengthens Theorem 7.3.11.

7.6.7. LEMMA. *Every regular space with a σ-locally finite base is normal.*

PROOF. Consider a regular space X with a base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where, for $n = 1, 2, \ldots$, $\mathcal{B}_n$ is a locally finite family, a closed set $F \subset X$ and an open set $W \subset X$ containing F. For each point $x \in F$ there is a natural number $n(x)$ and an open set $V(x) \in \mathcal{B}_{n(x)}$ such that $x \in V(x) \subset \operatorname{cl} V(x) \subset W$. Taking $W_n = \bigcup\{V(x) : n(x) = n\}$ we obtain a sequence of open sets $W_1, W_2, \ldots$ such that $F \subset \bigcup_{n=1}^{\infty} W_n$ and such that, by Theorem 7.6.1, $\operatorname{cl} W_n \subset W$ for $n = 1, 2, \ldots$ To complete the proof it is enough to appeal to Lemma 7.3.10. ■

7.6.8. THEOREM (Nagata, Smirnov). *A topological space is metrizable if and only if it is regular and has a σ-locally finite base.*

PROOF. In view of Corollary 7.6.6 it suffices to prove the sufficiency of the stated condition.

Consider a regular space X with base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ where $\mathcal{B}_n = \{V_t\}_{t \in T_n}$ is a locally finite family for each $n = 1, 2, \ldots$ Fix a pair of natural numbers m, n and for each $t \in T_n$ put

$$U_t = \bigcup\{V_s : s \in T_m \quad \text{and} \quad \operatorname{cl} V_s \subset V_t\}.$$

It follows from Theorem 7.6.1 that $\operatorname{cl} U_t \subset V_t$ and so by Lemma 7.6.7 and Theorem 7.3.9 there exists a continuous function $f_t \colon X \to I$ such that $f_t(X \setminus V_t) \subset \{0\}$ and $f_t(\operatorname{cl} U_t) \subset \{1\}$. For each point $(x, y) \in X \times X$ there exists a neighbourhood $U(x) \times V(y) \subset X \times X$ of the point and a finite set $T(x, y) \subset T_n$ such that $[U(x) \cup V(y)] \cap V_t = \emptyset$ for $t \in T_n \setminus T(x, y)$. Hence the formula

$$g_{m,n}(x, y) = \sum_{t \in T_n} |f_t(x) - f_t(y)|$$

validly defines a continuous function $g_{m,n} \colon X \times X \to \mathbf{R}_+$.

It is easily checked that by taking $x R_{m,n} y$ if and only if $g_{m,n}(x, y) = 0$ we define an equivalence relation on the set X and that the formula $\rho_{m,n}([x], [y]) = g_{m,n}(x, y)$ defines a metric on the set $X_{m,n}$ of equivalence classes of the relation $R_{m,n}$. It follows from the continuity of the function $g_{m,n}$ that the map $f_{m,n} \colon X \to X_{m,n}$ taking each point $x \in X$ to its equivalence class $[x] \in X_{m,n}$ is a continuous map of the space X into the space $X_{m,n}$ when the latter is endowed with the topology induced by the metric $\rho_{m,n}$. In view of Theorems 7.4.23 and 7.4.40 to complete the proof it is enough to check that the family $\{f_{m,n}\}_{m,n=1}^{\infty}$ separates points from closed sets.

Consider an arbitrary point $x \in X$ and a closed set $F \subset X$ with $x \notin F$. Since the family $\mathcal{B}$ is a base of the space X there exist natural numbers m and n and indices $t \in T_n$ and $s \in T_m$ such that $x \in V_s \subset \operatorname{cl} V_s \subset V_t \subset X \setminus F$. It follows from the definition of the functions $g_{m,n}$ that $g_{m,n}(x, y) \geq 1$ for each $y \in F$ and hence $\rho_{m,n}(f_{m,n}(x), f_{m,n}(y)) \geq 1$ for each $y \in F$. We deduce that $f_{m,n}(x) \notin \operatorname{cl} f_{m,n}(F)$, that is the family $\{f_{m,n}\}_{m,n=1}^{\infty}$ separates points from closed sets. ■

Theorems 7.6.5 and 7.6.8 imply the following.

7.6.9. THEOREM (Bing). *A topological space is metrizable if and only if it is regular and has a σ-discrete base.* ■

The notion of local finiteness leads us to distinguish an important class of topological spaces namely the paracompact spaces. We call a topological space X a *paracompact* space if X is a Hausdorff space and every open covering of the space X has a locally finite open refinement.

We note that in the definition of paracompactness the words "every open covering of the space X has a locally finite open refinement" cannot be replaced by the words "every open covering of the space X contains a locally finite covering". For instance, as may easily be seen, the space of natural numbers $\mathbf{N}$ with discrete topology is paracompact (the covering consisting of all singletons is open, locally finite and refines any open covering of the space $\mathbf{N}$) but the open covering $\{\mathbf{N} \cap [1,n)\}_{n=1}^{\infty}$ of the space does not contain any locally finite covering (cf. Exercise (g)).

The following is immediate from the definition of paracompactness.

7.6.10. THEOREM. *Every compact space is paracompact.* ■

On the other hand Theorem 7.6.4 implies the following.

7.6.11. THEOREM. *Every metrizable space is paracompact.* ■

Theorem 7.6.10 may be essentially strengthened; the following holds.

7.6.12. THEOREM. *If a regular space is the union of a countable sequence $A_1, A_2, \ldots$ of compact subspaces, then X is paracompact.*

PROOF. Consider any open covering $\mathcal{U}$ of the space X. Using the regularity of the space X we may find for each point $x \in X$ open sets U_x, V_x such that $x \in U_x \subset \operatorname{cl} U_x \subset V_x$ and the set V_x is contained in some member of $\mathcal{U}$. By Assertion 7.5.4 we may choose a countable number of open sets $U_{x_1}, U_{x_2}, \ldots$ with $X = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} U_{x_i}$. The sets $W_1, W_2, \ldots$, where $W_1 = W_{x_1}$ and $W_i = V_{x_i} \setminus (\operatorname{cl} U_{x_1} \cup \operatorname{cl} U_{x_2} \cup \ldots \cup \operatorname{cl} U_{x_{i-1}})$ for $i = 1, 2, 3, \ldots$, are open and form a covering of the space X since every point $x \in X$ belongs to the set $W_{i(x)}$, where $i(x)$ is the least natural number i such that $x \in V_{x_i}$. The covering thus obtained refines $\mathcal{U}$; since $U_{x_j} \cap W_i = \emptyset$ for $i > j$, it is locally finite. ■

It turns out that paracompact spaces are normal.

7.6.13. LEMMA. *Let X be a paracompact space and A and B a pair of disjoint closed subsets of the space X. If for each point $x \in A$ there exist open sets $U_x, V_x \subset X$ such that $x \in U_x$, $B \subset V_x$ and $U_x \cap V_x = \emptyset$ then there also exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

PROOF. The family $\{X \setminus A\} \cup \{U_x\}_{x \in A}$ is an open covering of the space X and so has a locally finite open refinement $\{W_t\}_{t \in T}$. Let $T_0 = \{t \in T : A \cap W_t \neq \emptyset\}$, evidently $B \cap \operatorname{cl} W_t = \emptyset$ for $t \in T_0$ and $A \subset U = \bigcup_{t \in T_0} W_t$. By Assertion 7.6.3 and Corollary 7.6.2 the set $V = X \setminus \bigcup_{t \in T_0} \operatorname{cl} W_t$ is open. It is easy to see that $B \subset V$ and $U \cap V = \emptyset$. ■

7.6.14. THEOREM. *Every paracompact space is normal.*

PROOF. Taking the set B of Lemma 7.6.13 to be a singleton we infer that every paracompact space is regular. Using this fact and applying Lemma 7.6.13 a second time we obtain the theorem. ■

Let X be a topological space. A family $\{f_t\}_{t\in T}$ of continuous functions where $f_t\colon X_t \to I$ for $t \in T$ is called a *partition of unity* if $\sum_{t\in T} f_t(x) = 1$ for each $x \in X$. The latter equation means that

$$\sup\{f_{t_1}(x) + f_{t_2}(x) + \ldots + f_{t_n}(x) : t_i \in T \text{ for } i \leq n \text{ and } n = 1, 2, \ldots\} = 1$$

for each $x \in X$; it follows in particular that the set $T(x) = \{t \in T : f_t(x) \neq 0\}$ is countable. We say that a partition of unity $\{f_t\}_{t\in T}$ is *locally finite* if the covering $\{f_t^{-1}((0,1])\}_{t\in T}$ is locally finite. If $\{f_t\}_{t\in T}$ is a locally finite partition of unity then for every point $x_0 \in X$ there is a neighbourhood U_0 of the point x_0 and a finite set $T_0 = \{t_1, t_2, \ldots, t_n\} \subset T$ such that $f_t(x) = 0$ for $x \in U_0$ and $t \in T\backslash T_0$, and $\sum_{i=1}^n f_{t_i}(x) = 1$ for $x \in U_0$. We say that the partition of unity $\{f_t\}_{t\in T}$ is *subordinate* to the covering $\mathcal{U}$ of the space X if the covering $\{f_t^{-1}((0,1])\}_{t\in T}$ refines the covering $\mathcal{U}$.

We now give an important characterization of paracompact spaces, which is useful both in topology and in analysis, based on the concept of a partition of unity. We precede the characterization with two lemmas.

7.6.15. LEMMA. *For every open covering $\{U_t\}_{t\in T}$ of a paracompact space X there exists a locally finite open covering $\{V_t\}_{t\in T}$ of the space such that* $\operatorname{cl} V_t \subset U_t$ *for* $t \in T$.

PROOF. Since X is a regular space we may define an open covering $\{W_s\}_{s\in S}$ of the space X such that for every index $s \in S$ there is an index $t(s) \in T$ with $\operatorname{cl} W_s \subset U_{t(s)}$. In view of the paracompactness of the space X we may assume that the covering $\{W_s\}_{s\in S}$ is locally finite. It is easily checked that the family $\{V_t\}_{t\in T}$ where $V_t = \bigcup\{W_s : t(s) = t\}$ for $t \in T$ is the desired covering. ■

7.6.16. LEMMA. *If a partition of unity $\{f_t\}_{t\in T}$ is subordinate to an open covering $\mathcal{U}$ of a topological space X, then the covering $\mathcal{U}$ has a locally finite open refinement.*

PROOF. Observe first that for every continuous function $g\colon X \to I$ and each point $x_0 \in X$ such that $g(x_0) > 0$ there exists a neighbourhood U_0 of the point x_0 and a finite set $T_0 \subset T$ such that $f_t(x) < g(x)$ for $x \in U_0$ and $t \in T\backslash T_0$. Indeed, it may easily be checked that the set $T_0 = \{t_1, t_2, \ldots, t_n\} \subset T$ for which $1 - \sum_{i=1}^n f_{t_i}(x_0) < g(x_0)$ and the neighbourhood $U_0 = \{x \in X : 1 - \sum_{i=1}^n f_{t_i}(x) < g(x)\}$ of the point x_0 have the desired properties.

For each point $x \in X$ there is an index $t(x) \in T$ such that $f_{t(x)}(x) > 0$. Taking $g = f_{t(x)}$ in the observation above we conclude that formula $f(x) = \sup\{f_t(x) : t \in T\}$ defines a continuous function $f\colon X \to (0,1]$. For each $t \in T$ the set $V_t = \{x \in X : f_t(x) > \frac{1}{2}f(x)\}$ is open. It is easy to see that the family $\mathcal{U} = \{V_t\}_{t\in T}$ is an open covering of the space X refining the covering $\mathcal{U}$. Appealing again to the observation made at the start of the proof and taking $g = \frac{1}{2}f$ we infer that the covering $\mathcal{U}$ is locally finite. ■

7.6.17. THEOREM. *For any T_1-space X the following conditions are equivalent:*

(1) *the space X is paracompact,*

(2) *for every open covering of the space X there exists a locally finite partition of unity subordinate to it,*

(3) *for every open covering of the space X there exists a partition of unity subordinate to it.*

PROOF. (1) $\Rightarrow$ (2). Consider a paracompact space X and an open covering $\mathcal{U} = \{U_t\}_{t\in T}$ of it. By Lemma 7.6.15 there exist locally finite coverings $\{V_t\}_{t\in T}$ and $\{W_t\}_{t\in T}$ of the space X with $\operatorname{cl} W_t \subset V_t \subset U_t$ for $t \in T$.

It follows from Theorems 7.6.14 and 7.3.9 that for every $t \in T$ there exists a continuous function $g_t\colon X \to I$ such that $g_t(X\setminus V_t) \subset \{0\}$ and $g_t(\operatorname{cl} W_t) \subset \{1\}$. In view of the local finiteness of the covering $\{V_t\}_{t\in T}$ we obtain a continuous function $g\colon X \to \mathbf{R}_+\setminus\{0\}$ by taking $g(x) = \sum_{t\in T} g_t(x)$ for each $x \in X$. It is easy to verify that the family $\{f_t\}_{t\in T}$ where $f_t = g_t/g$, is a locally finite partition of unity, subordinate to the covering $\mathcal{U}$.

(2) $\Rightarrow$ (3). The implication is obvious.

(3) $\Rightarrow$ (1). In view of Lemma 7.6.16 it remains to prove that every T_1-space X satisfying condition (3) is a Hausdorff space. Consider an arbitrary pair of distinct points $x, y \in X$. The family $\mathcal{U} = \{X\setminus\{x\}, X\setminus\{y\}\}$ is an open covering of the space X and so there exists a partition of unity $\{f_t\}_{t\in T}$ subordinate to $\mathcal{U}$. Choose any index $t_0 \in T$ such that $f_{t_0}(x) = a > 0$. Since $f_{t_0}(y) = 0$ the open sets $U = f_{t_0}^{-1}((\frac{1}{2}a, 1])$ and $V = f_{t_0}^{-1}([0, \frac{1}{2}a))$ satisfy the conditions $x \in U$, $y \in V$ and $U \cap V = \emptyset$. ■

We close our discussion by proving a simple theorem about subspaces of paracompact spaces.

7.6.18. THEOREM. *If X is a paracompact space and A is a closed subset of X, then the subspace A is paracompact.*

PROOF. Consider an arbitrary open covering $\{U_t\}_{t\in T}$ of the subspace A. For each $t \in T$ there exists an open set W_t of X such that $U_t = A \cap W_t$. The family $\{X\setminus A\} \cup \{W_t\}_{t\in T}$ is an open covering of the space X and so may be refined by a locally finite open covering $\{V_s\}_{s\in S}$ of the space X. The family $\{A \cap V_s\}_{s\in S}$ is a locally finite open covering of the space A refining the covering $\{U_t\}_{t\in T}$. To complete the proof it suffices to appeal to Theorem 7.4.12. ■

We might just notice that the topological product of two paracompact spaces need not be a paracompact space (see Problem 7.P.68; cf. Problem 7.P.67).

Exercises

a) Check that for every locally finite family $\{A_t\}_{t\in T}$ it is true that $\operatorname{bd}(\bigcup_{t\in T} A_t) \subset \bigcup_{t\in T} \operatorname{bd} A_t$.

b) Observe that every locally finite family of non-empty subsets of a compact space is finite.

c) Show that a T_1-space X has a locally finite base if and only if X is discrete.

d) Give an example of a Hausdorff space with a σ-locally finite base but which is not regular. (Hint: See Example 7.3.6).

e) Observe that Theorems 7.4.44 and 7.5.19 are implied by the Nagata-Smirnov Theorem (7.6.8).

f) Check that a Hausdorff space is compact if and only if every open covering of the space may be refined by a finite open covering.

g) Check that a Hausdorff space is compact if and only if every open covering of the space contains a locally finite open covering.

h) Show that for every closed subset A of a paracompact space X the quotient space X/A is paracompact (see Supplement 7.S.29).

i) Prove that if a paracompact space X has an open covering all of whose members are metrizable subspaces of the space X, then the space X is also metrizable.

7.S. Supplements

7.S.1. The term "topology" as a name for the family of open sets of a topological space is now universally accepted. Properly speaking, one should talk of the "topological structure" of a space and of endowing a set with a topological structure. Clearly the use of the same word for a family of sets and for a branch of mathematics cannot lead to any confusion. We should add that, from the point of view of topology, all the information about the space is already contained in its topology.

7.S.2. The first attempts at an axiomatic treatment of the concept of an abstract topological space were due to M. Fréchet (1906) and F. Riesz (1907). For Fréchet the primitive notion was that of a convergent sequence and for Riesz that of an accumulation point of a set. The definitions which they proposed, however, turned out to be inappropriate. The first satisfactory definition of a topological space was given by F. Hausdorff (1914). He took a space to be a set X equipped with a system of neighbourhoods $\{\mathcal{B}_x\}_{x\in X}$ with properties (BP1)–(BP4) (see Theorem 7.1.21 and Assertion 7.3.3). The more general notion of a topological space which we have taken in this book, which is now universally regarded as the correct definition, is due to K. Kuratowski (1922).

The class of topological spaces may be described in an equivalent way by three other systems of axioms. The primitive notion can be the family of closed sets, the closure operation or the interior operation (see Problems 7.P.1–7.P.3).

7.S.3. If two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on a set X satisfy $\mathcal{O}_2 \subset \mathcal{O}_1$ then we say that the topology $\mathcal{O}_1$ *finer* than the topology $\mathcal{O}_2$ or that the topology $\mathcal{O}_2$ is *coarser* than the topology $\mathcal{O}_1$. The finest topology on a set X is the discrete topology and the coarsest is the antidiscrete topology. The class of all topologies on a fixed set is ordered by the relation $\subset$. Evidently the relation is not in general a linear ordering in the class of all topologies, as two topologies on one given set need not be related by this relation.

These ideas lead at times to more succint and intuitive formulations. For instance the Tikhonov topology may be defined as the coarsest topology on the Cartesian product for which all the projections are continuous, while Theorem 7.5.12 may be expressed as saying: compact topologies are minimal in the class of Hausdorff topologies.

7.S.4. The notion of a $\mathcal{G}_\delta$- or $\mathcal{F}_\sigma$-set carries over without any change to topological spaces, a subset A of a topological space X is called a *$\mathcal{G}_\delta$-set* in X, if A may be expressed as the intersection of countably many open sets of X. Complements of $\mathcal{G}_\delta$-sets are known as *$\mathcal{F}_\sigma$-sets*. Clearly a subset A of a topological space X is an $\mathcal{F}_\sigma$-set in X if and only if it may be expressed as a union of countably many closed sets in X. The reader will have no difficulty in checking that Theorem 6.4.8 continues to hold in any topological space. However, Theorem 6.4.9 is not in general true without additional hypotheses about the topological space. If $\mathbf{m} > \aleph_0$, the closed set $\{x_0\}$, in the space $A(\mathbf{m})$ of Example 7.2.14, where x_0 is the unique accumulation point of the space, is not a $\mathcal{G}_\delta$-set. A normal space X in which every closed set is a $\mathcal{G}_\delta$-set is said to be *perfectly normal* (cf. Problem 7.P.14). A fragment of the theory of Borel sets may be developed in the context of perfectly normal spaces (see Supplement 6.S.7).

7.S.5. A topological space X is said to be a *Lindelöf* space, if X is a regular space and every open covering of the space contains a countable covering. Evidently countable regular spaces and compact spaces are Lindelöf spaces; more generally: regular spaces which are unions of a countable sequence of compact subspaces are Lindelöf spaces. It follows from Theorem 6.3.3 that a metrizable space is a Lindelöf space if and only if it satisfies the second axiom of countability, or, equivalently, is separable. A regular space satisfying the second axiom of countability is a Lindelöf space (see the proof of the implication (2) $\Rightarrow$ (3) of Theorem 6.3.3). The space $A(\mathbf{m})$ for $\mathbf{m} > \aleph_0$ is a Lindelöf space, but does not satisfy the second axiom of countability, nor is it a separable space. The Niemytzki plane is a separable space but is not a Lindelöf space because every Lindelöf space is normal (see Problem 7.P.15).

7.S.6. A set Σ is said to be *directed*, if a relation $\leq$ is defined on Σ that is transitive (that is, the conditions $\sigma \leq \rho$ and $\rho \leq \tau$ imply that $\sigma \leq \tau$), reflexive (that is, $\sigma \leq \sigma$ for every $\sigma \in \Sigma$) and has the property that for any pair σ, ρ of elements of Σ there is an element $\tau \in \Sigma$ satisfying $\sigma \leq \tau$ and $\rho \leq \tau$. A subset Σ_0 of a directed set Σ is said to be *cofinal* in Σ if for every element σ of the set Σ there exists an element ρ of the set Σ_0 satisfying the condition $\sigma \leq \rho$.

The analogous concept to that of a sequence of points in a metric space appropriate to the study of topological spaces is that of a net. A *net* of points in a topological space is an arbitrary map of some directed set Σ into the space X; a net is written in the form $S = \{x_\sigma, \sigma \in \Sigma\}$ where x_σ is the point of the space X assigned to the element σ of the directed set Σ. We say that the net $S = \{x_\sigma, \sigma \in \Sigma\}$ *converges* to a point x if for no neighbourhood U of x is the set $\{\sigma \in \Sigma : x_\sigma \in X \backslash U\}$ cofinal in Σ. Any point to which a net converges is said to be a *limit* of the net. A net $S = \{x_\sigma, \sigma \in \Sigma\}$ will in general have many limits, the set of all its limits is denoted by $\lim_{\sigma \in \Sigma} x_\sigma$. A point $x \in X$ is

said to be a *cluster point* of the net $S = \{x_\sigma, \sigma \in \Sigma\}$ if for every neighbourhood U of the point x the set $\{\sigma \in \Sigma : x_\sigma \in U\}$ is cofinal in Σ. The analogue of the concept of a subsequence in the theory of nets is that of a finer net. A net $S' = \{x'_{\sigma'}, \sigma' \in \Sigma'\}$ is said to be *finer* than the net $S = \{x_\sigma, \sigma \in \Sigma\}$ if there is a function $\varphi\colon \Sigma' \to \Sigma$ such that $x_{\varphi(\sigma')} = x'_{\sigma'}$ for $\sigma' \in \Sigma'$ and for each $\sigma_0 \in \Sigma$ there is $\sigma'_0 \in \Sigma'$ with the property that if $\sigma'_0 \leq \sigma'$ then $\sigma_0 \leq \varphi(\sigma')$.

Evidently every sequence $\{x_n\}$ may be regarded as a net $\{x_n, n \in \mathbf{N}\}$ where $\mathbf{N}$ is the set of natural numbers with the usual linear order $\leq$. To each subsequence $\{x_{k_n}\}$ of the sequence $\{x_n\}$ there corresponds the net $\{x'_n, n \in \mathbf{N}\}$ where $x'_n = x_{k_n}$ which is finer that $\{x_n, n \in \mathbf{N}\}$; the function $\varphi\colon \mathbf{N} \to \mathbf{N}$ here is given by $\varphi(n) = k_n$.

The concepts introduced above have the following five properties corresponding to the fundamental properties of sequences.

1) If a point x is a cluster point of a net S' that is finer than the net S, then x is a cluster point of S.

2) If x is a limit of the net S, then it is a limit of every net S' finer than S.

3) If a point x is a cluster point of a net S then it is a limit of some net S' finer than S.

4) A point x belongs to the closure of a set A in a space X if and only if there is a net $\{x_\sigma, \sigma \in \Sigma\}$ of points of X such that $x \in \lim_{\sigma\in\Sigma} x_\sigma$ and $x_\sigma \in A$ for every $\sigma \in \Sigma$.

5) A map $f\colon X \to Y$ is continuous at a point $x \in X$ if and only if for every net $\{x_\sigma, \sigma \in \Sigma\}$ of points of the space X such that $x \in \lim_{\sigma\in\Sigma} x_\sigma$ it is the case that $f(x) \in \lim_{\sigma\in\Sigma} f(x_\sigma)$.

Moreover it may be proved that a topological space is a Hausdorff space if and only if every net of points of the space X has at most one limit.

Proofs of the results mentioned above and additional material on nets may be found in [4], p. 49–56, where yet another treatment of convergence in topological spaces, namely the theory of filters, is presented and the equivalence of the two approaches demonstrated.

7.S.7. With reference to the closing remarks of Section 7.1 the problem arises of how to characterize the class of topological spaces where sequences suffice to describe the topology. The problem may be solved in two ways; two classes of spaces may be distinguished. X is a *Fréchet space* if for every set $A \subset X$ a point x belongs to the closure of the set if and only if there is a sequence $\{x_n\}$ of points of A convergent to x. A topological space is said to be *sequential* if a set $A \subset X$ is closed in X if and only if each sequence of points of A has all its limits also in A.

It is easy to verify that every space satisfying the first axiom of countability is a Fréchet space (see Exercise (f) of Section 7.1) and that every Fréchet space is sequential. The space X of Example 7.1.31 is not a sequential space. A slight modification of the construction leads to a sequential space which is not a Fréchet space. It is enough to take the set Y obtained by adding to the space X points of the form $(m, 0)$ where $m = 1, 2, \ldots$ and to declare the local base at the point $(m, 0)$ to be the family of sets of the form $\{(m, 0)\} \cup \{(m, n) : n = k, k+1, \ldots\}$ and to modify neighbourhoods of the

point $(0,0)$ by adding to each of them a set of the form $\{(m,0) : m = k, k+1, \ldots\}$. The simplest example of a Fréchet space that does not satisfy the first axiom of countability is the quotient space $\mathbf{R}/\mathbf{N}$ obtained from the real line $\mathbf{R}$ by pasting together into one point the set of natural numbers $\mathbf{N}$ (see Example 7.4.49).

7.S.8. The definitions of metrizable spaces and $T_{3\frac{1}{2}}$-spaces are essentially different from the definitions of the remaining classes of topological spaces discussed here. They are *extrinsic* definitions referring to objects existing outside the spaces under consideration (in this case reference is to the set of real numbers), as opposed to the *intrinsic* definitions of, for instance, separable spaces, spaces satisfying the first or second axiom of countability, the connected spaces, the T_i-spaces for $i \neq 3\frac{1}{2}$, as well as the compact spaces, locally compact and paracompact spaces. Intrinsic definitions are regarded as being better than extrinsic ones in particular because the invariance of the classes considered are then immediate from the form of the definition. Sometimes, however, the extrinsic definitions are simpler and much more natural. When a class of spaces is distinguished by means of an extrinsic definition it is usual and in accord with topological tradition, to search for an intrinsic characterization of the class. Theorem 7.6.8 contains an intrinsic characterization of metrizable spaces (cf. Supplement 7.S.27); an intrinsic characterization of $T_{3\frac{1}{2}}$-spaces is stated as Problem 7.P.9. In Supplement 7.S.25 we introduce one more class of topological spaces using an extrinsic definition namely the Čech-complete spaces. An intrinsic characterization of the class is given in Problem 7.P.53.

7.S.9. Apart from the class of T_i-spaces for $i = 1, 2, 3, 3\frac{1}{2}$ and 4 discussed in Section 7.3, the classes of T_i-spaces for $i = 0, 5$ and 6 are also studied. A topological space is a *T_0-space* if for every pair of distinct points $x, y \in X$ there is an open set $U \subset X$ containing just one of these points and omitting the other. Evidently every T_1-space is a T_0-space. An antidiscrete space containing at least two points is not a T_0-space. The two point set $F = \{0,1\}$ with topology consisting of $\emptyset$, $\{0\}$ and F constitutes an example of a T_0-space which is not a T_1-space. The name *T_5-space* is sometimes given to spaces which are hereditarily normal (see Supplement 7.S.11) and the name *T_6-space* is given to the perfectly normal spaces (see Supplement 7.S.4 and Problem 7.P.14).

7.S.10. By a *linear topological space* we mean a linear space $\mathcal{V}$ over the field of scalars $\mathbf{R}$ which is endowed with a topology $\mathcal{O}$ such that $(\mathcal{V}, \mathcal{O})$ is a Hausdorff space and such that the following two conditions are met:

(LT1) *addition of vectors is a continuous map from the topological product $\mathcal{V} \times \mathcal{V}$ into the space $\mathcal{V}$,*

(LT2) *multiplication by a scalar is a continuous map from the topological product $\mathbf{R} \times \mathcal{V}$ into the space $\mathcal{V}$.*

Clearly every normed space is a topological linear space relative to the topology induced by the norm metric in the space (see Supplement 1.S.1). A linear topological space is *locally convex* if it has a local base at every point consisting of convex sets. It is easy to see that a normed space is locally convex.

Apart from the Tietze-Urysohn Theorem another important generalization of Tietze's Theorem is *Dugundji's Theorem* which states that if A is a closed subset of a metric space X and $\mathcal{V}$ is a locally convex linear topological space then for every continuous map $f: A \to \mathcal{V}$ there exists a continuous extension $f^*: X \to \mathcal{V}$. It may also be assumed that the image $f^*(X)$ is contained in the convex hull of $f(A)$, that is the smallest convex subset of $\mathcal{V}$ to contain the image $f(A)$. The extension f^* may be defined by the formula given in the hint to Problem 7.P.61.

7.S.11. In studying the transfer of topological properties from a space to its subspaces and from the factors to the topological product, use is sometimes made of the concept of a hereditary property, or of a multiplicative property. A topological property $\mathcal{P}$ is said to be *hereditary* (or respectively, *hereditary relative to open subsets, closed subsets, dense subsets*, etc.) if every subspace A (every open subspace, closed subspace, dense subspace, etc.) of a space X having property $\mathcal{P}$ also has property $\mathcal{P}$. Likewise a topological property $\mathcal{P}$ is said to be *multiplicative* (m-*multiplicative, finitely multiplicative*) if a topological product $\bigtimes_{t\in T} X_t$ (where respectively $\operatorname{card} T \leq \mathfrak{m}$, $\operatorname{card} T < \aleph_0$) of spaces X_t having property $\mathcal{P}$ also has property $\mathcal{P}$. If a topological property $\mathcal{P}$ is not hereditary, but a space X and all its subspaces have property $\mathcal{P}$, then we say that the space X has *property $\mathcal{P}$ hereditarily*; it is in this sense that one speaks of *hereditarily normal* spaces (see Problem 7.P.18), *hereditarily separable* spaces etc.

7.S.12. Suppose given a family of pairwise disjoint topological spaces $\{X_t\}_{t\in T}$. The family $\mathcal{O}$ consisting of all subsets U of the set $\bigcup_{t\in T} X_t$ such that the intersection $U \cap X_t$ is open for each $t \in T$ satisfies the axioms (T1)–(T3); we may thus use it to endow the set $\bigcup_{t\in T} X_t$ with a topology. The resulting topological space is called the *topological union* of the spaces $\{X_t\}_{t\in T}$ and denoted $\bigoplus_{t\in T} X_t$, or $\bigoplus_{n=1}^{\infty} X_n$, if T is the set of natural numbers, or even $\bigoplus_{j=1}^{n} X_j$ or $X_1 \oplus X_2 \oplus \ldots \oplus X_n$ if $T = \{1, 2, \ldots, n\}$. The topological union may also be defined for a family of spaces $\{X_t\}_{t\in T}$ when these are not disjoint but the result of the operation is determined only up to homeomorphism. One takes any family of pairwise disjoint topological spaces $\{X_t'\}_{t\in T}$ such that the spaces X_t and X_t' are homeomorphic for each $t \in T$ (for instance we may take $X_t' = X_t \times \{t\}$) and the topological union $\bigoplus_{t\in T} X_t$ is defined to be the space $\bigoplus_{t\in T} X_t'$.

7.S.13. An important special case of the quotient space operation is the operation of the adjunction space. Suppose given two disjoint topological spaces X and Y and a map $f: X \to Y$ defined on a closed subset A of the space X. Consider the decomposition $\mathcal{R}$ of $X \oplus Y$ into the sets $f^{-1}(y) \cup \{y\}$ for $y \in Y$ and the singletons corresponding to the complement $X \setminus A$. The quotient space $(X \oplus Y)/R$ where R is the equivalence relation corresponding to the decomposition $\mathcal{R}$ is known as the *adjunction space of X and Y under the map f* and is denoted $X \cup_f Y$.

Suppose given spaces X and Y such that $(X \times I) \cap Y = \emptyset$ and a continuous map $f: X \to Y$. Consider the closed subset $A = X \times \{1\}$ of the space $X \times I$ and the continuous map $\bar{f}: A \to Y$ defined by the formula $\bar{f}(x, 1) = f(x)$ for $x \in X$. The adjunction space $(X \times I) \cup_{\bar{f}} Y$ is known as the *mapping cylinder of f*. The concept of a mapping cylinder is extensively used in homotopy theory.

It follows from Theorem 7.5.20 (and Theorem 7.5.22) that if the spaces X and Y are compact (and metrizable) then for every continuous map $f: X \to Y$ the adjunction space $X \cup_f Y$ and the mapping cylinder of f are compact (and metrizable).

7.S.14. An important operation on topological spaces is the operation of the limit of an inverse system. Suppose given for each element σ of a directed set Σ (see Supplement 7.S.6) a topological space X_σ and for every pair σ, ρ of elements of Σ satisfying $\rho \leq \sigma$ a continuous map $\pi_\rho^\sigma: X_\sigma \to X_\rho$. If for every triplet σ, ρ, τ of elements of Σ satisfying $\tau \leq \rho \leq \sigma$ we have $\pi_\tau^\rho \pi_\rho^\sigma = \pi_\tau^\sigma$ and moreover $\pi_\sigma^\sigma = \mathrm{id}_{X_\sigma}$ for each $\sigma \in \Sigma$ then the system $S = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ is called an *inverse system*. An inverse system $S = \{X_i, \pi_j^i, \mathbf{N}\}$, where $\mathbf{N}$ is the set of natural numbers with the usual linear order $\leq$, is called an *inverse sequence*; when writing down an inverse sequence the set $\mathbf{N}$ is suppressed, that is, it is written $\{X_i, \pi_j^i\}$. The subspace of the topological product $\bigtimes_{\sigma \in \Sigma} X_\sigma$ consisting of points $\{x_\sigma\}$ such that $\pi_{\sigma''}^{\sigma'}(x_{\sigma'}) = x_{\sigma''}$ for all $\sigma', \sigma'' \in \Sigma$ satisfying $\sigma'' \leq \sigma'$ is known as the *limit of the inverse system* $S = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$. A discussion of the properties of the operation of inverse limit may be found in [4], p. 98–105 and 141–142.

7.S.15. Suppose given any set X and a family $\{A_t\}_{t \in T}$ consisting of subsets of X. Assume that A_t is for each $t \in T$ a topological space and that the following conditions are satisfied:

(1) for every pair of indices $t, t' \in T$ both subspace topologies defined on the set $A_t \cap A_{t'}$, the one inherited from the space A_t and the other inherited from the space $A_{t'}$, coincide,

(2) for every pair of indices $t, t' \in T$ the set $A_t \cap A_{t'}$ is closed in both the space A_t and the space $A_{t'}$,

or, alternatively, condition (1) is satisfied together with the condition:

(2′) for every pair of indices $t, t' \in T$ the set $A_t \cap A_{t'}$ is open in both the space A_t and the space $A_{t'}$.

It is easily verified that the family $\mathcal{O} = \{A \subset X$: the set $A \cap A_t$ is open in A_t for each $t \in T\}$ is a topology on the set X; it is called the *weak topology determined by the family of subspaces* $\{A_t\}_{t \in T}$. Clearly a set $A \subset X$ is closed in the space X if and only if the set $A \cap A_t$ is closed in A_t for every $t \in T$. It follows from condition (1) that for each $t \in T$ the subspace topology inherited by the set A_t from the space X coincides with the original topology on the set A_t. If condition (2) is satisfied the sets A_t are closed in X whereas if (2′) is satisfied the sets A_t are open in X. Spaces with weak topologies are considered both in geometric and in algebraic topology (see Supplements 7.S.16 and 7.S.17) as well as in general topology (see Supplement 7.S.20).

7.S.16. Let $(V; K)$ be an arbitrary abstract complex and let $|K|$ be its underlying space (see Supplement 2.S.7). For each abstract simplex $A \in K$ the subspace $|A|$ of $|K|$ consisting of those functions $r: V \to \mathbf{R}_+$ for which $\{a \in V : r(a) \neq 0\} \subset A$ is homeomorphic to the unit simplex Δ^k where k is the dimension of the abstract simplex A. It is not difficult to check that the family $\{|A| : A \in K\}$ of subsets of $|K|$ satisfies conditions (1) and (2) of Supplement 7.S.15. On the set $|K|$ we may study not only

the metric ρ defined in Supplement 2.S.7 but also the weak topology defined by the family of subspaces $\{|A| : A \in K\}$; this topology is known as the *weak topology* of the underlying space of the abstract complex $(V; K)$. The set $|K|$ with topology induced by the metric ρ is denoted by $|K|_m$; the same set with weak topology is denoted by $|K|_w$. Clearly the identity $\mathrm{id}_{|K|}: |K|_w \to |K|_m$ is a continuous map so the weak topology of $|K|$ is finer than the topology induced by the metric ρ (see Problems 7.P.26–7.P.28).

7.S.17. A topological space X with a distinguished family of pairwise disjoint subspaces $\{e_t^n : t \in T_n, n = 0, 1, \ldots\}$ such that $X = \bigcup_{n=0}^{\infty} \bigcup_{t \in T_n} e_t^n$ is called a *CW-complex*, if the following is satisfied:

(1) for every integer $n \geq 0$ and each $t \in T_n$ there is a continuous map $f_t^n: \bar{B}^n \to X$ such that $f_t^n | B^n$ is a homeomorphism of B^n onto e_t^n,

(2) for every integer $n \geq 0$ and each $t \in T_n$ the set $\mathrm{cl}\, e_t^n \backslash e_t^n$ is contained in a finite union of sets e_s^i where $s \in T_i$ and $i < n$,

(3) the topology of the space X coincides with the weak topology determined by the family of subspaces $\{\mathrm{cl}\, e_t^n : t \in T_n,\ n = 0, 1, 2, \ldots\}$.

The subsets e_t^n are known as the *n-dimensional cells* of the CW-complex X. It is easy to see that the underlying space of every simplicial complex has a natural CW-complex structure: the cells are the interiors of the simplices belonging to the given complex. Similarly the underlying space of any abstract complex with weak topology has a natural CW-complex structure. Evidently every curvilinear polyhedron is a CW-complex. It transpires, however, that in general for a curvilinear polyhedron a CW-complex structure can be defined that is simpler than the natural structure resulting from the existence of the curvilinear triangulation. This demonstrates the superiority of the notion of CW-complex as compared with the notion of a curvilinear polyhedron. For example, the sphere S^m may be represented as a CW-complex consisting of two cells e^m and e^0; it is enough to take an arbitrary continuous map $f: \bar{B}^m \to S^m$ which is a homeomorphism from B^m onto $S^m \backslash \{x_0\}$ and sends the boundary of the ball $\bar{B}^m$ to the point x_0 and then to put $e^m = S^m \backslash \{x_0\}$ and $e^0 = \{x_0\}$. The class of CW-complexes is the class of topological spaces for which homotopy theory may be developed in the most natural way.

7.S.18. As noted already in Supplement 1.S.20 the notion of compactness for topological spaces was independently introduced by L. Vietoris (1921) and by P. S. Alexandrov and P. S. Urysohn (1923). Attempts at transferring this concept from metric spaces to topological spaces illustrate perfectly the difficulties of choosing an appropriate definition to appear in such a generalization (cf. Supplement 6.S.3). Apart from the concept of a compact space there was also introduced the concept of a *countably compact* space, defined as a Hausdorff space in which every countable open covering contains a finite covering, and the concept of a *sequentially compact* space defined, as a Hausdorff space in which every sequence contains a convergent subsequence.

For a long time it was a moot point which of these classes of spaces should be adopted as the correct extension of the class of compact metric spaces. There are no doubts now, as it has turned out that the class of compact spaces is best behaved

under the operations and is most frequently found in applications; also it leads to more interesting questions. Further material on countably compact spaces and sequentially compact spaces may be found in [4], p. 202–213 (see also Problems 7.P.47 and 7.P.48).

We should add that the term "compact space" is sometimes applied in the mathematical literature to arbitrary topological spaces in which open coverings contain finite coverings, that is, it is not assumed that a compact space is a Hausdorff space; such spaces would be better termed *quasi-compact*. Some of the theorems about compact spaces, among them Tikhonov's Theorem, carry over to quasi-compact spaces.

7.S.19. The topological power $D^{\mathfrak{m}}$ of the two-point discrete space $D = \{0,1\}$ with $\mathfrak{m} \geq \aleph_0$ is called the *Cantor cube of weight* $\mathfrak{m}$. A compact space X is called *dyadic* if there is a cardinal number $\mathfrak{m} \geq \aleph_0$ such that X is the image of the Cantor cube $D^{\mathfrak{m}}$ under a continuous map. It follows from Problem 6.P.39 that every metrizable compact space is dyadic. The dyadic spaces form an interesting class of compact spaces. Exhaustive information on these may be found in [4], p. 231–232 and 291; in particular it turns out that neither the space $A(\mathfrak{m})$ for $\mathfrak{m} > \aleph_0$ nor the compactification $\beta\mathbf{N}$ are dyadic.

7.S.20. A Hausdorff space X is said to be a *k-space* if the topology of X coincides with the weak topology generated by the family of compact subspaces of the space X (see Supplement 7.S.15). It may be proved that every locally compact space and every Hausdorff space satisfying the first axiom of countability is a k-space (see Problems 7.P.49 and 7.P.50). Further information about this class of spaces may be found in [4], p. 152–156.

7.S.21. Let $C(X,Y)$ denote the set of continuous maps from a topological space X into a topological space Y; we denote by $M(C,U)$ for $C \subset X$ and $U \subset Y$ the set of maps $f \in C(X,Y)$ for which $f(C) \subset U$. The family $\mathcal{B}$ consisting of finite intersections of sets of the form $M(C,U)$ where C is compact and U is open has the properties (B1)–(B2) of Theorem 7.1.20. The topology on the set $C(X,Y)$ generated by the base $\mathcal{B}$ is known as the *compact-open* topology. The space of maps $C(X,Y)$ with the compact-open topology is the most successful transfer to the context of topological spaces of the operation of forming the space of maps for metric spaces (see Supplement 7.S.22, Problem 7.P.45 and [4], p. 156–166).

7.S.22. Let F be a family of maps of a space X into a space Y. We say that the maps belonging to F are *evenly continuous* if for each point $x \in X$, every point $y \in Y$ and every neighbourhod $V \subset Y$ of the point y there exists a neighbourhood U of the point x and a neighbourhood W of the point y such that the conditions $f \in F$ and $f(x) \in W$ imply that $f(U) \subset V$. The notion introduced is the topological analogue of the notion of equicontinuous maps considered in Section 6.2. It may be shown that if X is a k-space (see Supplement 7.S.20) and Y is a regular space the closure of a set F in the space $C(X,Y)$ with compact-open topology is compact if and only if the maps in F are evenly continuous and for each point $x \in X$ the closure of the set $\{f(x) : f \in F\}$ in the space Y is compact (see [4], p. 163). (This is *Ascoli's Theorem.*)

7.S.23. Compactifications (of open sets in the plane) were first considered by C. Carathéodory (1913) in connection with some questions concerning analytic functions. Various theories of the real numbers effected a similar construction. The compactification βX was defined independently by E. Čech (1937) and M. H. Stone (1937). The compactifications of completely regular spaces, and particularly the Stone-Čech compactification, may be obtained by the general process of adjoining "ideal points" to various mathematical objects (see Problems 7.P.41 and 7.P.44). The concept of compactification plays an important rôle in analysis and in algebra.

7.S.24. An ordering relation $\leq$ may be defined on the family $\mathcal{C}(X)$ of all compactifications of a completely regular space X. Put $c_2X \leq c_1X$ if there is a continuous map $f : c_1X \to c_2X$ satisfying $fc_1 = c_2$ (see Problem 7.P.35). It follows from Theorem 7.5.34 that the Stone-Čech compactification βX is the largest element of the family $\mathcal{C}(X)$. It turns out that a smallest element exists in $\mathcal{C}(X)$ if and only if X is a locally compact space (see Problem 7.P.38); this element is the one-point compactification ωX (see Theorem 7.5.40).

7.S.25. We call a topological space X *Čech complete* if X is completely regular and the image $\beta(X)$ is a $\mathcal{G}_\delta$-set in the Stone-Čech compactification βX of the space X. The concept of Čech-completeness is a successful transfer to the context of topological spaces of the concept of complete metrizability (see Problems 7.P.52–7.P.54).

7.S.26. The concept of local finiteness was introduced by P. S. Alexandrov (1924); it is one of the most important concepts of modern general topology. Its significance became fully appreciated when J. Dieudonné introduced the class of paracompact spaces (1944), which turned out to be very useful in analysis, and after the discovery by Nagata (1950), Smirnov (1951) and Bing (1951) of the metrization theorems which bear their names. The theorem on the paracompactness of metric spaces (Theorem 7.6.4) is due to A. H. Stone (1948), who should not be confused with M. H. Stone. Some results of the latter were reviewed in Section 7.5. The proof of Theorem 7.6.4 given here is due to M. E. Rudin.

7.S.27. Apart from the Nagata-Smirnov and Bing Theorems several other metrization theorems for topological spaces are known. The more important ones may be found in [4], p. 280–288. The search for metrizability conditions for topological spaces played a large part in the development of general topology, since along the way several important and useful concepts of that theory were discovered.

7.S.28. Metrizability of a space is not preserved under either closed or open maps (see Problem 7.P.57 and 7.P.58). It turns out that if f is a closed map of a metrizable space X onto a space Y, then the space Y is metrizable if and only if Y satisfies the first axiom of countability, or equivalently, if for each $y \in Y$ the set $\operatorname{bd} f^{-1}(y)$ is a compact subspace of the space X. (This is the *Hanai-Morita-Stone Theorem*; see [4], p. 285.)

7.S.29. Further material on paracompact spaces, in particular several characterizations of this class of spaces, may be found in [4], p. 299–315. Of the theorems proved there we mention only *Michael's Theorem* which asserts that if f is a closed map of a paracompact space X onto a space Y, then the space Y is also paracompact (cf. Problem 7.P.58).

7.P. Problems

7.P.1. Show that if a family $\mathcal{C}$ of subsets of a set X has the properties (C1)–(C3) of Theorem 7.1.4, then the family $\mathcal{O}$ consisting of the sets whose complements belong to $\mathcal{C}$ is a topology on the set X. Observe that the family $\mathcal{C}$ is the family of closed subsets of the space obtained by endowing X with the topology $\mathcal{O}$. The topology $\mathcal{O}$ on the set X is called the *topology generated* by *the family $\mathcal{C}$ of closed sets.*

7.P.2. Show that if an operation cl carrying subsets of X into subsets of X has the properties (CO1)–(CO4) of Theorem 7.1.7, then the family $\mathcal{O} = \{A \subset X : \mathrm{cl}(X \backslash A) = X \backslash A\}$ is a topology on the set X. Observe that for each $A \subset X$ the set $\mathrm{cl}\, A$ is the closure of the set A in the space obtained by endowing X with the topology $\mathcal{O}$. The topology $\mathcal{O}$ on the set X is called the *topology generated by the closure operation* cl.

7.P.3. Show that if the operation int carrying subsets of X into subsets of X has the properties (IO1)–(IO4) of Theorem 7.1.9 then the family $\mathcal{O} = \{A \subset X : \mathrm{int}\, A = A\}$ is a topology on the set X. Observe that for every $A \subset X$ the set $\mathrm{int}\, A$ is the interior of the set A in the space obtained by endowing X with the topology $\mathcal{O}$. The topology $\mathcal{O}$ on the set X is called the *topology generated by the interior operation* int.

7.P.4. A subset A of a topological space X is said to be *dense-in-itself* if every point $x \in A$ is an accumulation point of the set A. Show that the union of any family of sets each of which is dense-in-itself and also the closure of a set which is dense-in-itself is again dense-in-itself. Deduce that every space is the union of two disjoint sets one of which is closed and dense-in-itself (such a set is termed *perfect*) and the other does not contain any non-empty set that is dense-in-itself (such a set is termed *scattered*).

7.P.5. Suppose given a subset A of a topological space X. A point $x \in X$ is said to be a *condensation point* of the set A, if every neighbourhood of the point x contains uncountably many points of the set. We denote the set of condensation points of the set A by A^0. Show that

$$A^0 = \mathrm{cl}\, A^0 \quad \text{and} \quad (A \cup B)^0 = A^0 \cup B^0,$$

and, assuming that the given space satisfies the second axiom of countability, that the set $A \backslash A^0$ is countable and $(A^0)^0 = A^0$. Deduce that every space satisfying the second axiom of countability is the union of two disjoint sets one of which is perfect (see Problem 7.P.4) and the other countable. (This is the *Cantor-Bendixson Theorem.*)

7.P.6. Let (X,ρ) be a pseudometric space (cf. Supplement 1.S.1). Show that the collection $\{\mathcal{B}_x\}_{x\in X}$ of families of subsets of X where $\mathcal{B}_x = \{B(x;1/n) : n = 1,2,\ldots\}$ and $B(x;1/n) = \{y \in X : \rho(x,y) < 1/n\}$ has the properties (BP1)–(BP3) of Theorem 7.1.21. The topology $\mathcal{O}$ on the set X generated by the neighbourhood system $\{\mathcal{B}_x\}_{x\in X}$ is called the *topology generated by the pseudometric* ρ. Show that the space X endowed with the topology generated by the pseudometric ρ is a T_1-space if and only if ρ is a metric.

7.P.7. Suppose given a set X which is linearly ordered by a relation $\leq$ and contains at least two points. For $a,b \in X$ put $a < b$ if $a \leq b$ and $a \neq b$. By the *intervals* on the set X we shall mean sets of the form

$$(a,b) = \{x \in X : a < x < b\}, \quad \text{where} \quad a,b \in X \quad \text{and} \quad a < b,$$

and sets of the form

$$(\leftarrow,a) = \{x \in X : x < a\} \quad \text{or} \quad (a,\rightarrow) = \{x \in X : a < x\}, \quad \text{where} \quad a \in X.$$

Show that the family $\mathcal{B}$ of all intervals on the linearly ordered set X has the properties (B1)–(B2) of Theorem 7.1.20. The topology $\mathcal{O}$ on the set X generated by the base $\mathcal{B}$ is called the *topology generated by the linear order* $\leq$.

A topological space whose topology may be generated by some linear order is known as a *linearly ordered topological space*. Prove that every linearly ordered topological space is normal. (Hint: Observe first that every linearly ordered topological space is a T_1-space and that for every set $C \subset X$ with empty derived set there is a family $\{U_x\}_{x\in C}$ of pairwise disjoint open sets in X such that $x \in U_x$ for $x \in C$. Next for disjoint closed sets $A,B \subset X$ consider the open sets

$$U_A = \bigcup\{(x,y) : x,y \in A \quad \text{and} \quad (x,y) \cap B = \emptyset\}$$

and

$$U_B = \bigcup\{(x,y) : x,y \in B \quad \text{and} \quad (x,y) \cap A = \emptyset\},$$

and check that $(\operatorname{cl} U_A) \cap B = (\operatorname{cl} U_B) \cap A = \emptyset$ and that the set $(A\backslash U_A) \cup (B\backslash U_B)$ has empty derived set.)

Recall that an element a of a linearly ordered set X satisfying $x \leq a$ for each element x of a subset A of the set X is called an upper bound of the set A in X. If a set $A \subset X$ has an upper bound in X, then we say that the set A is *bounded from above* in X. If a set $A \subset X$ has a least upper bound in X we call it the *supremum* of the set A in X. Check that if the empty set has a supremum in X, then this is the least element of the set X. Prove that a space X with topology generated by a linear order $\leq$ is connected if and only if every non-empty set $A \subset X$ which is bounded from above has a supremum in X, and further, for every pair a,b of elements of X with $a < b$ the interval (a,b) is non-empty. Show that a space X with topology generated by a linear order $\leq$ is compact if and only if every set $A \subset X$ has a supremum in X.

7.P.8. Let $M_0 = \{(x^1,x^2) \in \mathbf{R}^2 : x^1 \geq 0 \text{ and } 0 \leq x^2 \leq 2\}$, $x_0 = (-1,0)$ and $M = M_0 \cup \{x_0\}$; further let $L = \{(x^1,0) : x^1 \geq 0\}$ and for each $x = (x^1,0) \in L$

let $U(x) = \{(x^1,x^2) : 0 \leq x^2 \leq 2\} \cup \{(x^1+x^2,x^2) : 0 \leq x^2 \leq 2\}$. Check that the collection $\{\mathcal{B}_x\}_{x\in M}$ where $\mathcal{B}_x = \{U(x)\backslash B : B \subset M \text{ and } \operatorname{card} B < \aleph_0\}$ for $x \in L$, while $\mathcal{B}_x = \{\{x\}\}$ for $x \in M_0\backslash L$ and $\mathcal{B}_{x_0} = \{\{x_0\} \cup \{(x^1,x^2) \in M_0 : x^1 \geq i\}\}_{i=1}^{\infty}$ has the properties (BP1)–(BP4) and prove that the space M with topology generated by the neighbourhood system $\{\mathcal{B}_x\}_{x\in M}$ is regular but not completely regular (this example, considerably simpler than earlier ones, was recently given by A. Mysior in a paper *A regular space which is not completely regular*, Proc. Amer. Math. Soc. 81 (1981), 852–853). (Hint: Show that if a continuous map $f: M \to I$ satisfies $f(\{(x^1,0) : 0 \leq x^1 \leq 1\}) = \{1\}$ then the set $L_i = \{(x^1,0) : i-1 \leq x^1 \leq i \text{ and } f(x^1,0) = 1\} \subset L$ is infinite for $i = 1,2,\ldots$; observe for this purpose that if $f(x) = 1$ for $x \in L$ then there exists a countable set $B \subset U(x)$ such that $f(U(x)\backslash B) = \{1\}$.)

7.P.9. Show that a topological space X is completely regular if and only if X is a T_1-space and there exists a base $\mathcal{B}$ of the space X with the following properties:

(1) for each point $x \in X$ and each set $U \in \mathcal{B}$ containing x there is a set $V \in \mathcal{B}$ with $x \notin V$ and $U \cup V = X$,

(2) if the sets $U,V \in \mathcal{B}$ satisfy $U \cup V = X$, then there are sets $U',V' \in \mathcal{B}$ such that $X\backslash V \subset U'$, $X\backslash U \subset V'$ and $U' \cap V' = \emptyset$.

(Hint: Modify the proof of Urysohn's Lemma.)

7.P.10. Show directly that for the disjoint closed sets $A = \{(x^1,0) : x^1 \text{ is an irrational number}\}$ and $B = \{(x^1,0) : x^1 \text{ is a rational number}\}$ of the Niemytzki plane L there do not exist disjoint open sets $U,V \subset L$ such that $A \subset U$ and $B \subset V$. (Hint: See Problem 6.P.14).

7.P.11. Show that for every countable discrete family $\{F_i\}_{i=1}^{\infty}$ of closed subsets of a normal space X there exists a family $\{U_i\}_{i=1}^{\infty}$ of open subsets of the space X such that $F_i \subset U_i$ for $i = 1,2,\ldots$, and $\operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset$ for $i \neq j$. (Hint: Make use of the Tietze-Urysohn Theorem.)

7.P.12. We call a family $\{A_t\}_{t\in T}$ of subsets of a set X *point-finite* if for every point $x \in X$ the set $\{t \in T : x \in A_t\}$ is finite. Prove that for every point-finite open covering $\{U_t\}_{t\in T}$ of a normal space X there exists an open covering $\{V_t\}_{t\in T}$ of the space such that $\operatorname{cl} V_t \subset U_t$ for every $t \in T$. (Hint: Consider the set of all maps G of the set T into the family of open subsets of the space X such that $\bigcup_{t\in T} G(t) = X$ and such that for each $t \in T$ either $G(t) = U_t$ or $\operatorname{cl} G(t) \subset U_t$. Define an order $\leq$ on this set by putting $G_1 \leq G_2$ if $G_2(t) = G_1(t)$ for each index $t \in T$ such that $G_1(t) \neq U_t$. Use the Kuratowski-Zorn Theorem.)

7.P.13. Show that a subset A of a normal space X is an open $\mathcal{F}_\sigma$-set (a closed $\mathcal{G}_\delta$-set) if and only if there is a continuous function $f: X \to I$ with $f^{-1}(I\backslash\{0\}) = A$ (with $f^{-1}(0) = A$).

7.P.14. Observe that every metrizable space is perfectly normal (see Supplement 7.S.4). Check that the space $A(\mathbf{m})$ is perfectly normal if and only if $\mathbf{m} = \aleph_0$. Show that the Sorgenfrey line is perfectly normal.

Prove that for any T_1-space X the following conditions are equivalent:

(1) the space X is perfectly normal,

(2) for every open set $U \subset X$ there is a continuous function $f: X \to I$ such that $f^{-1}(I \setminus \{0\}) = U$,

(3) for every closed set $F \subset X$ there is a continuous function $f: X \to I$ such that $f^{-1}(0) = F$,

(4) for every pair of disjoint closed sets $A, B \subset X$ there is a continuous function $f: X \to I$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$,

(5) for every open set $W \subset X$ there is a sequence $W_1, W_2, \ldots$ of open sets of X such that $W = \bigcup_{n=1}^{\infty} W_n$ and $\operatorname{cl} W_n \subset W$ for $n = 1, 2, \ldots$

7.P.15. Deduce from Lemma 7.3.10 that every Lindelöf space (see Supplement 7.S.5) is normal. Observe that if X is a Lindelöf space and A is a closed subset of X, then the subspace A is a Lindelöf space. Show that in order that all the subspaces of a Lindelöf space X themselves be Lindelöf spaces it is necessary and sufficient that X is perfectly normal (see Supplement 7.S.4).

7.P.16. Prove that the Sorgenfrey line is a Lindelöf space (see Supplement 7.S.5) and observe that the Cartesian product of two Lindelöf spaces need not be a Lindelöf space. (Hint: Show that for every open covering $\{U_t\}_{t \in T}$ of the Sorgenfrey line K the difference $K \setminus \bigcup_{t \in T} V_t$ where V_t is the interior of the set U_t in the usual topology of the real line is countable.)

7.P.17. Prove that if a space X is normal and F is an $\mathcal{F}_\sigma$-set in the space X then the subspace F is normal. (Hint: Use Lemma 7.3.10.)

7.P.18. A topological space X is hereditarily normal if every subspace of X is normal (see Supplement 7.S.11). Observe that every metrizable space is hereditarily normal. Deduce from the equivalence of conditions (1) and (2) of Problem 7.P.14 that every perfectly normal space is hereditarily normal (see Supplement 7.S.4). Check that the space $A(\mathfrak{m})$ is hereditarily normal for every $\mathfrak{m} \geq \aleph_0$. Give an example of a hereditarily normal space which is not perfectly normal.

Prove that for any T_1-space X the following conditions are equivalent:

(1) the space X is hereditarily normal,

(2) for every open subset $U \subset X$ the subspace U is normal,

(3) for every pair of sets $A, B \subset X$ satisfying $A \cap \operatorname{cl} B = \emptyset = B \cap \operatorname{cl} A$ there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

7.P.19. Let X be any topological space. Every non-empty subset of the space X which is the intersection of open-and-closed sets and is inclusion-minimal with respect to this property is called a *quasi-component* of the space X. In other words, a non-empty set $K \subset X$ is a quasi-component of the space X if K is the intersection of some family of open-and-closed sets and for every open-and-closed set $W \subset X$ with $K \cap W \neq \emptyset$ we have the inclusion $K \subset W$. Check that the quasi-components of a space are closed and pairwise disjoint and that their union is the whole space. Prove that if X is a compact space the components and the quasi-components of X coincide.

7.P.20. Prove that the topological product of $\mathfrak{c}$ many separable spaces is separable. (Hint: Applying Corollary 7.4.27 and Theorem 7.2.7 observe that the problem reduces to proving the separability of the topological power $[D(\aleph_0)]^{\mathfrak{c}}$. Next, in the topological product $\bigtimes_{t\in I} N_t$ where N_t is the set of natural numbers $\mathbf{N}$ with discrete topology for each $t \in T$, consider the set A consisting of all functions $f: I \to \mathbf{N}$ which are constant on a finite number of disjoint open intervals with rational endpoints and take the value 1 everywhere else.)

7.P.21. Prove that the topological product of more than $\mathfrak{c}$ Hausdorff spaces each of weight exceeding 1 is not a separable space. Give an example of a T_1-space X with $\operatorname{card} X > 1$ such that all topological powers of X are separable.

7.P.22. Let A be any countable set that is dense in the Tikhonov cube $I^{\mathfrak{c}}$ (see Problem 7.P.20). Show that the subspace A of the space $I^{\mathfrak{c}}$ does not have a countable local base at any point.

7.P.23. Prove that the space $\mathbf{N}^{\aleph_1}$, where $\mathbf{N}$ is the set of natural numbers with the discrete topology is not normal. (Hint: Let $\mathbf{N}^{\aleph_1} = \bigtimes_{t\in T} N_t$ where $N_t = \mathbf{N}$ for each $t \in T$ and $\operatorname{card} T = \aleph_1$. Consider the sets $A_1, A_2 \subset \mathbf{N}^{\aleph_1}$ where A_i consists of those points $\{x_t\}$ such that for each $j \neq i$ the equation $x_t = j$ holds for at most one value of $t \in T$. Show that the sets A_1 and A_2 are disjoint and closed in $\mathbf{N}^{\aleph_1}$ but that for any open sets $U_1, U_2 \subset \mathbf{N}^{\aleph_1}$ with $A_1 \subset U_1$ and $A_2 \subset U_2$ the intersection $U_1 \cap U_2$ is non-empty).

7.P.24. Observe that every zero-dimensional space is a Tikhonov space and that if X is separable and its topology is induced by a metric ρ, then X is zero-dimensional if and only if the metric space (X, ρ) satisfies the equation $\operatorname{ind} X = 0$. Observe that the Cantor cube $D^{\aleph_0}$ (see Supplement 7.S.19) is homeomorphic to the Cantor set. Prove that for every cardinal number $\mathbf{m} \geq \aleph_0$ the Cantor cube $D^{\mathbf{m}}$ is a universal space for the zero-dimensional spaces of weight not exceeding $\mathbf{m}$. Show that $\chi(x, D^{\mathbf{m}}) = \mathbf{m}$ for every point $x \in D^{\mathbf{m}}$ and deduce that $\chi(x, I^{\mathbf{m}}) = \mathbf{m}$ for every point $x \in I^{\mathbf{m}}$.

7.P.25. The *Helly space* is the subspace X of the Tikhonov cube $I^{\mathfrak{c}} = \bigtimes_{i\in I} I_t$, where $I_t = I$ for $t \in I$, consisting of the non-decreasing functions $f: I \to I$. Prove that:

a) the Helly space is compact,
b) the Helly space contains a subspace homeomorphic to the discrete space of cardinality $\mathfrak{c}$ and a subspace homeomorphic to the Sorgenfrey line,
c) the Helly space is not hereditarily normal,
d) the Helly space satisfies the first axiom of countability,
e) the Helly space is separable.

(Hint: In proving (c) observe that the space X^2 is imbeddable in the space X; in the proof of (d) use the fact that each function in X has countably many points of discontinuity; to prove (e) use the hint for Problem 7.P.20).

7.P.26. Let X be a topological space with the weak topology generated by a family of subspaces $\{A'_t\}_{t\in T}$ (see Supplement 7.S.15). Show that a map f of the space X into a topological space Y is continuous if and only if the restriction $f|A_t$ is continuous for each $t \in T$.

7.P.27. Show that for every abstract complex $(V; K)$ the underlying space $|K|_w$ of the complex with weak topology is a normal space. (Hint: For every pair of disjoint closed sets $A, B \subset |K|_w$ construct a continuous function $f: |K|_w \to I$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.)

7.P.28. Prove that for every abstract complex $(V; K)$ the following conditions are equivalent:

(1) the complex $(V; K)$ is locally finite,
(2) the space $|K|_w$ is locally compact,
(3) the map $\mathrm{id}_{|K|}: |K|_w \to |K|_m$ is a homeomorphism,
(4) the space $|K|_w$ is metrizable.

7.P.29. A family $\mathcal{M}$ of subsets of a topological space X is called a *network of the space* X if every non-empty open set of the space may be expressed as a union of some collection of members of $\mathcal{M}$. The smallest cardinal number of the form card $\mathcal{M}$, where $\mathcal{M}$ is a network of the space X, is called the *network weight of the space* X and is denoted $nw(X)$. Observe that $nw(X) \leq \operatorname{card} X$ and $nw(X) \leq w(X)$ for any space X. Check that if X is a metrizable space then $nw(X) = w(X)$. Prove also that the equation $nw(X) = w(X)$ holds for every compact space X (this is *Arkhangelskii's Theorem*). Observe that if $f: X \to Y$ is a continuous map and $f(X) = Y$, then for every network $\mathcal{M}$ of the space X the family $\{f(M) : M \in \mathcal{M}\}$ is a network of the space Y. Deduce from this Theorem 7.5.21. (Hint: To prove the inequality $w(X) \leq nw(X)$ for a compact space X consider a network $\mathcal{M}$ of the space such that card $\mathcal{M} = nw(X)$ and for each pair of sets $M_1, M_2 \in \mathcal{M}$ for which this is possible choose disjoint open sets $U_1, U_2 \subset X$ such that $M_i \subset U_i$ for $i = 1, 2$; generate a topology on the set X with the base $\mathcal{B}$ consisting of all finite intersections of the selected open sets.)

7.P.30. Give an example of a perfectly normal Lindelöf space X for which $nw(X) < w(X)$ (see Supplement 7.S.5 and Problem 7.P.29). (Hint: Consider the topology on the plane generated by the neighbourhood system $\{\mathcal{B}_{(x^1,x^2)}\}_{(x^1,x^2)\in\mathbf{R}^2}$, where $\mathcal{B}_{(x^1,x^2)}$ is any local base in the plane $\mathbf{R}^2$ at the point (x^1, x^2) with $x^2 \neq 0$, while $\mathcal{B}_{(x^1,0)} = \{\{(x^1, 0)\} \cup U_n(x^1) : n = 1, 2, \ldots\}$ and $U_n(x^1)$ consists of the points which both lie in the interior of a disc of radius $1/n$ centred at $(x^1, 0)$ and at the same time are outside of the two discs of radius n tangent to the axis of the abscissae at $(x^1, 0)$.)

7.P.31. Prove that a Hausdorff space X is compact if and only if every covering of the space X whose members belong to a fixed subbase $\mathcal{P}$ of the space contains a finite covering. (This is *Alexander's subbase theorem.*) (Hint: Apply the Kuratowski-Zorn Theorem and observe that if a T_2-space X is not compact, then the family of those open coverings of the space X which do not contain finite coverings, ordered by inclusion, has a minimal element $\mathcal{R}$; show that $\mathcal{R} \cap \mathcal{P}$ is a covering of the space X.)

7.P.32. Derive Tikhonov's Theorem (7.5.16) from Alexander's subbase theorem (see Problem 7.P.31).

7.P.33. A topological space X is called a *continuum* if X is both compact and connected. Prove that

a) if f is a continuous map of a continuum X onto a Hausdorff space Y, then the space Y is also a continuum,

b) the topological product of continua is a continuum,

c) if $X = \bigcup_{i=1}^{m} X_i$, where each of the subspaces X_i is a continuum for $i = 1, 2, \ldots, m$ and $\bigcap_{i=1}^{m} X_i \neq \emptyset$, then the space X is a continuum,

d) if the non-empty continua X_n for $n = 1, 2, \ldots$ form a decreasing sequence then the intersection $X = \bigcap_{n=1}^{\infty} X_n$ is a non-empty continuum (see Theorem 1.8.20),

e) no continuum can be expressed as a union of countably many pairwise disjoint closed sets at least two of which are non-empty (see Theorem 6.5.8).

7.P.34. Let A be a dense subset of a topological space X and let Y be a compact space. Prove that a continuous map $f: A \to Y$ has a continuous extension $f^*: X \to Y$ if and only if for each pair of disjoint closed sets $B_1, B_2 \subset Y$ the closures of the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ in the space X are disjoint. (Hint: Check that for every point $x \in X$ the intersection $\bigcap\{\operatorname{cl} f(A \cap U) : U \in \mathcal{B}_x\}$, where $\mathcal{B}_x$ is the family of all neighbourhoods of the point x, is a singleton in the space Y.)

7.P.35. Show that the relation $\leq$ in the family $\mathcal{C}(X)$ of all compactifications of a completely regular space X is an ordering relation (see Supplement 7.S.24).

7.P.36. Prove that the compactifications c_1X and c_2X of a completely regular space X are equivalent if and only if for every pair of disjoint closed sets $A, B \subset X$ the conditions $\operatorname{cl} c_1(A) \cap \operatorname{cl} c_1(B) = \emptyset$ and $\operatorname{cl} c_2(A) \cap \operatorname{cl} c_2(B) = \emptyset$ are equivalent. (Hint: Use Problem 7.P.34).

7.P.37. Show that if a compactification cX of a normal space X has the property that for every pair of disjoint closed sets $A, B \subset X$ the closures $\operatorname{cl} c(A)$ and $\operatorname{cl} c(B)$ of the sets $c(A)$ and $c(B)$ in the compactification cX are disjoint, then the compactification cX and the Stone-Čech compactification βX are equivalent.

7.P.38. Let X be any non-compact completely regular space. Prove that if in the family $\mathcal{C}(X)$ of all compactifications of the space X ordered by the relation $\leq$ there exists a smallest element, then the space X is locally compact (see Supplement 7.S.24).

7.P.39. Prove that for the space $\mathbf{N}$ of natural numbers with discrete topology we have $w(\beta\mathbf{N}) = \mathfrak{c}$ and $\operatorname{card} \beta\mathbf{N} = 2^{\mathfrak{c}}$. (Hint: Using Problem 7.P.20 construct a continuous map of the compactification $\beta\mathbf{N}$ onto the Tikhonov cube $I^{\mathfrak{c}}$.

7.P.40. Let X be any completely regular space. A proper subset Δ of a ring $R \subset C(X)$ is called an *ideal* in the ring R if it satisfies the two conditions:

(IR1) if $f, g \in \Delta$ then $f + g \in \Delta$,

(IR2) if $f \in \Delta$ and $g \in R$, then $f \cdot g \in \Delta$.

An ideal Δ in the ring R is said to be *maximal* if Δ is not a proper subset of any ideal in the ring R. Show that every ideal in the ring R is contained in a maximal ideal in R. Prove that for a space X to be compact it is necessary and sufficient that for

every maximal ideal Δ in the ring $C(X)$, or, equivalently for every maximal ideal Δ in the ring $C^*(X)$, there exists a point $x \in X$ such that $f \in \Delta$ if and only if $f(x) = 0$. (Hint: To prove the necessity of this condition assume that there is an ideal Δ which for each point $x \in X$ contains a function f_x with values in the non-negative reals such that $f_x(x) = 1$ and consider the covering $\{f_x^{-1}(\frac{1}{2}, \frac{3}{2})\}_{x \in X}$ of the space X. To prove the sufficiency of the condition assume that the space X is non-compact and consider an open covering $\{U_t\}_{t \in T}$ which contains no finite covering, for every $x \in X$ choose an index $t(x) \in T$ such that $x \in U_{t(x)}$ and a function $f_x \in C^*(X)$ such that $f_x(x) = 1$ and $f_x(X \setminus U_{t(x)}) = \{0\}$; then consider an ideal Δ containing all the selected functions.)

7.P.41. Let X be any completely regular space, R the ring $C(X)$ or $C^*(X)$ and $\mathcal{M}$ the set of all maximal ideals in the ring R (see Problem 7.P.40). Define a topology on the set $\mathcal{M}$ by taking as base the family of all sets of the form $U(f) = \{\Delta \in \mathcal{M} : f \notin \Delta\}$ and show that $\mathcal{M}$ is a compact space. Verify that the map taking each point $x \in X$ to the ideal $\Delta(x) = \{f \in R : f(x) = 0\} \in \mathcal{M}$ is a homeomorphic imbedding $\Delta: X \to \mathcal{M}$ such that $\operatorname{cl} \Delta(X) = \mathcal{M}$. Prove that the space $\mathcal{M}$ is the Stone-Čech compactification of the space X. Deduce that the compact spaces X and Y are homeomorphic if and only if the rings $C(X)$ and $C(Y)$ are isomorphic. (Hint: To prove that $\mathcal{M} = \beta M$ use Problem 7.P.34.)

7.P.42. Let X be any completely regular space. If a ring $R \subset C^*(X)$ is closed under uniform convergence, contains all the constant functions, and separates points from closed sets, then we say that R is a *complete ring of functions* on the space X. Prove that the construction in the last problem applied to the set $\mathcal{M}$ of maximal ideals of an arbitrary complete ring R of functions on the space X yields a compactification $c_R X$ of the space X. Verify that R consists of all those continuous functions $f: X \to \mathbf{R}$ for which there is a continuous function $f^*: c_R X \to \mathbf{R}$ such that $f^* c_R = f$. Deduce that assigning to each ring R the compactification $c_R X$ establishes a one-to-one correspondence between complete rings of functions on the space X and the compactifications of the space X. (Hint: To prove that $\mathcal{M}$ is a Hausdorff space, observe that no function $f \in R$ satisfying $\frac{1}{2} \leq f(x) \leq \frac{3}{2}$ for $x \in X$ can belong to any ideal; for this make use of the equality $1/f = \sum_{n=0}^{\infty}(1-f)^n$. Construct a compactification cX of the space X with the property that R consists of all continuous functions $f: X \to \mathbf{R}$ for which there is a continuous function $f^*: cX \to \mathbf{R}$ with $f^* c = f$ and show that the compactifications $c_R X$ and cX are equivalent.)

7.P.43. Deduce from the Stone-Weierstrass Theorem that every continuous function $f: \mathbf{R} \to \mathbf{R}$ with period 2π, that is satisfying $f(r + 2\pi) = f(r)$ for each $r \in \mathbf{R}$, is a uniform limit of a sequence of functions of the form $\sum_{n=0}^{m}(a_n \cos nr + b_n \sin nr)$, where m is a natural number, $a_n, b_n \in \mathbf{R}$ for $n = 0, 1, \ldots, m$ and $r \in \mathbf{R}$. (Hint: Consider the ring $C(S^1)$ of all continuous real-valued functions defined on the circle S^1.)

7.P.44. Prove that the Stone space of a Boolean algebra is determined uniquely up to homeomorphism. Observe that the Stone space of the Boolean algebra of all subsets of a set of cardinality $\mathfrak{m} \geq \aleph_0$ is the Stone-Čech compactification of the discrete space $D(\mathfrak{m})$. (Hint: Make use of Problem 7.P.37.)

7.P.45. Let X be a non-empty compact space and Y a metrizable space with a fixed metric ρ. Show that the metric $\tilde{\rho}$ on the set $C(X,Y)$ (see Section 6.2) determines the compact-open topology (see Supplement 7.S.21). Deduce that if X is a non-empty compact space then for any equivalent metrics ρ and ρ' on the set Y the metrics $\tilde{\rho}$ and $\tilde{\rho}'$ on the set $C(X,Y)$ are equivalent (cf. Problem 6.P.4).

7.P.46. Endow the square $X = I^2$ with a linear order by taking $(x^1,x^2) \leq (y^1,y^2)$ if $x^1 < y^1$ or if $x^1 = y^1$ and $x^2 \leq y^2$. The space X with topology generated by the linear order $\leq$ is called the *lexicographic square* (see Problem 7.P.7).

Prove that the lexicographic square is a compact space satisfying the first axiom of countability which is neither separable, nor perfectly normal (see Supplement 7.S.4). Observe that the Sorgenfrey line is imbeddable in the lexicographic square. The subspace $K_2 = \{(x^1,0) : 0 < x^1 \leq 1\} \cup \{(x^1,1) : 0 \leq x^1 < 1\}$ of the lexicographic square is called variously the *split interval*, the *double arrow space* or the *lexicographic double interval*. Prove that the space K_2 is compact, hereditarily separable (see Supplement 7.S.11) and perfectly normal. Observe that the space K_2 may be represented as the union of two subspaces both of which are homeomorphic to the Sorgenfrey line.

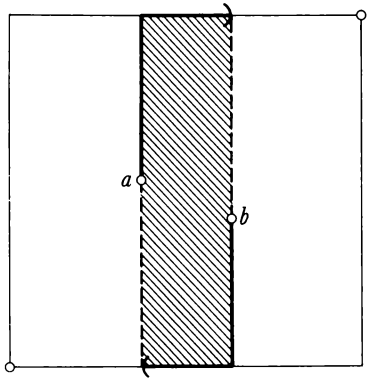


Fig.163. The interval (a,b) in the lexicographic square.

7.P.47. Observe that every compact space is countably compact. Show that every sequentially compact space is countably compact. Give an example of a compact space which is not sequentially compact and an example of a countably compact space which is not compact (see Supplement 7.S.18). (Hint: The space $\beta\mathbf{N}$ is not sequentially compact. The subspace of the Tikhonov cube I^c consisting of those points which have only countably many coordinates different from zero is countably compact (see also Problem 7.P.48).)

7.P.48. Let W be the set of ordinal numbers not exceeding the first uncountable ordinal ω_1 with the topology generated by the natural linear order $\leq$ on this set (see Problem 7.P.7) and let $W_0 = W\backslash\{\omega_1\}$ be the subspace of the space W. Prove that $\beta W_0 = W$. Observe that the space W_0 is countably compact.

7.P.49. Observe that every locally compact space is a k-space (see Supplement 7.S.20). Prove that a Hausdorff space is a k-space if and only if it is homeomorphic to a quotient space of a locally compact space.

7.P.50. Show that every Hausdorff space satisfying the first axiom of countability is a k-space (see Supplement 7.S.20).

7.S.51. Show that every locally compact space is Čech-complete (see Supplement 7.S.25).

7.S.52. Show that if X is a Čech-complete space and the sets $B_1, B_2, \ldots$ are closed in X and have empty interior then the union $B = \bigcup_{n=1}^{\infty} B_n$ has empty interior. (See Supplement 7.S.25 and Theorem 6.4.2.)

7.P.53. Prove that a completely regular space X is Čech-complete if and only if X has a countable family $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open coverings with the property that every centred family $\{F_t\}_{t\in T}$ of closed sets of X containing for each natural number n a set which is contained in some member of the covering $\mathcal{U}_n$, has non-empty intersection (see Supplement 7.S.25).

7.P.54. Prove that a topological space is completely metrizable if and only if it is metrizable and Čech-complete (see Supplement 7.S.25). (Hint: See Problems 6.P.13 and 6.P.22.)

7.P.55. Show that, in order for the topology of a metrizable space X to be induced by a metric ρ with the following property: each subspace $A \subset X$ is compact if and only if it is closed and bounded under ρ, it is necessary and sufficient for the space X to be locally compact and to satisfy the second axiom of countability.

7.P.56. Prove that for every cardinal number $\mathbf{m} \geq \aleph_0$ the topological power $[J(\mathbf{m})]^{\aleph_0}$ of the hedgehog with $\mathbf{m}$ spikes (see Example 6.3.5) is a universal space for metrizable spaces of weight not exceeding $\mathbf{m}$. (Hint: Modify the proof of Theorem 7.6.8; use the perfect normality of metrizable spaces (see condition (2) in Problem 7.P.14).)

7.P.57. Give an example of a closed map of the real line onto a non-metrizable space.

7.P.58. Give an example of an open map of a subspace of the plane onto a Hausdorff space which is not regular. (Hint: Consider the subspace $X = \{(0,0)\} \cup \{(1,1/n) : n = 1,2,\ldots\} \cup \{(i, 1/n + 1/n \cdot m) : i = 0,1,\ n = 1,2,\ldots,\ m = n, n+1, \ldots\}$ of the plane $\mathbf{R}^2$.)

7.P.59. Let X be a metric space. Prove that if every point $x \in X$ has a neighbourhood U which, regarded as a subspace of the space X, is completely metrizable, then the space X is completely metrizable. (Hint: Use Problem 6.P.22 and Theorem 7.6.11 to show that the space X is a $\mathcal{G}_\delta$-set in its completion $\tilde{X}$.)

7.P.60. Prove that every locally separable space (see Problem 6.P.8) which is paracompact may be expressed as the union of pairwise disjoint open separable subspaces. (Hint: Consider a locally finite open covering $\mathcal{U}$ of the space X whose members are

separable subspaces of the space X, define an equivalence relation R by having URU' if there is a sequence of open sets $U_0, U_1, \ldots, U_k \in \mathcal{U}$ such that $U_0 = U$, $U_k = U'$ and $U_{j-1} \cap U_j \neq \emptyset$ for $j = 1, 2, \ldots, k$ and investigate the unions of the equivalence classes of this relation.)

7.P.61. Let A be a closed subspace of a metric space X. Prove that to every continuous function $f: A \to \mathbf{R}$ we may assign a continuous extension $f^*: X \to \mathbf{R}$ in such a way that $\sup\{|f^*(x)| : x \in X\} = \sup\{|f(x)| : x \in A\}$ and $(t_1 f_1 + t_2 f_2)^* = t_1 f_1^* + t_2 f_2^*$ for all $f_1, f_2: A \to \mathbf{R}$ and $t_1, t_2 \in \mathbf{R}$. (Hint: Consider a locally finite open covering $\{V_t\}_{t \in T}$ of the subspace $X \setminus A$ that refines the covering $\{B(x; \frac{1}{4}\rho(x, A))\}_{x \in X \setminus A}$ of $X \setminus A$, and for each $t \in T$ choose a point $x_t \in X \setminus A$ such that $V_t \subset B(x_t; \frac{1}{4}\rho(x_t, A))$ and a point $a_t \in A$ such that $\rho(a_t, x_t) < \frac{5}{4}\rho(x_t, A)$. Define continuous functions $g_t: X \setminus A \to I$ such that $g_t(X \setminus V_t) \subset \{0\}$ for $t \in T$ and $\sum_{t \in T} g_t(x) = 1$ for each $x \in X \setminus A$ and take $f^*(x) = f(x)$ for $x \in A$ and $f^*(x) = \sum_{t \in T} f(a_t) g_t(x)$ for $x \in X \setminus A$.)

7.P.62. Show that every countably compact paracompact space is compact (see Supplement 7.S.18).

7.P.63. Give an example of a normal space which is not paracompact.

7.P.64. Show that every Lindelöf space (see Supplement 7.S.5) is paracompact.

7.P.65. Prove that if a paracompact space X contains a dense subspace which is a Lindelöf space (see Supplement 7.S.5) then the space X is a Lindelöf space too. Observe that every separable paracompact space is a Lindelöf space.

7.P.66. Prove that every locally compact paracompact space X may be expressed as a pairwise disjoint union of open subspaces which are locally compact Lindelöf spaces (see Supplement 7.S.5). (Hint: See the hint to Problem 7.P.60.)

7.P.67. Prove that if X is a compact spaces and Y is a paracompact space, then the topological product $X \times Y$ is a paracompact space.

7.P.68. Give an example of two paracompact spaces whose topological product is not a paracompact space. (Hint: See Problem 7.P.16.)

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